On the Properties of Acyclic Bilateral Phase Type Distributions

András Horváth
Dipartimento di Informatica, Università di Torino
horvath@di.unito.it

Miklós Telek
Department of Telecommunications, Technical University of Budapest
telek@hit.bme.hu

ABSTRACT
This paper aims to collect a set of properties of bilateral phase type (BPH) distributions that help their practical use. Similar to the relation of the phase type (PH) and the acyclic phase type (APH) distributions, it turns out that restricting the underlying Markov chain to be acyclic results in a more tractable class of distributions, the acyclic bilateral phase type (ABPH) distributions. Here we present a canonical representation, moment bounds and moment matching methods for ABPH distributions.

Keywords
phase type distributions, bilateral phase type distributions, moment bounds, moment fitting.

1. INTRODUCTION
Because of their simplicity and tractability Markovian models are widely used in applied stochastic modelling. For the first sight, it is a limitation of Markov models that the inter-event times of continuous time Markov chains (CTMC) are exponentially distributed, but a set of exponentially distributed events controlled by a background CTMC can be used to model non-exponential durations. The class of distributions composed this way is referred to as phase type (PH) distributions (see, e.g. [12, 11]). Phase type distributions find application in a wide range of fields and they are also effectively applied to approximate general (non-PH) continuous distributions (assuming no probability mass at zero) can have support on \((0, \infty), (\infty, 0)\) or \((-\infty, \infty)\).

Together with the definition of BPH distributions the authors posed several research problems in [1]. In this paper we consider some of these problems and extend the classification with the class of acyclic bilateral phase type (ABPH) distributions. Throughout this paper we consider the case with support on \((-\infty, \infty)\) with no probability mass at zero. The specialisation of the results to the other cases follows the same pattern as the similar treatment of PH distributions.

The rest of the paper is organised as follows. Section 2 introduces the class of bilateral phase type distributions. Section 3 presents the class of acyclic bilateral phase type distributions and some of its properties. Symbolic moment matching expressions and moment bounds are provided in Section 4 and 5, respectively. Section 6 presents a moment matching method, which generate BPH distributions based on their first 3 moments, and Section 7 demonstrates the behaviour of this method.

2. BILATERAL PHASE TYPE DISTRIBUTIONS
Following [1] we define the class of BPH distribution with the help of Markov reward models.

Let \(\{Z(t), t \geq 0\}\) be a continuous time Markov chain (CTMC) on state space \(S = \{1, 2, \ldots, N + 1\}\) with generator \(Q = \{q_{ij}\}\) and initial probability vector \(\gamma\). The state space is composed by \(N\) transient states \(\{1, 2, \ldots, N\}\) and an absorbing one \(\{N + 1\}\).

We assign a real valued constant, referred to as reward rate, to the states of \(S\) and a real valued reward function, \(B(t)\), to \(Z(t)\) such that \(B(t)\) describes the reward accumulated by \(Z(t)\) in the interval \((0, t)\). \(B(0) = 0\) and during the sojourn in state \(i \in S\) the amount of accumulated reward increases at rate \(r_i\), i.e., \(dB(t)/dt = r_i\), when \(Z(t) = i\). If \(r_i\) is negative \(B(t)\) decreases during the sojourn in \(i\). The amount of reward accumulated during the interval \((0, t)\) is

\[
B(t) = \int_0^t r_{Z(\tau)} d\tau
\]

Let \(R\) be the diagonal matrix composed of the reward rates of the transient states, \(R = \text{diag}(r_i)\).

The time to absorption of the CTMC, \(\tau = \inf\{t : Z(t) = N + 1\}\), is said to be phase type distributed with initial probability vector \(\alpha\), composed by the elements of \(\gamma\) associated with the transient states, and generator \(A\), composed by the
elements of $Q$ associated with the transient states:

$$
\gamma = [\alpha, \cdot] ; \quad Q = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.
$$

**Definition 1.** [1] The reward accumulated till absorption, $B(r)$, is bilateral phase type (BPH) distributed with initial probability vector $\alpha$, transient generator $A$ and reward matrix $R$.

[1] presents several properties of BPH distributions:

1. The class of BPH distributions inherits several useful properties of PH distributions like denseness, closure (for finite sum, mixture, linear combination), tractability.

