Numerical Inverse Laplace Transformation by concentrated matrix exponential distributions *

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ABSTRACT

We present effective numerical inverse Laplace transformation (ILT) method which belongs to the Abate–Whitt framework and exhibits some of the best properties among all the procedures of the framework. E.g., the proposed ILT method does not generate overshoot and undershoot (upward/downward jump exceeding the jump of the original function), numerically stable and gradually improving.

Keywords: numerical inverse Laplace transformation, Abate–Whitt framework, concentrated matrix exponential distribution.

1. INTRODUCTION

There are plenty of numerical inverse Laplace transformation methods published in the literature (for a relatively recent survey we refer to [6]). Among these methods one of the most widely applied and well characterized subset is Abate–Whitt framework defined in [1]. This framework implicitly defines function families in which various optimizations can be performed in order to obtain efficient inverse Laplace transformation methods.

We propose a procedure which is based on the most general function family of the Abate–Whitt framework (referred to as Class III in [1]) where we adopt a restriction that the inverse Laplace transformation should be non-overshooting.

It turns out that matrix exponential (ME) distribution applied in the Abate–Whitt framework ensures non-overshooting inverse Laplace transformation. In [5] low order inverse Laplace transformation is applied, using concentrated ME (CME) distributions, which were available at that time [2, 4]. Recent improvements in the computation of CME distributions [3] allow us to extend the numerical ILT method also to high orders. In this work we present the first numerical experiences about the ILT method based on high order CME distributions.

*This work is supported by the OTKA K-123914 project

2. INVERSE LAPLACE TRANSFORMATION AND THE ABATE–WHITT FRAMEWORK

For a real or complex valued function $h(t)$ the Laplace transform is defined as

$$h^*(s) = \int_{t=0}^{\infty} e^{-st} h(t)\mathrm{d}t. \tag{1}$$

and the inverse transform problem is to find an approximate value of h at point T (i.e., $h(T)$) based on $h^*(s)$.

Remark 1. *We assume that $\int_{t=0}^{\infty} e^{-st}h(t)\mathrm{d}t$ is finite for $\mathrm{Re}(s) > 0$ thus $h^*(s)$ is well-defined by (1) for $\mathrm{Re}(s) > 0$.*

Remark 2. *We assume that $h(t)$ is real in this work. As a result, $h^*(\bar{s}) = \bar{h}^*(s)$ and $h^*(\bar{s}) + h^*(s) = 2\mathrm{Re}(h^*(s))$.*

Among the wide range of inverse Laplace transformation methods, we restrict our attention to the Abate–Whitt framework which we summarize below.

2.1 The Abate–Whitt framework

The idea is to approximate h by a finite linear combination of the transform values, via

$$h(T) \approx h_n(T) := \sum_{k=1}^{n} \frac{\eta_k}{T} h^*\left(\frac{\beta_k}{T}\right), \ T > 0, \tag{2}$$

where the nodes β_k and weights η_k are complex numbers, which depend on n, but not on the transform h^* or the time argument T. This framework was introduced and investigated by Abate and Whitt in [1]. When $h(t)$ in (1) is real valued it can be approximated by the real part of the weighted transform values:

$$\mathrm{Re}(h(T)) \approx \mathrm{Re}(h_n(T)) = \sum_{k=1}^{n} \mathrm{Re}\left(\frac{\eta_k}{T} h^*\left(\frac{\beta_k}{T}\right)\right).$$

In the special case when there is a complex conjugate pair among the nodes and weights (that is, $\eta_i = \bar{\eta}_j$ and $\beta_i = \bar{\beta}_j$) then

$$\frac{\eta_i}{T} h^*\left(\frac{\beta_i}{T}\right) + \frac{\eta_j}{T} h^*\left(\frac{\beta_j}{T}\right) = 2\mathrm{Re}\left[\frac{\eta_i}{T} h^*\left(\frac{\beta_i}{T}\right)\right].$$

For numerical comparisons, we consider two classic algorithms of the Abate–Whitt framework: the Gaver–Stehfest method and the Euler method, which are investigated also in [1]. These two methods approximate $h(T)$ by $h_n(T)$, where $h_n(T)$ has the form (2) with weights η_k and nodes β_k, $k = 1, 2, \ldots n$ as follows.

