Matrix-analytic solution of second order Markov fluid models by using matrix-quadratic equations

[Extended Abstract]

Gábor Horváth
Budapest Univ. of Technology and Economics
Department of Networked Systems and Services
ghorvath@hit.bme.hu

Miklós Telek
MTA-BME Information Systems Research Group
Magyar Tudósok krt. 2, 1117 Budapest, Hungary
telek@hit.bme.hu

ABSTRACT

There are various solution methods available for the first-order Markovian fluid models, where the (state dependent) fluid rates are constant. Among these solution methods the recently developed matrix-analytic method provides an efficient, numerically stable way to determine the stationary fluid level distribution even if the number of states is high.

In second-order Markovian fluid models the process determining the change of the fluid level is a Brownian motion with state-dependent drift and variance parameters. This paper presents a matrix-analytic solution of second-order fluid models where the matrix parameter of the matrix-exponential solution is obtained as a minimal non-negative solution of a matrix-quadratic equation.

1. INTRODUCTION

First-order Markovian fluid flow models are popular modeling tools with many practical applications. The differential equation providing the steady state distribution of the fluid level has been solved by eigenvalue decomposition based methods (see [4]). Later, more efficient procedures appeared that can solve larger models without the need of the numerically demanding eigenvalue decomposition and complex arithmetic. Such a method is the matrix-analytic solution (appearing in [5]), that provides the stationary distribution in a matrix-exponential form. The crucial step of this procedure is obtaining the minimal non-negative solution of a matrix-Riccati equation. In [5], this step is reduced to the solution of the matrix-quadratic equation.

In this paper we consider second-order Markovian fluid flows, which are Markov-modulated Brownian motions with a boundary at level 0. The differential equations governing the system are provided in [2] together with an eigenvalue-based solution method. As the matrix-analytical approach turned out to be numerically superior to the eigenvalue-based one in the first-order case, the aim to generalize it to the second-order case is natural. The contribution of the paper is the introduction of a matrix-quadratic equation whose minimal non-negative solution provides the matrix parameter of the matrix-exponentially distributed stationary fluid level distribution.

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2. SECOND-ORDER FLUID MODELS

Second-order fluid flows are two-dimensional processes $\{\mathcal{X}(t), \mathcal{Z}(t),\ t \geq 0\}$, where $\mathcal{X}(t)$ is a continuous time Markov chain (CTMC) with generator $\mathbf{Q}$ and state space $\mathcal{S}$ (commonly referred to as background or modulating Markov chain), and $\mathcal{X}(t)$ represents the fluid level in a buffer. While the CTMC is in state i for an infinitesimal Δ long interval, the increment of $\mathcal{X}(t)$ is normally distributed with mean $r_i\Delta$ and variance $\sigma_i^2\Delta$. Diagonal matrices $\mathbf{R}$ and $\mathbf{S}$ contain the drift and variance parameters, hence $\mathbf{R} = \text{diag}\langle r_i\rangle$ and $\mathbf{S} = \text{diag}\langle \sigma_i^2/2\rangle$. We consider an infinite fluid buffer which has a lower boundary at level 0 and denote the stationary fluid density by row vector $f(x) = [f_i(x), i \in \mathcal{S}]$, defined by $f_i(x) = \lim_{t\to\infty} \frac{d}{dx} P(\mathcal{X}(t) < x, \mathcal{Z}(t) = i)$. The probability mass accumulating at level 0 and state i is denoted by $p_i = \lim_{t\to\infty} P(\mathcal{X}(t) = 0, \mathcal{Z}(t) = i)$. According to [3] $f(x)$ satisfies the following differential and boundary equations

$$\frac{d}{dx} f(x)\mathbf{R} - \frac{d^2}{dx^2} f(x)\mathbf{S} = f(x)\mathbf{Q}, \tag{1}$$

$$f(0)\mathbf{R} - f'(0)\mathbf{S} = p\mathbf{Q}, \tag{2}$$

where $f'(0) = \frac{d}{dx} f(x)|_{x=0}$.

The states in $\mathcal{S}$ are divided into first order states with $\sigma_i^2 = 0$ and second order states with $\sigma_i^2 > 0$. For first order states the probability of staying at the boundary is zero when the rate is positive, that is $p_i = 0,\ \forall i : r_i > 0$. Two boundary behaviors are distinguished in the literature for second order states [1, Section 5.7], the *reflecting*, and the *absorbing* boundary.