2. The $(\alpha, A, R)$ representation is not unique. E.g., if $C$ is a non-singular diagonal matrix with positive entries, then $(\alpha, CA, CR)$ defines the same BPH distribution. Thanks to this property all rates can be changed to 1 or -1 without changing the distribution.

3. One can eliminate the states with zero reward rates.

4. One of the most interesting properties of BPH distributions is that they can be composed as the mixture of a positive and a negative PH distribution.

A consequence of this composition, not mentioned in [1], is that the BPH class also inherits closure property for minimum and maximum operation.

### 2.1 Moments of BPH distributions

Based on property 4 we introduce probability $p$, which is the probability that the BPH distribution is positive, and 2 (positive) PH distributions, $PH^+$ and $PH^-$. The BPH takes the value of $PH^+$ with probability $p$ and $(-1)^i PH^-$ with probability $(1-p)$, where $i$ is the $i^{th}$ moment of the BPH distribution satisfy

$$
\mu_i = p\mu_i^+ + (1-p)\mu_i^-.
$$

### 2.2 Continuity behaviour at zero

The potential discontinuity of the probability density function (pdf) of BPH distributions is mentioned in [1]. Practical applications of BPH distributions might require that their pdf functions exhibit a given continuity behaviour at zero. In particular we pay special attention to those ABPH distributions whose pdf has equal left and right limit at zero. These distributions will be referred to as ABPH-C.

### 3. ACYCLIC BILATERAL PHASE TYPE DISTRIBUTIONS

To extend the applicability and tractability of BPH distributions we introduce the subclass with acyclic underlying Markov chain.

**Definition 2.** The BPH distribution, whose underlying Markov chain does not contain cycle (i.e., $A$ is an upper triangular matrix) is referred to as acyclic bilateral phase type (ABPH) distribution.

Indeed ABPH distributions inherit the properties of the APH distributions, such as denseness, closure (for finite sum, mixture, linear combination, min/max operation), tractability, real eigenvalues and we provide further consequences below.

ABPH($n^+, n^-$) denotes an acyclic bilateral phase type distribution for which the number of phases with positive (negative) rate is $n^+ (n^-)$, where $n^+ \geq 0$ ($n^- \geq 0$).

**Theorem 3.** Any ABPH($n^+, n^-$) distribution can be transformed into the canonical form depicted in Figure 1.

![Figure 1: Canonical form of an ABPH($n^+, n^-$) distribution with $n^+ = n$ and $n^- = m$](image)

**Proof.** Consider a path with $k^- \leq n^-$ negative and $k^+ \leq n^+$ positive phases leading from an initial state to the absorbing state as depicted in Figure 2. The path can be reorganised in such way, as illustrated in Figure 3, that at first the phases with positive rate are passed through. This transformation obviously does not modify the probability density of the fluid accumulated along the path. The same probability density function can be realized by the structure depicted in Figure 4. The initial probabilities of the negative phases have to be set as

$$
p_i^- = Pr(\text{fluid accumulated along the path of Figure 3})
$$

is negative and fluid level crosses level zero during the stay in the $i^{th}$ negative phase.}

The initial probabilities of the positive phases can be determined by interchanging the role of the negative and positive phases.

The decomposition into negative and positive parts described above can be performed for all possible paths of the ABPH($n^+, n^-$) distribution. By applying the results presented in [5], the probabilistic mixture of the negative (positive) parts of all paths can be realized by a single sequence of $n^+$ positive ($n^-$ negative) phases; this sequence is in canonical form. The combination of the sequence with negative
4. SYMBOLIC EXPRESSIONS FOR 3 MOMENT MATCHING

4.1 Moments matching ABPH(1,1)

Assuming there is no probability mass at zero the first 3 moments define the parameters of an ABPH(1,1) based on (2) which takes the form

\[ \mu_k = k!p(\lambda^+)^{-k} + k!(1-p)(\lambda^-)^{-k}, \quad k = 1, 2, 3. \quad (4) \]