Gaver–Stehfest method (for even n)

$$\beta_k = k\ln(2), \qquad \text{for } 1 \le k \le n,$$

$$\eta_k = (-1)^{n/2+k}\ln(2) \sum_{j=\lfloor (k+1)/2 \rfloor}^{\min(k,n/2)} \frac{j^{n/2+1}}{(n/2)!} \binom{n/2}{j}\binom{2j}{j}\binom{j}{k-j},$$

$$\text{for } 1 \le k \le n,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Euler method (for odd n)

$$\beta_k = \frac{(n-1)\ln(10)}{6} + \pi i(k-1), \quad 1 \le k \le n$$

$$\eta_k = 10^{(n-1)/6}(-1)^{k-1}\xi_k, \quad 1 \le k \le n$$

where

$$\xi_1 = \frac{1}{2}$$

$$\xi_k = 1, \quad 2 \le k \le (n+1)/2$$

$$\xi_n = \frac{1}{2^{(n-1)/2}}$$

$$\xi_{n-k} = \xi_{n-k+1} + 2^{-(n-1)/2}\binom{(n-1)/2}{k}$$

$$\text{for } 1 \le k < (n-1)/2.$$

Remark 3. *The set of real valued functions $\sum_k \eta_k e^{-\beta_k t}$ with potentially complex valued coefficients has the following real representations.*

Class I *If both η_k and β_k are real then $\sum_k \eta_k e^{-\beta_k t}$ is a real representation.*

Class II *If η_k is real and β_k is complex then*

$$\mathrm{Re}\left(\sum_k \eta_k e^{-\beta_k t}\right) = \sum_k \eta_k e^{-b_k t}\cos(\omega_k t)$$

is its real representation, where $\beta_k = b_k + i\omega_k$.

Class III *If both η_k and β_k are complex then*

$$\mathrm{Re}\left(\sum_k \eta_k e^{-\beta_k t}\right) = \sum_k a_k e^{-b_k t}\cos(\omega_k t + \phi_k)$$

is its real representation, where $\beta_k = b_k + i\omega_k$ and a_k, ϕ_k are real and obtained from the real and imaginary parts of η_k [4].

The Gaver–Stehfest method falls into Class I, the Euler method falls into Class II, the proposed ME distribution based method (described in detail in Section 3.2) falls into Class III.

For $\mathrm{Re}(\beta_k) > 0$, we can reformulate the inverse Laplace transformation methods of the Abate–Whitt framework as

$$h_n(T) = \frac{1}{T}\sum_{k=1}^{n} \eta_k h^*\left(\frac{\beta_k}{T}\right) = \frac{1}{T}\sum_{k=1}^{n} \eta_k \int_0^\infty e^{-\frac{\beta_k}{T}t} h(t)\mathrm{d}t$$
$$= \int_0^\infty h(t) f_T^n(t)\mathrm{d}t, \tag{3}$$

where the numerical approximation of the Laplace inverse at point T is obtained as the integral of the original function, $h(t)$, with

$$f_T^n(t) = \frac{1}{T}\sum_{k=1}^{n} \eta_k e^{-\frac{\beta_k}{T}t}. \tag{4}$$

If $f_T^n(t)$ was the Dirac impulse function at point T then the Laplace inversion would be perfect, but depending on the order of the approximation (n), the applied inverse transformation method (weights η_k, nodes β_k) and the time point (T), function $f_T^n(t)$ only approximates the Dirac impulse function with a given accuracy.

Remark 4. *$f_T^n(t)$ is a scaled version of*

$$f_1^n(t) = \sum_{k=1}^{n} \eta_k e^{-\beta_k t} \tag{5}$$

because, according to (4)*,*

$$f_T^n(t) = \frac{1}{T} f_1^n\left(\frac{t}{T}\right). \tag{6}$$

3. MATRIX EXPONENTIAL DISTRIBUTIONS

The class of matrix exponential distributions of order N, denoted $\mathrm{ME}(N)$, contains random variables with pdf of the form

$$f_X(t) = -\alpha \mathbf{A} e^{\mathbf{A}t}\mathbf{1}, \qquad t \ge 0, \tag{7}$$

where α is a row vector of length N, $\mathbf{A}$ is a matrix of size $N \times N$ and $\mathbf{1}$ is a column vector of ones of size N. As $f_X(t)$ is a pdf, $f_X(t)$ is non-negative for $t \ge 0$.

