Moment bounds of Phase type distributions based on the steepest increase property

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Outline

ECQT 2016: conjectures on the moment bounds of FPHs.

ECQT 2018: proofs for some of those conjectures and some related results.

Phase type (PH) distributions

Summary

- "time to absorption in a Markov chain of N transient states"
- If the initial probability vector is α and the transient generator matrix is $\boldsymbol{A}$ then

$$f_{\mathcal{Y}}(x) = \alpha e^{\boldsymbol{A}x}(-\boldsymbol{A})\mathbb{1},$$

  and its kth moment is

$$m_k = \int_x x^k f_{\mathcal{Y}}(x)dx = n!\alpha(-\boldsymbol{A})^{-n}\mathbb{1}$$

where $\mathbb{1}$ is the column vector of ones.

Matrix representation allows nice matrix analytic description and numerical procedures.

PH distributions (with infinite support)

Some properties

- non-unique representation (with invariant eigenvalues)

$$\hat{\alpha} = \alpha \boldsymbol{G}, \hat{\boldsymbol{A}} = \boldsymbol{G}^{-1}\boldsymbol{A}\boldsymbol{G}, \boldsymbol{G}\mathbb{1} = \mathbb{1},$$

- Bounded coefficient of variation (cv)

$$cv = \frac{m_2}{m_1^2} - 1 \geq \frac{1}{N},$$

by Aldous-Shepp (martingal), O'Cinneide (majorization).

- Equality is achieved by Erlang(N) distribution

$$f(x) = \frac{\lambda^N x^{N-1} e^{-\lambda x}}{(N-1)!}.$$

PH distributions with finite support

Introduced by Ramaswami and Viswanath.

Based on ordinary PH random variables $\mathcal{Y}_i \equiv$PH$(\alpha_i, \boldsymbol{A}_i)$ we define

- $\mathcal{Z}_1 = b + (\mathcal{Y}_1 | \mathcal{Y}_1 < B - b)$,
- $\mathcal{Z}_2 = B - (\mathcal{Y}_2 | \mathcal{Y}_2 < B - b)$
- convex combination of $\mathcal{Z}_1$ and $\mathcal{Z}_2$

with PDFs

- $f_{\mathcal{Z}_1}(x) = \frac{1}{1 - \alpha_1 e^{\boldsymbol{A}_1 (B-b)} \mathbb{1}} \alpha_1 e^{\boldsymbol{A}_1 (x-b)} (-\boldsymbol{A}_1) \mathbb{1}$,
- $f_{\mathcal{Z}_2}(x) = \frac{1}{1 - \alpha_2 e^{\boldsymbol{A}_2 (B-b)} \mathbb{1}} \alpha_2 e^{\boldsymbol{A}_2 (B-x)} (-\boldsymbol{A}_2) \mathbb{1}$,
- $f_{\mathcal{Z}_3}(x) = c f_{\mathcal{Z}_1}(x) + (1-c) f_{\mathcal{Z}_2}(x)$, with $0 < c < 1$.

for $b < x < B$ and 0 otherwise.

Moments of FPH

The nth moment of $\mathcal{Z}_1$ with parameters α, $\boldsymbol{A}$ over interval (b, B) is

$$m_n = \frac{\sum_{d=0}^{n} \binom{n}{d} d! \alpha (-\boldsymbol{A})^{-d} \left(b^{n-d} \boldsymbol{I} - (b+T)^{n-d} e^{\boldsymbol{A}T} \right) \mathbb{1}}{1 - \alpha e^{\boldsymbol{A}T} \mathbb{1}},$$

where $T = B - b$.

We focus on $b = 0$. The cases when $b > 0$ can be computed from this moment relation.

Some extreme FTP distributions

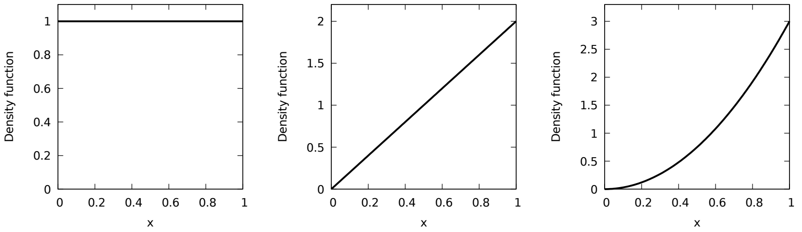
A truncated exponential distribution with a very high intensity trends to a unit impulse at b.