(4) simplify to a second order equation, whose relevant solution is

\[
p = \frac{-27\mu_1^2 - 18\mu_2 + 18\mu_3 (\mu_3 - \sqrt{c})}{2c} + \frac{12\mu_3 (-2\mu_3 + \sqrt{c}) + 9\mu_1 \mu_2 (-2\mu_3 + \sqrt{c})}{2c}, \quad \lambda^- = \frac{-3\mu_1 \mu_2 + \mu_3 + \sqrt{c}}{3\mu_2^2 - 2\mu_1 \mu_3}, \]

\[
\lambda^+ = \frac{3\mu_1 \mu_2 - \mu_3 + \sqrt{c}}{3\mu_2^2 - 2\mu_1 \mu_3},
\]

where \( c = -27\mu_1^2 - 18\mu_2 + 18\mu_3 + 24\mu_1^3 - 18\mu_1 \mu_2 + \mu_3 + \mu_2^2. \) The \( \mu_1, \mu_2, \mu_3 \) moments are feasible ABPH(1,1) moments (exists an ABPH(1,1), whose first 3 moments are \( \mu_1, \mu_2, \mu_3 \)), if the solution is feasible, i.e., \( 0 \leq p \leq 1, \lambda^+ > 0, \lambda^- > 0. \)

4.2 Moments matching with ABPH-C(1,1) distributions

The density of an ABPH(1,1) is

\[
f(t) = \begin{cases} 
(1-p)\lambda^- e^{-\lambda^+ t} & \text{if } t < 0, \\
p\lambda^+ e^{-\lambda^- t} & \text{if } t > 0.
\end{cases} \quad (5)
\]

where the \( \lambda^+ > 0 \) and \( \lambda^- > 0 \), are the parameters of the positive and the negative parts and \( p \) \( (0 < p < 1) \) is the mixing probability.

Applying (4) for \( k = 1, 2 \) and considering that the pdf has to have equal left and right limits, i.e. \( p\lambda^+ = (1-p)\lambda^- \), results in 3 equations for the three unknowns. There are two solutions of this set of equation and the relevant one is:

\[
p = \frac{3\mu_1^2 - 2\mu_2 - \mu_1 \sqrt{2\mu_2 - 3\mu_1^2}}{6\mu_1^2 - 4\mu_2}, \quad \lambda^+ = \frac{-\mu_1 + \sqrt{2\mu_2 - 3\mu_1^2}}{\mu_2 - 2\mu_1^2},
\]

\[
\lambda^- = \frac{\mu_1 + \sqrt{2\mu_2 - 3\mu_1^2}}{\mu_2 - 2\mu_1^2}.
\]

The \( \mu_1, \mu_2, \mu_3 \) moments are feasible ABPH-C(1,1) moments, if the solution is feasible, i.e., \( 0 \leq p \leq 1, \lambda^+ > 0, \lambda^- > 0. \)

5. MOMENT BOUNDS FOR 3 MOMENTS

The previous section presented results where the first 3 moments, or the moments and the continuity behaviour uniquely determine the ABPH distribution. In this section we extend the 3 moments based analysis for higher order BHP distributions, but in this case the 3 moments do not uniquely define the distribution. First we provide results on the permissible range of the tree moments and than in the next section a fitting procedure to create BPH distributions with a given first 3 moments.

5.1 Moment bounds for ABPH distributions

Let \( n_2 = \frac{\mu_2}{\mu_1} \) and \( n_3 = \frac{\mu_3}{\mu_1} \) denote the second and third normalized moments of a distribution with ordinary moments \( \mu_1, \mu_2, \mu_3 \). The following theorem is from [3].

**Theorem 4.** The second and third normalized moments of the APH(n) class are subjects to the following constraints

\[
n + \frac{1}{n} \leq n_2 < \infty, \quad (6)
\]

\[
n_3 \text{ lower bound:} \quad \left\{ \begin{array}{l}
\lambda_n \leq n_3 \quad \text{if } \frac{n + 1}{n} \leq n_2 \leq \frac{n + 3}{n + 1}, \\
n + \frac{1}{n} < n_2 \quad \text{if } \frac{n + 4}{n + 1} < n_2.
\end{array} \right. \quad (7)
\]

\[
n_3 \text{ upper bound:} \quad \left\{ \begin{array}{l}
n_3 \leq u_n \quad \text{if } \frac{n + 1}{n} \leq n_2 \leq \frac{n}{n - 1}, \\
n_3 < \infty \quad \text{if } \frac{n}{n - 1} < n_2.
\end{array} \right. \quad (8)
\]

