Explicit evaluation of ME(3) membership

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***Abstract*—We present an explicit method to evaluate the non-negativity of order 3 Matrix exponential functions.**

***Index Terms*—Matrix Exponential distributions;**

I. Introduction

Matrix exponential (ME) distributions [1] gain attention in various application fields due to the availability of efficient methods for the analysis of stochastic models with ME distributions. Unfortunately, it is hard to decide if a ME function defines a distribution (non-negative in $(0, \infty)$) or not. The class of order 3 ME (ME(3)) functions has been analyzed in [2] and [3], but none of them proposed explicit methods for all cases. In this paper we propose explicit methods instead of the numerical solutions of transcendent equations proposed in [3].

II. General principles of the method

Our goal is to explicitly determine whether a vector-square matrix pair of size 3, $(\alpha, \mathbf{A})$, determines a matrix exponential distribution with density $f(t) = \alpha e^{\mathbf{A}t}(-\mathbf{A})\mathbb{1}$ or not. We assume that the necessary condition $\lim_{t\to\infty} f(t) = 0$ ($\Leftrightarrow$ the real parts of the eigenvalues of $\mathbf{A}$ are negative) holds and focus only on the non-negativity of $f(t)$ in $(0, \infty)$.

The general approach

Let us consider a matrix exponential function of order n with distinct real eigenvalues, $f(t) = \sum_{i=1}^{n} a_i e^{\lambda_i t}$, where λ_i are the eigenvalues and $a_i \neq 0$ are real constants. The idea is to divide the inequality by one of the $e^{\lambda_i t}$ terms:

$$f(t) \geq 0 \quad \rightsquigarrow \quad \tilde{f}(t) = \sum_{i=1}^{n-1} a_i e^{(\lambda_i - \lambda_n)t} \geq -a_n.$$

which is a modified problem of one dimension less. This gives the motivation to study the following problems simultaneously:

$$\tilde{f}(t) \geq 0, \quad \forall t \geq 0, \tag{1}$$

$$\tilde{f}(t) \geq b, \quad \forall t \geq 0, \tag{2}$$

$$\tilde{f}(t) = 0, \tag{3}$$

$$\tilde{f}(t) = b. \tag{4}$$

Our approach will be to first solve (2), (3), (4) for $n = 2$ and then to trace back the order 3 problem of (1) to an order 2 problem of (2).

III. ME(3) distributions

In the case of ME(3) distributions we can distinguish four different cases according to the eigenvalue structure of $\mathbf{A}$:

1) three different negative real eigenvalues,
2) two different negative real eigenvalues,
3) one negative real eigenvalue,
4) one negative real and a complex conjugate pair.

In [3] explicit formulas were given to decide ME(3) membership only in the cases of 1) and 3). With the help of the mentioned general approach we provide explicit formulas for cases 2) and 4) below.

A. Two different eigenvalues

We have to consider two cases. Assume that the eigenvalues are $\lambda_2 < \lambda_1 < 0$ (λ_1 is referred to as dominant eigenvalue). In the case when the multiplicity of λ_1 is one, the general form of the density function is

$$f_1(t) = a_1 e^{\lambda_1 t} + (a_2 + a_{21} t) e^{\lambda_2 t}, \text{ where } a_1, a_{21} \neq 0. \tag{5}$$

In the other case when the multiplicity of the dominant eigenvalue is two, we can write

$$f_2(t) = (a_1 + a_{11} t) e^{\lambda_1 t} + a_2 e^{\lambda_2 t}, \text{ where } a_2, a_{11} \neq 0. \tag{6}$$

Dividing (5) or (6) by the exponential term of the single eigenvalue gives the following problem of type (2):

$$\hat{f}(t) = (g_1 + g_2 t) e^{\gamma t} \geq b \ \ \forall t \geq 0, \text{ where } b, g_2 \neq 0. \tag{7}$$

Elementary calculations gives us the root of the function $\hat{f}(t)$, its extreme point (the root of $d\hat{f}(t)/dt$) and its extreme value

$$t^* = \frac{-g_1}{g_2},$$

$$t_{opt} = -\frac{g_2 + g_1 \gamma}{g_2 \gamma} = \frac{-1}{\gamma} + t^*,$$

$$f_{opt} = \hat{f}(t_{opt}) = -\frac{g_2}{\gamma} \cdot \exp\left(-\frac{1}{\gamma} - \frac{g_1}{g_2}\right).$$

t^* and t_{opt} coincide iff $g_2 = 0$, thus $t^* \neq t_{opt}$. Depending on the sign of γ and g_2 there are four cases to consider.