- In case of a *reflecting* boundary, the probability mass is zero at the boundary, that is $p_i = 0,\ \forall i : \sigma_i^2 > 0$.
- In case of an *absorbing* boundary, the density at level 0 is zero, that is $f_i(0) = 0,\ \forall i : \sigma_i^2 > 0$.

3. THE STATIONARY SOLUTION

Throughout this paper we restrict our attention to the case where the mean fluid rate is non-zero, that is $r_i \neq 0, \forall i \in \mathcal{S}$. The state space $\mathcal{S}$ is partitioned according to the sign of the rates and variances as follows:

- $\mathcal{S}^{+} = \{i : r_i > 0, \sigma_i^2 = 0\}$, $\mathcal{S}^{-} = \{i : r_i < 0, \sigma_i^2 = 0\}$,
- $\mathcal{S}^{\sigma+} = \{i : r_i > 0, \sigma_i^2 > 0\}$, $\mathcal{S}^{\sigma-} = \{i : r_i < 0, \sigma_i^2 > 0\}$.

Hence, the set of states is decomposed as $\mathcal{S} = \mathcal{S}^{+} \cup \mathcal{S}^{\sigma+} \cup \mathcal{S}^{\sigma-} \cup \mathcal{S}^{-} = \mathcal{S}^{\bullet} \cup \mathcal{S}^{-}$, where $\mathcal{S}^{\bullet} = \mathcal{S}^{+} \cup \mathcal{S}^{\sigma+} \cup \mathcal{S}^{\sigma-}$. In the rest of the paper it is assumed that the states of the CTMC are numbered according to the $\mathcal{S}^{+}, \mathcal{S}^{\sigma+}, \mathcal{S}^{\sigma-}, \mathcal{S}^{-}$ order of subsets.

Similar to the solution of first order Markov fluid models, $f(x)$ can be expressed in a matrix-exponential form (see e.g. [2]). According to [2, Theorem 4.] the order of this matrix-exponential equals $|\mathcal{S}^{\bullet}|$ (the number of states in $\mathcal{S}^{\bullet}$). Taking this fact into consideration the solution can be transformed into the following form

$$f(x) = \pi e^{\mathbf{K}x} \begin{bmatrix} \mathbf{I} & \mathbf{\Psi} \end{bmatrix}, \tag{3}$$

where π is a row vector of size $|\mathcal{S}^{\bullet}|$, the size of $\mathbf{K}$ and $\mathbf{\Psi}$ are $|\mathcal{S}^{\bullet}| \times |\mathcal{S}^{\bullet}|$ and $|\mathcal{S}^{\bullet}| \times |\mathcal{S}^{-}|$, respectively. Because the form of the solution is the same as in the case of first order fluid models we use the same matrix notations. It is important to note, however, that matrices $\mathbf{K}$ and $\mathbf{\Psi}$ do not have the same elegant probabilistic interpretations as they have in [5] for the first order case.

In order to fully characterize the stationary behavior, it remains to solve

- matrices $\mathbf{K}$ and $\mathbf{\Psi}$,
- vector π,
- and the vector of probability masses at level 0 p.

3.1 Computing matrices $\mathbf{K}$ and $\mathbf{\Psi}$

Substituting (3) into the differential equation (1) gives

$$\mathbf{K}\mathbf{R}_{\bullet} - \mathbf{K}^2\mathbf{S}_{\bullet} = \mathbf{Q}_{\bullet\bullet} + \mathbf{\Psi}\mathbf{Q}_{-\bullet}, \tag{4}$$

$$\mathbf{K}\mathbf{\Psi}\mathbf{R}_{-} - \underbrace{\mathbf{K}^2\mathbf{\Psi}\mathbf{S}_{-}}_{\mathbf{0}} = \mathbf{Q}_{\bullet-} + \mathbf{\Psi}\mathbf{Q}_{--}, \tag{5}$$

where $\mathbf{S}_{-} = \mathbf{0}$ has been exploited.