where \( l_n \) and \( u_n \) are defined as follows:

\[
l_n = \frac{(3 + a_n)(n - 1) + 2a_n}{(n - 1)(1 + a_n p_n)} - 2a_n(n + 1)
\]

\[
u_n = \frac{1}{n^2 n_2} \left( 2(n - 2)(n_2 - n - 1) \sqrt{1 + \left( \frac{n(n - 2)}{n - 1} \right)} \right.
\]

\[
+ \left. (n + 2)(3n_2 - 2n - 2) \right)
\]

with

\[
p_n = \frac{(n + 1)(n - 2)}{3n_2(n - 1)} \frac{\sqrt{2 + n - 1}}{\left( \frac{n + 1}{1 + a_n p_n} \right) - \frac{n_2 - 2}{n} - 1},
\]

\[
a_n = \frac{n_2 - 2}{p_n(1 - n_2) + \frac{n_2 p_n n_2 - 2}{n - 1}}.
\]

Let \( ABPH^+(n, n^-) \) denote an ABPH\((n^+, n^-)\) distribution with \( n^+ = n, n^- \geq 1 \) (at least one negative phase), and \( \mu_1 > 0 \) (positive first moment).

**Theorem 5.** The second and third normalised moments of an ABPH\((n, n^-)\) distribution are subject to the following constraints:

\[
n + \frac{1}{n} \leq n_2 < \infty, \quad (10)
\]
\[\begin{align*}
-\infty < n_3 &\leq u_n \quad \text{if } \frac{n+1}{n} \leq n_2 \leq \frac{n}{n-1} , \\
-\infty < n_3 &< \infty \quad \text{if } \frac{n}{n-1} < n_2 ,
\end{align*}\]

where \(u_n\) is given in (9) in Theorem 4. Note that

- the bounds for the second normalised moment are identical for both the APH(\(n\)) class and the ABPH\(^+(n, n^-)\) class,
- the upper bound for the third normalised moment is identical for both the APH(\(n\)) and the ABPH\(^+(n, n^-)\) class,
- the lower bound for the third normalised moment for the ABPH\(^+(n, n^-)\) class is \(-\infty\),
- the bounds do not depend on the number of negative phases, i.e., a single negative phase is sufficient to reach the bounds.

**Proof.** Thanks to Theorem 3 we can assume that the ABPH\(^+(n, n^-)\) distribution is given by a mixture of an ABPH\((n^+, 0)\) with \(n^+ = n\) and an ABPH\((0, n^-)\) distribution. Let \(\mu_1, \mu_2, \mu_3, n_1, n_2, n_3\) denote the moments and the normalised moments of the ABPH\((n, 0)\) (ABPH\((0, n^-)\)) distribution. Then the moments of the ABPH\(^+(n, n^-)\) distribution satisfy (2). By introducing \(a = \mu_1^+ / \mu_1^-\) the normalised moments of the ABPH\(^+(n, n^-)\) distribution are

\[\begin{align*}
n_2 &= \frac{n_2 (1 - p) + a^2 n_2 p}{(-1 - p + ap)^2} , \\
n_3 &= \frac{-n_2^{-a} n_1 (1 - p) + a^3 n_2^+ n_2 p}{(-1 - p + ap)(n_2 (1 - p) + a^2 n_2 p)} .
\end{align*}\]

Since the first moment of ABPH\(^+(n, n^-)\) is positive we must have

\[p \mu_1^+ - (1 - p) \mu_1^- > 0 \quad \text{and since } p > \frac{1}{1 + a} .\]

The second normalised moment given in (12) is monotone decreasing in the interval \(p \in (1/(1 + a), 1]\) and its limits are

\[\lim_{p \to 1} n_2 = \infty , \quad \lim_{p \to 0} n_2 = n_2^+ .\]

According to (6) \(n_2^+\) is bounded by \((n + 1)/n \leq n_2^+ < \infty\) and since (15) implies (10). The only value of \(p\) that satisfies (12) and (14) is

\[p = \frac{2(1 + a)^2 n_2}{2(1 + a) n_2 - n_2^+ + a^2 n_2^+ + \sqrt{n_2^+ / 4(1 + a)n_2^+}} .\]

