Moment bounds of Phase type distributions based on the steepest increase property

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Outline

ECQT 2016: conjectures on the moment bounds of FPHs.
ECQT 2018: proofs for some of those conjectures and some related results.

1 Introduction
2 Steepest increase property
3 PH Moment bounds
4 FPH Moment bounds
5 Final remarks
Phase type (PH) distributions

Summary

- “time to absorption in a Markov chain of $N$ transient states”
- If the initial probability vector is $\alpha$ and the transient generator matrix is $A$ then

$$f_Y(x) = \alpha e^{Ax}(-A)1,$$

and its $k$th moment is

$$m_k = \int x^k f_Y(x) dx = n! \alpha (-A)^{-n} 1,$$

where $1$ is the column vector of ones.

Matrix representation allows nice matrix analytic description and numerical procedures.
Some properties

- non-unique representation (with invariant eigenvalues)

\[ \hat{\alpha} = \alpha \mathbf{G}, \hat{\mathbf{A}} = \mathbf{G}^{-1} \mathbf{A} \mathbf{G}, \mathbf{G} \mathbf{1} = \mathbf{1}, \]

- Bounded coefficient of variation (cv)

\[ cv = \frac{m_2}{m_1^2} - 1 \geq \frac{1}{N}, \]

by Aldous-Shepp (martingal), O’Cinneide (majorization).

- Equality is achieved by Erlang(N) distribution

\[ f(x) = \frac{\lambda^N x^{N-1} e^{-\lambda x}}{(N - 1)!}. \]
**PH distributions with finite support**

Introduced by Ramaswami and Viswanath.

Based on ordinary PH random variables $\mathcal{Y}_i \equiv \text{PH}(\alpha_i, A_i)$ we define

1. $\mathcal{Z}_1 = b + (\mathcal{Y}_1 | \mathcal{Y}_1 < B - b)$,
2. $\mathcal{Z}_2 = B - (\mathcal{Y}_2 | \mathcal{Y}_2 < B - b)$
3. convex combination of $\mathcal{Z}_1$ and $\mathcal{Z}_2$

with PDFs

- $f_{\mathcal{Z}_1}(x) = \frac{1}{1 - \alpha_1 e^{A_1(B-b)}} \alpha_1 e^{A_1(x-b)(-A_1)} 1$
- $f_{\mathcal{Z}_2}(x) = \frac{1}{1 - \alpha_2 e^{A_2(B-b)}} \alpha_2 e^{A_2(B-x)(-A_2)} 1$
- $f_{\mathcal{Z}_3}(x) = cf_{\mathcal{Z}_1}(x) + (1 - c)f_{\mathcal{Z}_2}(x)$, with $0 < c < 1$

for $b < x < B$ and 0 otherwise.
Moments of FPH

The $n$th moment of $Z_1$ with parameters $\alpha, A$ over interval $(b, B)$ is

$$m_n = \frac{\sum_{d=0}^{n} \binom{n}{d} d! \alpha(-A)^{-d} \left( b^{n-d} I - (b + T)^{n-d} e^{AT} \right) 1}{1 - \alpha e^{AT} 1},$$

where $T = B - b$.

We focus on $b = 0$. The cases when $b > 0$ can be computed from this moment relation.
Some extreme FTP distributions

A truncated exponential distribution with a very high intensity trends to a unit impulse at $b$.

A truncated exponential with a very small intensity tends to uniform distribution on $(b, B)$, since

$$
\lim_{\lambda \to 0} f^{Exp}_{Z_1}(x) = \lim_{\lambda \to 0} \lambda e^{\lambda x} / (1 - e^{-\lambda}) = 1.
$$
Some extreme FPH distributions

Similarly, a truncated Erlang-$N$ distribution with very small intensity gives $\lim_{\lambda \to 0} f_{\mathcal{Z}_1}^{\text{Erl}-N}(x) = N x^{N-1}$, yielding linear, quadratic, cubic distributions,

These FPHs are very hard to approximate with ordinary PHs.
In 1999, O’Cinneide published the steepest increase lemma:

**Lemma**

**For a PH distribution of order** \( m \)

\[
\frac{f'(t)}{f(t)} \leq \frac{m - 1}{t} - \lambda < \frac{m - 1}{t} \quad \text{for} \ t > 0,
\]

where \( \lambda > 0 \) is the dominant eigenvalue of \( A \).