We define the following diagonal matrixes with strictly positive diagonal elements $\mathbf{C}_{\bullet} = \begin{bmatrix} \mathbf{R}_{+} & & \\ & \mathbf{R}_{\sigma+} & \\ & & -\mathbf{R}_{\sigma-} \end{bmatrix}$ and $\mathbf{C}_{-} = -\mathbf{R}_{-}$, and choose constant c such that

$$c > \max\left(\max_{i \in \mathcal{S}^{+}} \frac{-q_{ii}}{r_i}, \max_{i \in \mathcal{S}^{\sigma-} \bigcup \mathcal{S}^{\sigma+}} \frac{-r_i + \sqrt{r_i^2 - 2\sigma_i^2 q_{ii}}}{\sigma_i^2} \right). \tag{6}$$

Using these we transform the original analysis problem as follows. Let $\hat{\mathbf{K}} = \frac{1}{c}\mathbf{C}_{\bullet}^{-1}\mathbf{K}\mathbf{C}_{\bullet}$, $\hat{\mathbf{\Psi}} = \frac{1}{c}\mathbf{C}_{\bullet}^{-1}\mathbf{\Psi}\mathbf{C}_{-}$, $\hat{\mathbf{S}}_{\bullet} = c\mathbf{C}_{\bullet}^{-1}\mathbf{S}_{\bullet}$, and $\hat{\mathbf{Q}} = \frac{1}{c}\mathbf{C}^{-1}\mathbf{Q}$. Equations (4) and (5) simplify to

$$\hat{\mathbf{K}}\hat{\mathbf{I}}_{\bullet} - \hat{\mathbf{K}}^2\hat{\mathbf{S}}_{\bullet} = \hat{\mathbf{Q}}_{\bullet\bullet} + \hat{\mathbf{\Psi}}\hat{\mathbf{Q}}_{-\bullet}, \tag{7}$$

$$-\hat{\mathbf{K}}\hat{\mathbf{\Psi}} = \hat{\mathbf{Q}}_{\bullet-} + \hat{\mathbf{\Psi}}\hat{\mathbf{Q}}_{--}, \tag{8}$$

where $\hat{\mathbf{I}}_{\bullet} = \mathbf{C}_{\bullet}^{-1}\mathbf{R}_{\bullet} = \begin{bmatrix} \mathbf{I}_{+} & & \\ & \mathbf{I}_{\sigma+} & \\ & & -\mathbf{I}_{\sigma-} \end{bmatrix}$.

In the first-order case, when $\mathcal{S}^{\sigma+} = \mathcal{S}^{\sigma-} = \emptyset$, the identities $\hat{\mathbf{I}}_{\bullet} = \mathbf{I}$ and $\hat{\mathbf{S}}_{\bullet} = \mathbf{0}$ hold, which make equations (7) and (8) easy to solve: $\hat{\mathbf{K}}$ is given in (7) and inserting it into (8) leads to the well-known matrix Riccati equation for the matrix $\hat{\mathbf{\Psi}}$. In the second-order case, however, $\hat{\mathbf{\Psi}}$ and $\hat{\mathbf{K}}$ can not be obtained this way. Instead, a special quasi birth-death Markov chain (QBD) is introduced, and the fundamental matrix of this QBD will provide matrices $\hat{\mathbf{\Psi}}$ and $\hat{\mathbf{K}}$.

THEOREM 1. *The minimal non-negative solution of the matrix-quadratic equation* $\mathbf{F} + \mathbb{R}\mathbf{L} + \mathbb{R}^2\mathbf{B} = \mathbf{0}$ *defined by the QBD with forward, local and backward matrix blocks*

$$\mathbf{F} = \begin{bmatrix} \hat{\mathbf{Q}}_{\bullet\bullet} + \hat{\mathbf{I}}_{\bullet} + \hat{\mathbf{S}}_{\bullet} & \hat{\mathbf{Q}}_{\bullet-} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{9}$$

$$\mathbf{L} = \begin{bmatrix} -\hat{\mathbf{I}}_{\bullet} - 2\hat{\mathbf{S}}_{\bullet} & \mathbf{0} \\ \hat{\mathbf{Q}}_{-\bullet} & \hat{\mathbf{Q}}_{--} - \mathbf{I}_{-} \end{bmatrix}, \tag{10}$$

$$\mathbf{B} = \begin{bmatrix} \hat{\mathbf{S}}_{\bullet} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{-} \end{bmatrix} \tag{11}$$

is

$$\mathbb{R} = \begin{bmatrix} \hat{\mathbf{K}} + \mathbf{I}_{\bullet} & \hat{\mathbf{\Psi}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{12}$$

PROOF. First we show that $\mathbf{F}$, $\mathbf{L}$ and $\mathbf{B}$ are proper QBD matrix blocks, that is for $\forall i, j \in \mathcal{S}$ $\mathbf{F}_{ij}$ and $\mathbf{B}_{ij}$ are non-negative, $\mathbf{L}_{ij}$ when $i \neq j$ are non-negative, $\mathbf{L}_{ii}$ are negative and the row sums of $\mathbf{F}+\mathbf{L}+\mathbf{B}$ is zero, that is $(\mathbf{F}+\mathbf{L}+\mathbf{B})\mathbb{1} = 0$, where $\mathbb{1}$ is the column vector of ones.