Substituting (16) into (13), it can be shown that \(n_3\) is a monotone increasing function of \(a\) in the interval \(a \in (0, \infty)\) with limits

\[\lim_{a \to 0^+} n_3 = -\infty , \quad \lim_{a \to \infty} n_3 = \frac{n_2 n_3^+}{n_2^+} \]

which implies the lower bound in (11) for the third normalised moments of an ABPH\(^+(n, n^-)\) distribution. For what concerns the upper bound, based on Theorem 4 and (15) we have \((n + 1)/n \leq n_1^+ \leq n_2\). Theorem 4 implies also that in the range \(n_2 \in [(n + 1)/n, n_2]\) the quantity \(n_2 n_3^+/n_2^+\) is maximal if \(n_2 = n_2^+\) and \(n_2^+\) is the maximal third normalised moment realizable by an APH(\(n\)) distribution. This implies the upper bound in (11) for the third normalised moment of an ABPH\(^+(n, n^-)\) distribution which is identical to that of an APH(\(n\)) distribution. □

The numerical values of the bounds of the APH(\(n\)) and the ABPH\(^+(n, n^-)\) class are illustrated in Figure 5 and 6. The darkest shaded area contains those normalised moments pairs that can be realized with two phases. Points in the second darkest shaded area can be realized with three phases. The dashed dotted line corresponds to \(n_2 = n_2^+\) which is the lower limit of non-negative random variables.

The structure of these bounds are presented in Figure 7 and 8. These figures indicate the normalised moments of the Erlang(\(n\)) \((n \geq 1)\) distributions lie on this line.

Let ABPH\(^-(n^+, n^-)\) denote an ABPH\((n^+, n^-)\) distribution with \(n^- = n\), \(n^+ \geq 1\) (at least one positive phase) and \(\mu_1 < 0\) (negative first moment).

**Theorem 6.** The second and third normalised moments of an ABPH\(^+(n^+, n^-)\) distribution are subject to the following constraints:

\[\begin{align*}
\frac{n+1}{n} &\leq n_2 < \infty , \\
-\infty &< n_3 \leq u_n , \\
-\infty &< n_3 < \infty .
\end{align*}\]

**Proof.** The theorem is the trivial counterpart of Theorem 5. □

The normalised moments cannot be used in the case when the mean is zero. The following theorem gives the bounds for the second and third moments for this case.

**Theorem 7.** The bounds for the second and the third moments of a ABPH\((n^+, n^-)\) distribution with zero mean are

\[0 < \mu_2 < \infty \quad \text{and} \quad -\infty < \mu_3 < \infty .\]

**Proof.** We prove the theorem only for the most restricted case. Consider the ABPH(1,1) structure depicted in Figure 9. From \(\mu_1 = 0\) we have \(p/\lambda^{-} - (1 - p)/\lambda^{+} = 0\), from which \(p^{+} = \lambda^{+} / (\lambda^{+} + \lambda^{-})\). The second moment then can be written as

\[\mu_2 = \frac{2}{\lambda^{+} - \lambda^{-}} = \frac{2\mu_2}{(\lambda^{-})^2}\]

where \(a = \lambda^{-} / \lambda^{+}\). The limits for the second moment are

\[\lim_{\lambda^{+} \to \infty} \mu_2 = 0 , \quad \lim_{\lambda^{-} \to 0} \mu_2 = \infty .\]

From (21) we have

\[\lambda^{-} = \sqrt{\frac{2\mu_2}{\mu_2}}\]
Figure 5: Bounds of normalised moments of the APH(n) class (n=2,3,4,8,25)

Figure 6: Bounds of normalised moments of the ABPH+(n+,n−) class (n=2,3,4,8,25)

Figure 7: Structure of bounds of normalised moments of the APH(n) class

Figure 8: Structure of bounds of normalised moments of the ABPH+(n+,n−) class

Figure 9: ABPH(1,1) distribution

with which the third moment is

\[ \mu_3 = \frac{3(a - 1)\mu_2^{3/2}}{\sqrt{2a}}. \]  

(24)

The limits for the third moment are

\[ \lim_{a \to 0} \mu_3 = -\infty, \quad \lim_{a \to \infty} \mu_3 = \infty. \]  

(25)