A truncated exponential with a very small intensity tends to uniform distribution on (b, B), since

$$\lim_{\lambda \to 0} f_{\mathcal{Z}_1}^{Exp}(x) = \lim_{\lambda \to 0} \lambda e^{\lambda x}/(1 - e^{-\lambda}) = 1.$$

Some extreme FPH distributions

Similarly, a truncated Erlang-N distribution with very small intensity gives $\lim_{\lambda\to 0} f_{\mathcal{Z}_1}^{Erl-N}(x) = N\,x^{N-1}$, yielding linear, quadratic, cubic distributions,



These FPHs are very hard to approximate with ordinary PHs.

Steepest increase property

In 1999, O'Cinneide published the steepest increase lemma:

Lemma

For a PH distribution of order m

$$\frac{f'(t)}{f(t)} \leq \frac{m-1}{t} - \lambda < \frac{m-1}{t} \qquad \text{for } t > 0,$$

where $\lambda > 0$ is the dominant eigenvalue of $\boldsymbol{A}$.
The equality holds when $\mathcal{Y}$ is Erlang(m, λ) distributed.

It has several equivalent forms

- $\frac{\mathrm{d}}{\mathrm{d}t}(tf(t)) \leq (m - \lambda t)f(t) < mf(t)$,
- $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f(t)}{t^{m-1}}\right) \leq -\frac{\lambda f(t)}{t^{m-1}} < 0$.

The steepest increase of $f(t)$ is t^{m-1}.

Steepest increase property

Proof by O'Cinneide(1999) based on a conjecture, which is proved by Yao (2002).

Proof.

For CTMC generator $\boldsymbol{Q}$ of size m: $\boldsymbol{Q}e^{\boldsymbol{Q}} \leq (m-1)e^{\boldsymbol{Q}}$.

If $\boldsymbol{A}$ is a transient generator with dominant eigenvalue λ, then it gives $\boldsymbol{A}e^{\boldsymbol{A}} \leq (m-1-\lambda)e^{\boldsymbol{A}}$.

Setting $\boldsymbol{A} =: \boldsymbol{A}t$ we have $e^{\boldsymbol{A}t}\boldsymbol{A}t \leq (m-1-\lambda t)e^{\boldsymbol{A}t}$ for $t > 0$.

Pre-multiplying and post-multiplying by α and $-\boldsymbol{A}\mathbb{1}$, respectively, we obtain $f'(t)t \leq (m-1-\lambda t)f(t)$. □

PH moment bounds based on steepest increase

Lemma

For $n = 0, 1, \ldots$, the $n+1$-st moment of $\mathcal{Y}$ (of order m and with dominant eigenvalue λ) is bounded by

$$\mathbb{E}\left(\mathcal{Y}^{n+1}\right) \leq \frac{m+n}{\lambda}\mathbb{E}(\mathcal{Y}^n),$$

and the equality holds when $\mathcal{Y}$ is Erlang(m, λ).

For $n = 0$ and $n = 1$ it gives

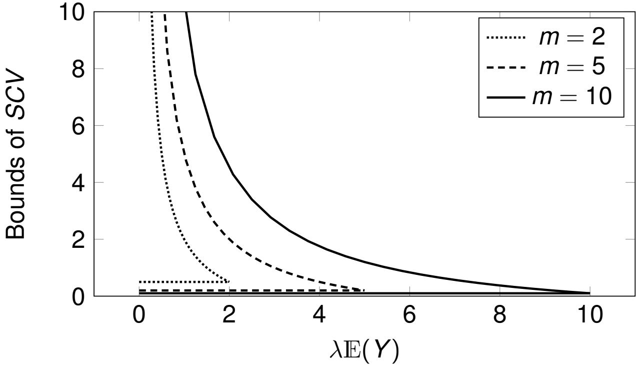
$$\mathbb{E}(\mathcal{Y}) \leq \frac{m}{\lambda}, \quad \text{and} \quad \mathbb{E}\left(\mathcal{Y}^2\right) \leq \frac{m+1}{\lambda}\mathbb{E}(\mathcal{Y}).$$