Due to (6)

$$\hat{\mathbf{Q}}_{++} + \mathbf{I}_{+} \geq 0, \tag{13}$$

$$\hat{\mathbf{Q}}_{\sigma+\sigma+} + \mathbf{I}_{\sigma+} + \hat{\mathbf{S}}_{\sigma+} \geq 0, \tag{14}$$

$$\hat{\mathbf{Q}}_{\sigma-\sigma-} - \mathbf{I}_{\sigma-} + \hat{\mathbf{S}}_{\sigma-} \geq 0 \tag{15}$$

hold, hence $\hat{\mathbf{Q}}_{\bullet\bullet} + \hat{\mathbf{I}}_{\bullet} + \hat{\mathbf{S}}_{\bullet}$ (and therefore $\mathbf{F}$) is non-negative. Additionally, the diagonal elements of $\mathbf{I}_{\sigma-} - 2\hat{\mathbf{S}}_{\sigma-}$ are negative according to (6), hence $-\hat{\mathbf{I}}_{\bullet} - 2\hat{\mathbf{S}}_{\bullet}$ is also a diagonal matrix with strictly negative diagonal elements (and therefore $\mathbf{L}$ has negative diagonal and non-negative off-diagonal elements) and the non-negativity of $\mathbf{B}$ is straightforward. Further more, due to $(\hat{\mathbf{Q}}_{\bullet\bullet} + \hat{\mathbf{Q}}_{\bullet-})\mathbb{1} = 0$ and $(\hat{\mathbf{Q}}_{-\bullet} + \hat{\mathbf{Q}}_{--})\mathbb{1} = 0$ we also have $(\mathbf{F} + \mathbf{L} + \mathbf{B})\mathbb{1} = 0$.

To prove that (12) gives a solution of the matrix-quadratic equation is straightforward, since substituting $\mathbb{R}$ into the matrix-quadratic equation gives identity. When $\mathbb{R}$ is the minimal non-negative solution of the matrix-quadratic equation then its eigenvalues are on the unit disk and consequently the eigenvalues of $\hat{\mathbf{K}}$ have non-positive real parts. □

$\mathbf{K}$ and $\mathbf{\Psi}$ can be computed from $\hat{\mathbf{K}}$ and $\hat{\mathbf{\Psi}}$, and the eigenvalues of $\mathbf{K}$ are identical with the eigenvalues of $c\hat{\mathbf{K}}$, and they have non-positive real parts.

3.2 Computing vectors π and p

In this section we focus on the cases when all states have the same boundary behavior. The analysis of mixed cases is also possible, but it falls out of the scope of the current paper.

3.2.1 *Reflecting boundary*

If the boundary is reflecting in all second order states then $p_{\bullet} = 0$ holds. Inserting the matrix-exponential solution into (2) and taking the state partitioning into account leads to equations

$$\pi\mathbf{R}_\bullet - \pi\mathbf{K}\mathbf{S}_\bullet = p_-\mathbf{Q}_{-\bullet}, \tag{16}$$

$$\pi\mathbf{\Psi}\mathbf{R}_- = p_-\mathbf{Q}_{--}, \tag{17}$$

since $\mathbf{S}_- = 0$. Obtaining p_- from (17) and substituting into (16) give

$$\pi(\mathbf{R}_\bullet - \mathbf{K}\mathbf{S}_\bullet - \mathbf{\Psi}\mathbf{R}_-(\mathbf{Q}_{--})^{-1}\mathbf{Q}_{-\bullet}) = 0, \tag{18}$$

$$p_- = \pi\mathbf{\Psi}\mathbf{R}_-(\mathbf{Q}_{--})^{-1}, \tag{19}$$

where $\mathbf{R}_-(\mathbf{Q}_{--})^{-1}$ is a non-negative matrix. The normalization condition of this system of linear equations comes from $\int_x f(x)\mathbb{1} + p_-\mathbb{1} = 1$, that is

$$\pi\left((-\mathbf{K})^{-1}\begin{bmatrix}\mathbf{I} & \mathbf{\Psi}\end{bmatrix}\mathbb{1} + \mathbf{R}_-(\mathbf{Q}_{--})^{-1}\mathbb{1}\right) = 1. \tag{20}$$