Since for any ABPH(n+,n−) with \( \mu_2 > 0 \), Theorem 7 implies that in case of zero mean, the combination of a single negative and a single positive phase gives complete flexibility with respect to the possible range of the second and third normalised moments. Based on (24), given second and third moments, \( \mu_2 \) and \( \mu_3 \), can be realized by

\[
a = \begin{cases} 
\frac{9\mu_2^2 + \mu_3^2 - \mu_3\sqrt{18\mu_2^2 + \mu_3^2}}{9\mu_2^2} & \text{if } \mu_3 < 0, \\
\frac{9\mu_2^2 + \mu_3^2 + \mu_3\sqrt{18\mu_2^2 + \mu_3^2}}{9\mu_2^2} & \text{if } 0 \leq \mu_3.
\end{cases} \]  

(26)

Then \( \lambda^- \) is given by (23), \( \lambda^+ = \lambda^- / a \) and, based on the zero mean assumption, \( p = 1/(1+a) \).

5.2 Moment bounds for ABPH-C distributions

In this section we deal with ABPH distributions whose pdf has equal left and right limits at zero. A simple sufficient but not necessary condition for having this property is that
every basic path contains both positive and negative phases.

As shown by Theorem 5, a single negative phase is enough to have whole flexibility for the first three moments in case of the ABPH$^+(n^+, n^-)$ class. The following theorem defines a region of 2nd and 3rd normalised moments that can be realized with a special ABPH-C$^+(n, 1)$ structure.

**Theorem 8.** Any point of the following second and third normalised moment range can be obtained by an ABPH$^+(n, 1)$ distribution, depicted in Figure 10, composed of $n$ positive phases and a single negative phase with zero initial probability:

\[
\frac{n+1}{n} \leq n_2 < \infty, \tag{27}
\]

\[
\frac{1}{n_2} < u_n, \quad \text{if} \quad \frac{n+1}{n} < n_2 < \frac{n}{n-1}, \tag{28}
\]

where

\[
l_n = \frac{3}{1-a_1} - \frac{(1-a) n_2 - 2}{n_2 a_1 (1-a)} \left( 2 - \frac{a_1^2}{(1-a_1)(1-a_1) n_2 - 2} \right),
\]

\[
a_1 = \frac{n_2 - 1 + \sqrt{(n_2 - 1) \frac{n+1}{n} - 1}}{n_2 - \frac{n+1}{n}},
\]

and $l_n$ and $u_n$ are given in (8) and (9), respectively.

**Proof.** The moments of the given ABPH$^+(n, 1)$ structure are

\[
\mu_1 = \mu_1^+ - \mu_1^-, \quad \mu_2 = \mu_2^+ + \mu_2^- - 2 \mu_1^+ \mu_1^-,
\]

\[
\mu_3 = \mu_3^+ - 3 \mu_2^+ \mu_1^- + 3 \mu_1^+ \mu_2^- + \mu_3^-,
\]

where $\mu_1^+ > 0$ and $\mu_1^- = k! \mu_1^+$. Introducing $a = \mu_1^+ / \mu_1^-$ and the normalised moments we obtain

\[
n_2 = \frac{2(1-a) + a^2 n_2^+}{(1-a)^2}, \tag{29}
\]

\[
n_3^3 = \frac{6(a-1) - 3a^2 n_2^+ + a^3 n_2^+ n_2^+}{(1-a)^2 (2(1-a) + a^2 n_2^+)}.
\]

From $\mu_1^+ > 0$, we have $a > 1$.

In (29), $n_2^+$ is a monotone decreasing function of $a$ for $a > 1$, such that $\lim_{a \to 1^+} n_2^+ = \infty$ and $\lim_{a \to \infty} n_2^+ = n_2^+$. As $a \to 1$, $n_2^+$ is bounded from above, because $n_2^+$ is unbounded from above when $n_2^+ > n/n_1$.