The equality holds when \( \gamma \) is Erlang(\( m, \lambda \)) distributed.

It has several equivalent forms

- \( \frac{d}{dt} (tf(t)) \leq (m - \lambda t)f(t) < mf(t), \)
- \( \frac{d}{dt} \left( \frac{f(t)}{t^{m-1}} \right) \leq -\frac{\lambda f(t)}{t^{m-1}} < 0. \)

The steepest increase of \( f(t) \) is \( t^{m-1} \).
Proof by O’Cinneide (1999) based on a conjecture, which is proved by Yao (2002).

**Proof.**

For CTMC generator \( Q \) of size \( m \): \( Q e^Q \leq (m - 1) e^Q \).

If \( A \) is a transient generator with dominant eigenvalue \( \lambda \), then it gives \( A e^A \leq (m - 1 - \lambda) e^A \).

Setting \( A =: A t \) we have \( e^{A t} A t \leq (m - 1 - \lambda t) e^{A t} \) for \( t > 0 \).

Pre-multiplying and post-multiplying by \( \alpha \) and \(-A 1\), respectively, we obtain \( f'(t) t \leq (m - 1 - \lambda t) f(t) \).
PH moment bounds based on steepest increase

**Lemma**

For $n = 0, 1, \ldots$, the $n+1$-st moment of $\mathcal{Y}$ (of order $m$ and with dominant eigenvalue $\lambda$) is bounded by

$$E\left(\mathcal{Y}^{n+1}\right) \leq \frac{m+n}{\lambda} E(\mathcal{Y}^n),$$

and the equality holds when $\mathcal{Y}$ is Erlang($m, \lambda$).

For $n = 0$ and $n = 1$ it gives

$$E(\mathcal{Y}) \leq \frac{m}{\lambda}, \quad \text{and} \quad E(\mathcal{Y}^2) \leq \frac{m+1}{\lambda} E(\mathcal{Y}).$$

That is, we have two bounds for $SCV_{\mathcal{Y}}$

$$\frac{1}{m} \leq SCV_{\mathcal{Y}} \leq \frac{m+1}{\lambda E(\mathcal{Y})} - 1.$$
PH moment bounds based on steepest increase

Bounds of SCV for ordinary PH distributions. The lower bound is the $1/m$ by Aldous-Shepp, the upper bound is from the steepest increase property.
**PH moment bounds based on steepest increase**

**Proof.**

Multiplying \( \frac{d}{dt}(tf(t)) \leq (m - \lambda t)f(t) \) by \( t^n \) and integrating from 0 to \( \infty \) gives

\[
LHS = \int_{t=0}^{\infty} t^n d(tf(t)) = [t^{n+1} f(t)]_{t=0}^{\infty} - \int_{t=0}^{\infty} tf(t)dt^n
\]

\[
= -n \int_{t=0}^{\infty} tf(t)t^{n-1}dt = -n \mathbb{E}(Y^n);
\]

\[
RHS = \int_{t=0}^{\infty} t^n (m - \lambda t)f(t)dt = m \int_{t=0}^{\infty} t^n f(t)dt - \lambda \int_{t=0}^{\infty} t^{n+1} f(t)dt
\]

\[
= m \mathbb{E}(Y^n) - \lambda \mathbb{E}(Y^{n+1}),
\]

from which we have \( -n \mathbb{E}(Y^n) \leq m \mathbb{E}(Y^n) - \lambda \mathbb{E}(Y^{n+1}) \).
**FPH upper moment bounds**

Let $\mathcal{W} = \mathcal{Y} | \mathcal{Y} < T$, where $\mathcal{Y}$ is PH distributed. Its moments are

$$E(\mathcal{W}^i) = \frac{E_i(T)}{E_0(T)},$$

where $E_i(T) = \int_{t=0}^{T} t^i f(t)dt$.