Assuming that $\mathbf{A}$ is diagonalizable, with spectral decomposition $\mathbf{A} = \sum_{i=1}^n u_i \lambda_i v_i$, the pdf can be written as

$$f_X(t) = \sum_{i=1}^{n} \underbrace{-\alpha \mathbf{A} u_i v_i \mathbf{1}}_{c_i} e^{\lambda_i t} = \sum_{i=1}^{n} c_i e^{\lambda_i t}, \tag{8}$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $\mathbf{A}$. Comparing (8) and (5) indicates that ME distributions with diagonalizable matrix $\mathbf{A}$ can be used in the place of $f_1^n(t)$.

Remark 5. *Due to the non-negativity of $f_X(t)$, the integral in* (3) *results in an inverse Laplace transformation without overshoot.*

3.1 Concentrated ME distributions

Concentrated ME(N) distributions with low coefficient of variation has been calculated in [4] up to $N = 47$ and in [3] for up to $N = 2001$, using the following form (for odd N):

$$f_{\mathrm{ME}}(t) = c\,\mathrm{e}^{-\lambda t} \prod_{i=0}^{(N-1)/2} \cos^2(\omega t - \phi_i) \tag{9}$$

with real values of c, λ, ω and $\phi_1, \ldots, \phi_{(N-1)/2}$ obtained from numerical optimization.

3.2 ME distribution-based inverse Laplace transformation

In order to apply the CME distributions for inverse Laplace transformation (9) needs to be rewritten in a form consistent with (5):

$$f_{\mathrm{ME}}(t) = c\,\mathrm{e}^{-\lambda t} \prod_{i=0}^{(N-1)/2} \cos^2(\omega t - \phi_i) = \sum_{i=1}^{N} \eta_i e^{-\beta_i t}$$

where $n = (N+1)/2$, η_1, β_1 are real, and the values $\beta_2, \ldots, \beta_n$ have positive imaginary parts. For the details of this transformation, see the Appendix of [4].

4. NUMERICAL COMPARISON WITH THE ME BASED METHOD

In order to investigate the properties of the considered inverse Laplace transformation methods (Euler, Gaver–Stehfest (Gaver in short), Concentrated ME based (CME)), we performed a set of numerical inverse Laplace transformations for the 6 functions of Table 1 using our Matlab implementation, where we applied standard double precision floating point The arithmetic for CME and 100 digit precision arithmetic with the Matlab Symbolic Math Toolbox for Euler and Gaver. Numerical properties of the 6 test functions are rather similar; we demonstrate them using mainly the test function $\lfloor t \rfloor \mod 2$.

$h(t)$	e^{-t}	$\sin t$	$\mathbb{1}(t > 1)$
$h^*(s)$	$\frac{1}{1+s}$	$\frac{1}{s^2+1}$	$\frac{1}{s}e^{-s}$
$h(t)$	$\mathbb{1}(t > 1)e^{1-t}$	$\lfloor t \rfloor$	$\lfloor t \rfloor \mod 2$
$h^*(s)$	$\frac{e^{-s}}{1+s}$	$\frac{1}{s}\frac{1}{e^s-1}$	$\frac{1}{s}\frac{1}{e^s+1}$

Table 1: Set of test functions

Figures 1 and 2 investigate the dependency of the Gaver and the Euler methods on the order. The Gaver method fails to follow the alternating feature of the original function for low order ($n = 10$). It produces a smooth curve with some overshoot for medium order ($n = 50$), and reaches its limit of numerical stability, despite using 100 digit precision, at $n = 64$. The Euler method for low order ($n = 11$) follows the alternating feature of the original function for longer; it produces a smooth curve with more dominant overshoot for medium order ($n = 51$), and reaches its limit of numerical stability, despite using 100 digit precision, at $n = 101$.

In comparison, the inverse Laplace transformation obtained by the CME method is depicted in Figure 3. The CME method does not produce overshoot at any order. Similar to the Euler method, the CME method follows the alternating feature of the original function for low order ($n = 10$). It produces a smooth curve for low, medium ($n = 50$) and high ($n = 500$) orders using double precision arithmetic. The accuracy of the inverse Laplace transformation continuously increases with the order.