The remaining limit ($\lim_{a \to a}$) is based on the lower bound of the $n_3^3$ ratio. According to Theorem 4

\[
\min \frac{n_3^3}{n_2^3} \leq \frac{2 n_2^3 - 1}{n_2^3}, \quad \text{when} \quad \frac{n+1}{n} < n_2^+, \quad \text{i.e.,} \quad a \in (a_1, \infty). \tag{31}
\]

Taking the upper bound of the minimal $n_3^3$ ratio and substituting it into (33) gives:

\[
\min n_3^3 \leq \frac{3}{1-a} \left( \frac{1-a n_2^3 - 2}{n_2^3 a(1-a)} \left( 2 - \frac{1}{n_2^3} \right) \right) = \frac{3}{1-a} \left( \frac{1-a n_2^3 - 2}{n_2^3 a(1-a)} \left( 2 - \frac{1}{n_2^3} \right) \right) \tag{36}
\]

(36) is a monotone increasing function of $a$ (when $a > 1$ and $n_2 > 1$). Substituting the minimal value of $a$, $a = a_1$, gives the missing lower bound of $n_3$. □

For $n_2 > n/(n-1)$ and $n = 3$ the lower bound of $n_3$ is depicted in Figure 11. For small $n_2$ values we have a large $a$ (which eliminates the effect of the negative part) and the curve follows the increasing tendency of (7). Instead for large $n_2$ values the negative part makes its effect on the
minimal $n_3$ value and at a given point it becomes negative and diverges to $-\infty$ as $n_2$ tends to infinity.

6. MATCHING ALGORITHM FOR THREE MOMENTS

6.1 Matching 3 moments with ABPH distribution

Hereinafter we briefly sketch a procedure for the construction of an ABPH distribution with given first three moments, $\mu_i$, $1 \leq i \leq 3$. The mean is assumed to be positive. If the mean is negative the trivial counterpart of the procedure can be applied.

From Theorem 5 we know that a single negative phase is sufficient. With a single negative phase we have that $n_2 = 2$ and $n_3 = 3$. The number of phases on the positive side, $n^+$, can be easily determined based on the bounds provided in (10) and (11). It can be seen from (15) and (17) that, in order to construct an ABPH with given 2nd and 3rd normalised moments, $n_2$ and $n_3$, we must have

$$n_2^+ < n_2 \quad \text{and} \quad \frac{n_2^+ n_3^+}{n_2} < n_3^+.$$ 

The lower bound for $n_2^+$ and the upper bound for $n_3^+$ for a given $n^+$ are provided instead by Theorem 4. Having chosen any pair $(n_2^+, n_3^+)$ satisfying the above bounds, it is possible to construct an ABPH distribution with given first three moments, $\mu_i$, $1 \leq i \leq 3$. This can be done by solving numerically (12) and (13) for $a$ and $p$. Then $\mu_1^+$ can be obtained from $p \mu_1^- - (1-p) \mu_1^+/a = \mu_1$ and $\mu_1^+ = \mu_1^- / a$. As a result, on the negative side we have a single phase with mean $\mu_1^-$ while on the positive side we have $n^+$ phases with mean $\mu_1^+$ and normalised moments $n_2^+$ and $n_3^+$. The distribution on the positive side can be constructed by the algorithm presented in [3].

The actual values of $n_2^+$ and $n_3^+$ are chosen based on how closely the resulting distribution fits the mean of the negative and the positive side of the original distribution. I.e., we aim to minimise the relative error

$$\frac{\left| \mu_1^+ - \mu_1^- \right|}{\mu_1^+} + \frac{\left| \mu_1^+ - \mu_1^- \right|}{\mu_1^-},$$

(37)

where $\mu_1^+$ and $\mu_1^-$ denote the mean of the original distribution on the positive and the negative side, respectively. This relative error is minimised through a numerical search in that region of the $(n_2^+, n_3^+)$ pairs which allows for the realization of the given 2nd and 3rd normalised moments, $n_2$ and $n_3$.

6.2 Matching 3 moments with ABPH-C distribution

An algorithm for matching 3 moments with ABPH-C distributions can be defined based on Theorem 8. We can proceed as follows. The range for $a$ is given in (33). Having chosen a given value of $a$, based on (29), (30) and $\mu = \mu_1^+ - \mu_1^- / a$ we can obtain $\mu_1^+$, $n_2^+$, and $n_3^+$; and we also have that $\mu_1^+ = \mu_1^- / a$. The parameters of the positive phases (Figure 10) then can be obtained based on $\mu_1^+$, $n_2^+$ and $n_3^+$ by applying the algorithm given in [3]. The exit rate of the negative phase instead is simply $1/\mu_1^-$. The actual value of $a$ is chosen to minimise the error given in (37).