3.2.2 Absorbing boundary

In case of the absorbing boundary, the density is zero in the second order states, hence, $f_{\sigma+}(0) = 0$ and $f_{\sigma-}(0) = 0$. Since the density at zero is expressed by $f(0) = \pi\begin{bmatrix}\mathbf{I} & \mathbf{\Psi}\end{bmatrix}$, this implies that $\pi_{\sigma+} = 0$ and $\pi_{\sigma-} = 0$, given that $\pi = \begin{bmatrix}\pi_+ & \pi_{\sigma+} & \pi_{\sigma-}\end{bmatrix}$. With such a π vector the terms in (2) are

$$f(0)\mathbf{R} = \begin{bmatrix}\pi_+\mathbf{R}_+ & 0 & 0 & \pi_+\mathbf{\Psi}_{+-}\mathbf{R}_-\end{bmatrix}, \tag{21}$$

$$f'(0)\mathbf{S} = \begin{bmatrix}0 & \pi_+\mathbf{K}_{+,\sigma_+}\mathbf{S}_{\sigma_+} & \pi_+\mathbf{K}_{+,\sigma_-}\mathbf{S}_{\sigma_-} & 0\end{bmatrix}, \tag{22}$$

since $\mathbf{S}_+ = \mathbf{S}_- = 0$. Hence, for our partitioned vectors and block matrices (2) can be rewritten as

$$\begin{aligned}
\pi_+\mathbf{R}_+ &= p_{\sigma+}\mathbf{Q}_{\sigma+,+} + p_{\sigma-}\mathbf{Q}_{\sigma-,+} + p_-\mathbf{Q}_{-,+},\\
-\pi_+\mathbf{K}_{+,\sigma_+}\mathbf{S}_{\sigma_+} &= p_{\sigma+}\mathbf{Q}_{\sigma+,\sigma_+} + p_{\sigma-}\mathbf{Q}_{\sigma-,\sigma_+} + p_-\mathbf{Q}_{-,\sigma_+},\\
-\pi_+\mathbf{K}_{+,\sigma_-}\mathbf{S}_{\sigma_-} &= p_{\sigma+}\mathbf{Q}_{\sigma+,\sigma_-} + p_{\sigma-}\mathbf{Q}_{\sigma-,\sigma_-} + p_-\mathbf{Q}_{-,\sigma_-},\\
\pi_+\mathbf{\Psi}_{+-}\mathbf{R}_- &= p_{\sigma+}\mathbf{Q}_{\sigma+,-} + p_{\sigma-}\mathbf{Q}_{\sigma-,-} + p_-\mathbf{Q}_{-,-},
\end{aligned}$$

which, in matrix form, defines

$$\begin{bmatrix}\pi_+ & p_{\sigma+} & p_{\sigma-} & p_-\end{bmatrix}\cdot \begin{bmatrix}-\mathbf{R}_+ & \mathbf{K}_{+,\sigma_+}\mathbf{S}_{\sigma_+} & \mathbf{K}_{+,\sigma_+}\mathbf{S}_{\sigma_-} & -\mathbf{\Psi}_{+-}\mathbf{R}_-\\ \mathbf{Q}_{\sigma+,+} & \mathbf{Q}_{\sigma+,\sigma_+} & \mathbf{Q}_{\sigma+,\sigma_-} & \mathbf{Q}_{\sigma+,-}\\ \mathbf{Q}_{\sigma-,+} & \mathbf{Q}_{\sigma-,\sigma_+} & \mathbf{Q}_{\sigma-,\sigma_-} & \mathbf{Q}_{\sigma-,-}\\ \mathbf{Q}_{-,+} & \mathbf{Q}_{-,\sigma_+} & \mathbf{Q}_{\sigma-,-} & \mathbf{Q}_{-,-}\end{bmatrix} = 0. \tag{23}$$

Finally, the normalization condition to be added to the above set of equations is

$$\begin{bmatrix}\pi_+ & p_{\sigma+} & p_{\sigma-} & p_-\end{bmatrix}\cdot\begin{bmatrix}(-\mathbf{K})^{-1}_{+\bullet}\begin{bmatrix}\mathbf{I} & \mathbf{\Psi}\end{bmatrix}\mathbb{1}\\ \mathbb{1}\\ \mathbb{1}\\ \mathbb{1}\end{bmatrix} = 1. \tag{24}$$

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