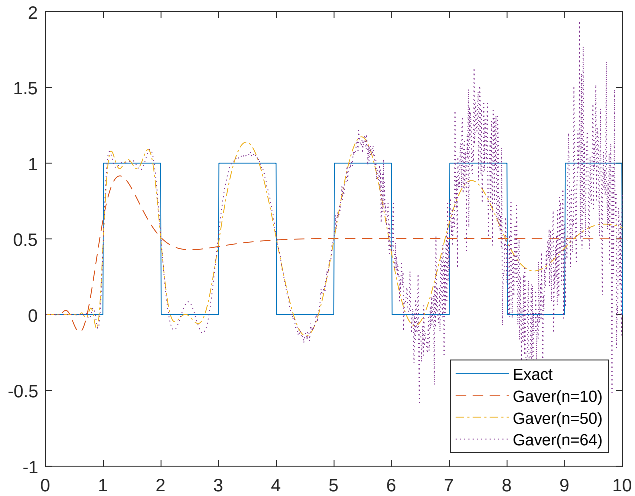

Figure 1: $h(t) = \lfloor t \rfloor \mod 2$ with Gaver method

Figures 4 and 5 compare the methods for $h(t) = \lfloor t \rfloor$ with low and medium orders, while Figures 6 and 7 compare the methods for $h(t) = \lfloor t \rfloor \mod 2$ with the same orders. In each case, the benefit of the non-overshooting inverse Laplace transformation is dominant. Especially, the figures with medium orders indicate the uncertainties coming from overshooting inverse Laplace transformation using the more alternating Euler method.

The sharpest increase of the Euler and the CME methods are similar for the same orders. Approximating discontinuity, the Euler method provides a bit sharper increase/decrease than the CME method at a given order, but at the cost of significant overshoots before and after the discontinuity.

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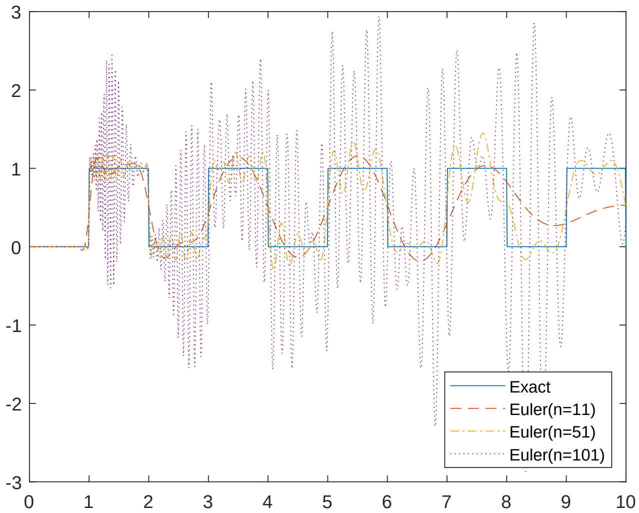