7. NUMERICAL EXAMPLE

In this section we consider a few simple examples. The distribution whose normalised moments we match with an ABPH distribution is the normal distribution. Parameters of the distributions are given in Table 1 which lists: the parameters of the normal distribution (the mean and the variance); the second and the third normalised moments of the normal distribution, $n_2$ and $n_3$; probability of positive values in the original distribution, $p^+$; the mean of the normal distribution on the negative and on the positive side, $\mu_1^-$ and $\mu_1^+$.

The distributions are matched by both procedures proposed. Numerical characteristics of the resulting distributions are summarised in Table 2 which lists for both methods: the necessary number of phases on the positive side, $n^+$; probability of positive values in the matching distribution, $p^+$; the mean of the matching distribution on the negative and the positive side, $\mu_1^-$ and $\mu_1^+$.

Figures 12 and 13 depict the pdf of two cases, D1 and D6. In Figure 12 it can be seen that we have jump at 0 in the pdf of the matching ABPH distribution. In Figure 13 one can observe instead a case in which the ABPH distribution is not able to fit closely $p^+$. This happens because the region of the $(n_2^+, n_3^+)$ pairs from which $n_2$ and $n_3$ can be realized is very narrow ($n_2^+$ has to be between 1.5 and 1.5555). In both
Table 1: Parameters of normal distributions matched by ABPH distributions

<table>
<thead>
<tr>
<th></th>
<th>( n^+ )</th>
<th>( p^+ )</th>
<th>( \mu_1^+ )</th>
<th>( \mu_2^+ )</th>
<th>( \mu_3^+ )</th>
<th>( n^- )</th>
<th>( p^- )</th>
<th>( \mu_1^- )</th>
<th>( \mu_2^- )</th>
<th>( \mu_3^- )</th>
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</thead>
<tbody>
<tr>
<td>D1</td>
<td>2</td>
<td>0.9231</td>
<td>1.1601</td>
<td>0.9206</td>
<td></td>
<td></td>
<td></td>
<td>5</td>
<td>0.8685</td>
<td>1.2524</td>
</tr>
<tr>
<td>D2</td>
<td>2</td>
<td>0.7401</td>
<td>1.7324</td>
<td>1.0855</td>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>0.6902</td>
<td>1.6791</td>
</tr>
<tr>
<td>D3</td>
<td>2</td>
<td>0.6984</td>
<td>2.0613</td>
<td>1.4576</td>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>0.7245</td>
<td>1.9953</td>
</tr>
<tr>
<td>D4</td>
<td>10</td>
<td>0.9973</td>
<td>3.0138</td>
<td>2.1362</td>
<td>18</td>
<td></td>
<td></td>
<td>18</td>
<td>0.9947</td>
<td>3.0180</td>
</tr>
<tr>
<td>D5</td>
<td>4</td>
<td>0.9802</td>
<td>3.1037</td>
<td>2.1406</td>
<td>8</td>
<td></td>
<td></td>
<td>8</td>
<td>0.9697</td>
<td>3.1221</td>
</tr>
<tr>
<td>D6</td>
<td>2</td>
<td>0.9986</td>
<td>3.0189</td>
<td>10.9965</td>
<td>6</td>
<td></td>
<td></td>
<td>6</td>
<td>0.9210</td>
<td>3.3532</td>
</tr>
<tr>
<td>D7</td>
<td>26</td>
<td>0.9999</td>
<td>5.0005</td>
<td>10.7464</td>
<td>38</td>
<td></td>
<td></td>
<td>38</td>
<td>1 - 2 \times 10^{-6}</td>
<td>5</td>
</tr>
<tr>
<td>D8</td>
<td>9</td>
<td>0.9988</td>
<td>5.0128</td>
<td>5.8747</td>
<td>17</td>
<td></td>
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<td>17</td>
<td>0.9984</td>
<td>5.0119</td>
</tr>
<tr>
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<td>0.9914</td>
<td>5.0731</td>
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<td></td>
<td>12</td>
<td>0.9678</td>
<td>5.0168</td>
</tr>
</tbody>
</table>

Table 2: Characteristics of the matching ABPH and ABPH-C distributions

8. REFERENCES