**Lemma**

$$E(\mathcal{W}^n) \leq \frac{(m + n - 1)T}{m + n} E(\mathcal{W}^{n-1}).$$

**Proof.**

Multiplying $\frac{d}{dt}(tf(t)) \leq mf(t)$ by $t^{n-1}(T - t)$ and integrating from 0 to $T$ gives the lemma by the same steps.
Corollary

$E(W^n)$ is bounded by

$$E(W^n) \leq \frac{mT^n}{m + n}.$$

and the equality holds when $Y$ is Erlang($\lambda, m$) and $\lambda \to 0$.

Proof.

Recursively applying the previous lemma for moments 1, \ldots, $n$ gives the upper bound.

For $n = 1$, $E(W) \leq mT/(m + 1)$ indicates that no FPH distribution with can have a mean close to the upper bound $T$. 
For $n = 2$

\[ SCV_{W} = \frac{\mathbb{E}(W^2)}{\mathbb{E}(W)^2} - 1 \leq \frac{(m + 1)T}{(m + 2)\mathbb{E}(W)} - 1. \]

Upper bounds of $SCV$ for $W$ distributions with $T = 1$
Lemma

For \( n = 1, 2, \ldots \), the \( n+1 \)-st moment of \( \mathcal{W} \) is bounded by

\[
\frac{m + n + \lambda T}{\lambda} \mathbb{E}(\mathcal{W}^n) - \frac{(m + n - 1) T}{\lambda} \mathbb{E}(\mathcal{W}^{n-1})
\]

\[
\leq \mathbb{E}(\mathcal{W}^{n+1}) \leq \frac{m + n}{\lambda} \mathbb{E}(\mathcal{W}^n) - \frac{T^{n+1} f(T)}{E_0(T)} < \frac{m + n}{\lambda} \mathbb{E}(\mathcal{W}^n)
\]

Proof.

The lower bound is obtained by multiplying

\[
\frac{d}{dt} (t f(t)) \leq (m - \lambda t) f(t)
\]

with \( (T - t) t^{n-1} \) and integrating from 0 to \( T \),

the lower bound is obtained by multiplying

\[
\frac{d}{dt} (t f(t)) \leq (m - \lambda t) f(t)
\]

with \( t^n \) and integrating from 0 to \( T \).
Corollary

\[ \text{SCV}_{\mathcal{W}} = \frac{E(\mathcal{W}^2)}{E(\mathcal{W})^2} - 1 \] is bounded by

\[ \frac{m + 1 + \lambda T}{\lambda E(\mathcal{W})} - \frac{mT}{\lambda (E(\mathcal{W}))^2} - 1 \leq \text{SCV}_{\mathcal{W}} < \frac{m + 1}{\lambda E(\mathcal{W})} - 1. \]

Proof.

For \( n = 1 \) the previous lemma gives

\[ \frac{m + 1 + \lambda T}{\lambda} E(\mathcal{W}) - \frac{mT}{\lambda} \leq E(\mathcal{W}^2) < \frac{m + 1}{\lambda} E(\mathcal{W}) \]

from which the corollary comes by dividing with \((E(\mathcal{W}))^2\) and subtracting 1.
The feasible range of the mean values and SCVs of $\mathcal{W}$ with $b = 0$ and $B = 2$
Summary

- Steepest increase property is also a tool to obtain moments bounds.
  It gives
    - $\lambda$ dependent upper bounds for ordinary PH distributions,
    - $\lambda$ dependent lower bounds and $\lambda$ independent upper bounds for finite PH distributions.

Plans

- Analysis of queueing models with finite PH distributions.