Figure 2: $h(t) = \lfloor t \rfloor \mod 2$ **with Euler method**

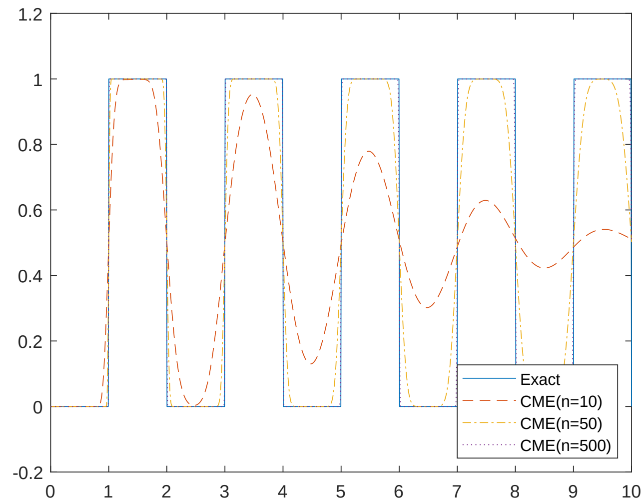

Figure 3: $h(t) = \lfloor t \rfloor \mod 2$ **with CME method**

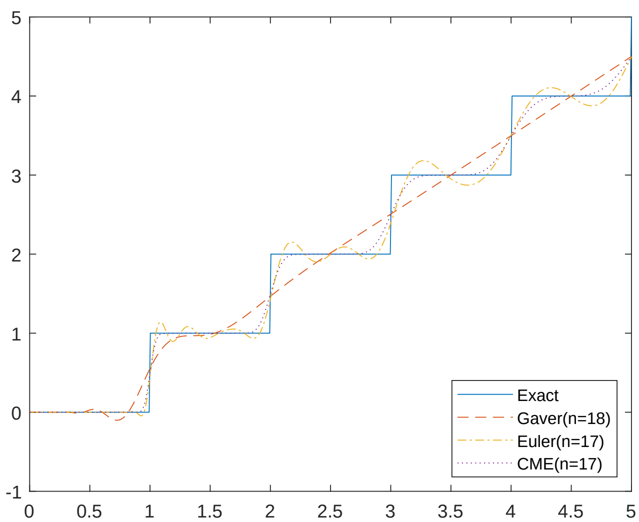

Figure 4: $h(t) = \lfloor t \rfloor$ **with low orders**

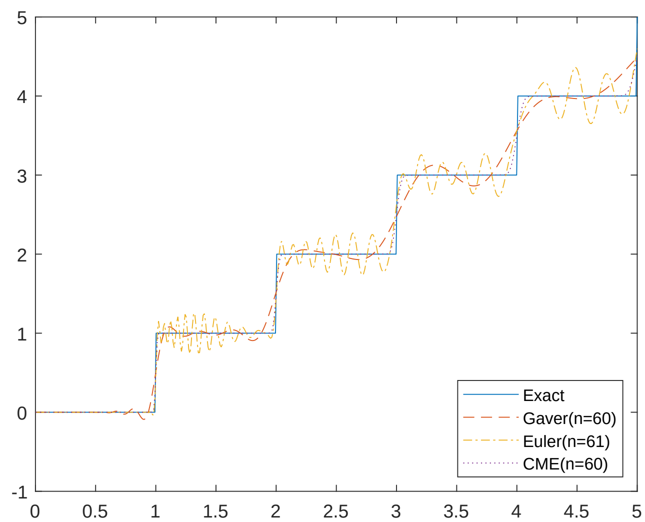

Figure 5: $h(t) = \lfloor t \rfloor$ **with medium orders**

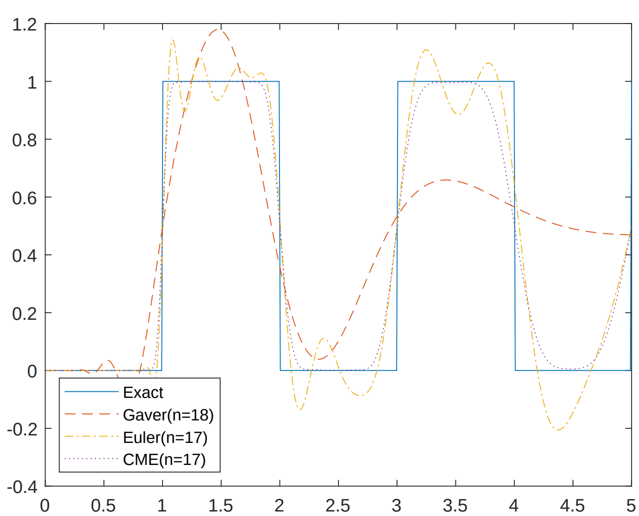

Figure 6: $h(t) = \lfloor t \rfloor \mod 2$ **with low orders**

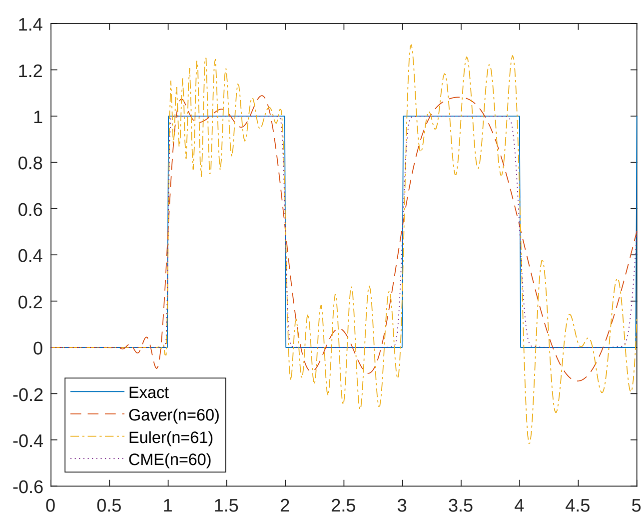

Figure 7: $h(t) = \lfloor t \rfloor \mod 2$ **with medium orders**