- $\gamma < 0,\ g_2 < 0$. The possible values of b depend on the sign of t_{opt}. If $t_{opt} \leq 0$, i.e. $g_1 \leq -g_2/\gamma$ then $b \leq \hat{f}(0) =$

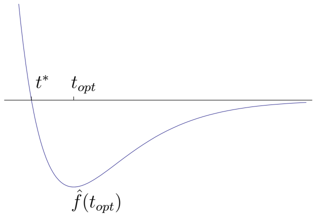

Fig. 1. Structure of $\hat{f}(t)$ when $\gamma < 0$ and $g_2 < 0$

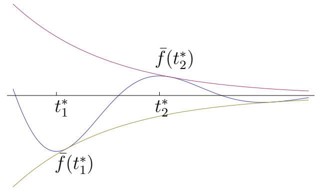

Fig. 2. Structure of $\bar{f}(t)$ when $\bar{f}(t_1^*) < 0$

g_1. Otherwise $b \leq f_{opt}$ (see Figure 1). This gives us two possible necessary and sufficient conditions for $\hat{f}(t) \geq b$:

$$\gamma < 0,\, g_2 < 0,\, b \leq g_1 \leq \frac{-g_2}{\gamma} \tag{8}$$

$$\gamma, g_2 < 0,\, g_1 > \frac{-g_2}{\gamma},\, b \leq -\frac{g_2}{\gamma} \cdot e^{-\frac{1}{\gamma} - \frac{g_1}{g_2}} \tag{9}$$

- $\gamma < 0,\ g_2 > 0$. This time the possible values of b depend on the sign of t^*. If $t^* \leq 0$, i.e. $g_1 \geq 0$ then $b \leq 0$. Otherwise $b \leq \hat{f}(0) = g_1$. We thus gain two more conditions:

$$\gamma < 0,\ b \leq g_1 < 0 < g_2 \tag{10}$$

$$\gamma < 0,\ g_2 > 0,\ g_1 \geq 0,\ b \leq 0 \tag{11}$$

- $\gamma > 0,\ g_2 < 0$. $\hat{f}(t) \geq b$ can't hold for any b since $\lim\limits_{t \to \infty} \hat{f}(t) = -\infty$.
- $\gamma > 0,\ g_2 > 0$. The possible values of b again depend on the sign of t_{opt}. If $t_{opt} \leq 0$, i.e. $g_1 \geq -g_2/\gamma$ then $b \leq \hat{f}(0) = g_1$. Otherwise $b \leq f_{opt}$. So the last two conditions are:

$$\gamma > 0,\, g_2 > 0,\, g_1 \geq \frac{-g_2}{\gamma},\, b \leq g_1 \tag{12}$$

$$\gamma, g_2 > 0,\, g_1 < \frac{-g_2}{\gamma},\, b \leq -\frac{g_2}{\gamma} \cdot e^{-\frac{1}{\gamma} - \frac{g_1}{g_2}} \tag{13}$$

Conditions (8), (9), (10) and (11) will be used after the trace back of (5). Similarly (12) and (13) will be used for (6). After these preparations we can prove the following.

Theorem 1: $f_1(t)$ as defined in (5) is non-negative for $t \geq 0$ if and only if $\lambda_2 < \lambda_1 < 0$ and one of the following hold

i) $a_{21} < 0,\ a_2 \leq \frac{-a_{21}}{\lambda_2 - \lambda_1},\ a_1 \geq -a_2$;
ii) $a_{21} < 0,\ a_2 > \frac{-a_{21}}{\lambda_2 - \lambda_1},\ a_1 \geq \frac{a_{21}}{\lambda_2 - \lambda_1} e^{\left(-\frac{1}{\lambda_2 - \lambda_1} - \frac{a_2}{a_{21}}\right)}$;
iii) $a_{21} > 0,\ a_2 < 0,\ a_1 \geq -a_2$;
iv) $a_{21} > 0,\ a_2 \geq 0,\ a_1 > 0$.