That is, we have two bounds for $SCV_{\mathcal{Y}}$

$$\frac{1}{m} \leq SCV_{\mathcal{Y}} \leq \frac{m+1}{\lambda \mathbb{E}(\mathcal{Y})} - 1.$$

PH moment bounds based on steepest increase

Bounds of SCV for ordinary PH distributions. The lower bound is the $1/m$ by Aldous-Shepp, the upper bound is from the steepest increase property.



PH moment bounds based on steepest increase

Proof.

Multiplying $\frac{\mathrm{d}}{\mathrm{d}t}(tf(t)) \le (m - \lambda t)f(t)$ by t^n and integrating from 0 to ∞ gives

$$LHS = \int_{t=0}^{\infty} t^n \mathrm{d}(tf(t)) = [t^{n+1}f(t)]_0^{\infty} - \int_{t=0}^{\infty} tf(t)\mathrm{d}t^n$$
$$= -n \int_{t=0}^{\infty} tf(t)t^{n-1}\mathrm{d}t = -n\mathbb{E}(\mathcal{Y}^n)\,;$$
$$RHS = \int_{t=0}^{\infty} t^n(m - \lambda t)f(t)\mathrm{d}t = m\int_{t=0}^{\infty} t^n f(t)\mathrm{d}t - \lambda \int_{t=0}^{\infty} t^{n+1} f(t)\mathrm{d}t$$
$$= m\mathbb{E}(\mathcal{Y}^n) - \lambda\mathbb{E}\left(\mathcal{Y}^{n+1}\right),$$

from which we have $-n\mathbb{E}(\mathcal{Y}^n) \le m\mathbb{E}(\mathcal{Y}^n) - \lambda\mathbb{E}\left(\mathcal{Y}^{n+1}\right)$. □

FPH upper moment bounds

Let $\mathcal{W} = \mathcal{Y} | \mathcal{Y} < T$, where $\mathcal{Y}$ is PH distributed. Its moments are $\mathbb{E}(\mathcal{W}^i) = \frac{E_i(T)}{E_0(T)}$, where $E_i(T) = \int_{t=0}^{T} t^i f(t) \mathrm{d}t$.

Lemma

$$\mathbb{E}(\mathcal{W}^n) \le \frac{(m+n-1)T}{m+n} \mathbb{E}\left(\mathcal{W}^{n-1}\right).$$

Proof.

Multiplying $\frac{\mathrm{d}}{\mathrm{d}t}(tf(t)) \le mf(t)$ by $t^{n-1}(T-t)$ and integrating from 0 to T gives the lemma by the same steps. □

FPH upper moment bounds

Corollary

$\mathbb{E}(\mathcal{W}^n)$ is bounded by

$$\mathbb{E}(\mathcal{W}^n) \leq \frac{mT^n}{m+n}.$$

and the equality holds when $\mathcal{Y}$ is Erlang(λ, m) and $\lambda \to 0$.

Proof.

Recursively applying the previous lemma for moments $1, \ldots, n$ gives the upper bound. $\square$

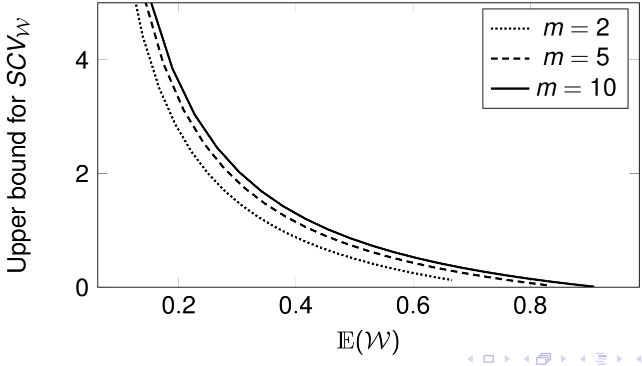
For $n = 1$, $\mathbb{E}(\mathcal{W}) \leq mT/(m+1)$ indicates that no FPH distribution with can have a mean close to the upper bound T.