Proof: Dividing $f_1(t)$ by $e^{\lambda_1 t}$ results in an inequality of type (7). Substituting its parameters into (8), (9), (10) and (11) we obtain the theorem. ■

Theorem 2: $f_2(t)$ as defined in (6) is non-negative for $t \geq 0$ if and only if $\lambda_2 < \lambda_1 < 0$ and one of the following hold

i) $a_{11} > 0,\ a_1 \geq \frac{-a_{11}}{\lambda_1 - \lambda_2},\ a_2 \geq -a_1$;
ii) $a_{11} > 0,\ a_1 < \frac{-a_{11}}{\lambda_1 - \lambda_2},\ a_2 \geq \frac{a_{11}}{\lambda_1 - \lambda_2} e^{\left(-\frac{1}{\lambda_1 - \lambda_2} - \frac{a_1}{a_{11}}\right)}$.

Proof: Dividing $f_2(t)$ by $e^{\lambda_2 t}$ results in an inequality of type (7). Substituting its parameters into (12) and (13) results the statement of the theorem. ■

B. One real and a complex conjugate pair of eigenvalues

The general form of the density function in this case is

$$f_3(t) = a_1 e^{\lambda_1 t} + a_2 \cos(\omega t + \phi) e^{\lambda_2 t}, \tag{14}$$

where $t \geq 0,\ a_2 > 0,\ -\pi < \phi < \pi,\ \lambda_1, \lambda_2 < 0$. We want $f_3(t) \geq 0$ to hold so $\lambda_2 \leq \lambda_1 < 0$ should hold. The result of the order reduction step is

$$f_3(t) \geq 0 \Leftrightarrow \bar{f}(t) \geq b \quad (\forall t \geq 0),$$

where $\bar{f}(t) = \cos(\omega t + \phi) e^{\lambda t}$, $\lambda = \lambda_2 - \lambda_1$, $b = \frac{-a_1}{a_2}$ and $a_1 > 0$. The extreme points of $\bar{f}(t)$ are obtained at $\bar{f}(t)' = 0$ which are

$$\lambda \cos(\omega t + \phi) = \omega \sin(\omega t + \phi) \qquad \Leftrightarrow$$

$$t_k^* = \frac{\tan^{-1}\left(\frac{\lambda}{\omega}\right) - \phi + k\pi}{\omega},$$

where $k \in \mathbb{Z}$. Note that $\cos(\omega t + \phi)$ and $\sin(\omega t + \phi)$ cannot be zero at the same time and $\tan(\omega t + \phi)$ is π/ω periodic. Since $\cos(\omega t + \phi)$ is $2\pi/\omega$ periodic and $e^{\lambda t}$ is monotone decreasing it is enough to consider the t_k^* which fall into $[0, 2\pi/\omega]$ (see Figure 2). If $k^* = -\lfloor (\tan^{-1}\left(\frac{\lambda}{\omega}\right) - \phi)/\pi \rfloor$, then the extreme points in $[0, 2\pi/\omega]$ are $t_i^* = \left(\tan^{-1}\left(\frac{\lambda}{\omega}\right) - \phi + (k^* + i - 1)\pi\right)/\omega$, $i = 1, 2$. It only remans to check if $\bar{f}(t_i^*) \geq b$, $i = 1, 2$.

Theorem 3: $f_3(t)$ as defined in (14) is non-negative for $t \geq 0$ if and only if one of the following hold

- $\lambda_1 = \lambda_2$ and $a_1 \geq a_2 > 0$,
- $\lambda_2 < \lambda_1 < 0,\ a_1 > 0$ and $\bar{f}(t_i^*) \geq b,\ i = 1, 2$.

Proof: If $\lambda_1 = \lambda_2$ then $f_3(t) \geq 0$ simplifies to $\cos(\omega t + \phi) \geq \frac{-a_1}{a_2}$. It follows that $\frac{-a_1}{a_2} \leq -1$, i.e. $a_1 \geq a_2$. If $\lambda_2 < \lambda_1 < 0$ then we proceed according to the analysis of $\bar{f}(t) \geq b$. ■

IV. Conclusions

The most difficult part of checking the validity of ME distributions is the analysis of the non-negativity of ME functions. We propose a general order reduction approach for the analysis of the non-negativity of order n ME functions. In case of order 3 ME functions this approach results in explicit expressions for all possible cases. In case of higher order ME functions the proposed order reduction approach might or might not result in explicit expressions depending on the properties of the obtained lower order ME functions. Future research plans contain the analysis of higher order ME functions using the proposed approach.

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