FPH upper moment bounds

For $n = 2$

$$SCV_{\mathcal{W}} = \frac{\mathbb{E}(\mathcal{W}^2)}{\mathbb{E}(\mathcal{W})^2} - 1 \leq \frac{(m+1)T}{(m+2)\mathbb{E}(\mathcal{W})} - 1.$$

Upper bounds of SCV for $\mathcal{W}$ distributions with $T = 1$



FPH λ dependent moment bounds

Lemma

For $n = 1, 2, \ldots$, the $n+1$-st moment of $\mathcal{W}$ is bounded by

$$\frac{m+n+\lambda T}{\lambda}\mathbb{E}(\mathcal{W}^n) - \frac{(m+n-1)T}{\lambda}\mathbb{E}\left(\mathcal{W}^{n-1}\right)$$
$$\leq \mathbb{E}\left(\mathcal{W}^{n+1}\right) \leq \frac{m+n}{\lambda}\mathbb{E}(\mathcal{W}^n) - \frac{T^{n+1}f(T)}{E_0(T)} < \frac{m+n}{\lambda}\mathbb{E}(\mathcal{W}^n)$$

Proof.

The lower bound is obtained by multiplying $\frac{\mathrm{d}}{\mathrm{d}t}(tf(t)) \leq (m-\lambda t)f(t)$ with $(T-t)t^{n-1}$ and integrating from 0 to T,

the lower bound is obtained by multiplying $\frac{\mathrm{d}}{\mathrm{d}t}(tf(t)) \leq (m-\lambda t)f(t)$ with t^n and integrating from 0 to T. □

FPH λ dependent moment bounds

Corollary

$SCV_{\mathcal{W}} = \frac{\mathbb{E}(\mathcal{W}^2)}{\mathbb{E}(\mathcal{W})^2} - 1$ is bounded by

$$\frac{m+1+\lambda T}{\lambda \mathbb{E}(\mathcal{W})} - \frac{mT}{\lambda (\mathbb{E}(\mathcal{W}))^2} - 1 \leq SCV_{\mathcal{W}} < \frac{m+1}{\lambda \mathbb{E}(\mathcal{W})} - 1.$$

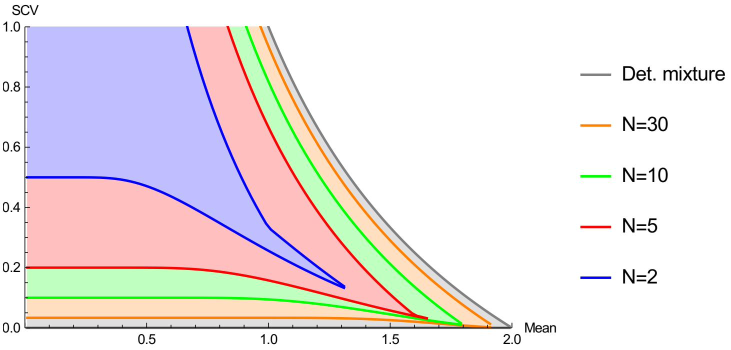
Proof.

For $n = 1$ the previous lemma gives

$$\frac{m+1+\lambda T}{\lambda}\mathbb{E}(\mathcal{W}) - \frac{mT}{\lambda} \leq \mathbb{E}\left(\mathcal{W}^2\right) < \frac{m+1}{\lambda}\mathbb{E}(\mathcal{W})$$

from which the corollary comes by dividing with $(\mathbb{E}(\mathcal{W}))^2$ and subtracting 1. □

FPH bounds on the second moment



The feasible range of the mean values and SCVs of $\mathcal{W}$ with $b = 0$ and $B = 2$

Final remarks

Summary

- Steepest increase property is also a tool to obtain moments bounds.
  It gives
  - λ dependent upper bounds for ordinary PH distributions,
  - λ dependent lower bounds and λ independent upper bounds for finite PH distributions.

Plans

- Analysis of queueing models with finite PH distributions.