On the Minimal Coefficient of Variation of Matrix Exponential Distributions

T. Éltető, S. Rácz, M. Telek
Technical University of Budapest

Aug. 2006.

1 ME with complex poles

The density of a ME distribution has the following general form

$$f(t) \;=\; \sum_{i=1}^{N_r} g_i^{n_i}(t)\; e^{p_i t} \;+ \sum_{i=1}^{N_c/2} a_i^c e^{p_i^c t} + \bar{a}_i^c e^{\bar{p}_i^c t}$$

where N_r is the the number different of real poles and N_c is the number of complex poles of $f^*(s)$, n_i is the multiplicity of the real eigenvalue p_i, $g_i^{\# n_i}(t)$ is a real polynomial of order $n_i - 1$.

The complex poles can also be represented as

$$\begin{aligned}\sum_{i=1}^{N_c/2} a_i^c e^{p_i^c t} + \bar{a}_i^c e^{\bar{p}_i^c t} &= \sum_{i=1}^{N_c/2} 2e^{-\Re(p_i^c)t}(\Re(a_i^c)\cos(\Im(p_i^c)t) + \Im(a_i^c)\sin(\Im(p_i^c)t)) \\ &= \sum_{i=1}^{N_c/2} 2|a_i^c|e^{-\Re(p_i^c)t}(\cos(\phi)\cos(\Im(p_i^c)t) + \sin(\phi)\sin(\Im(p_i^c)t)) \\ &= \sum_{i=1}^{N_c/2} 2|a_i^c|e^{-\Re(p_i^c)t}\cos(\Im(p_i^c)t - \phi)\end{aligned}$$

where $a_i^c = |a_i^c|e^{i\phi}$.

In the rest we only check the non-negativity of $f(t)$ and do not normalize and scale it. The coefficient of variation is computed as $cv = \frac{m_2 m_0}{m_1^2} - 1$, where $m_i = \int_0^\infty t^i f(t)dt$ for $i = 0, 1, 2$.

2 Conjectures

2.1 $n = 3$

The minimal coefficient of variation is obtained when there is a pair of complex poles ($p^c = p \pm ip_c$) and a real one whose decay is identical with the complex one (p). The coefficient of the real pole is such that the density function hits the x axes infinitely many times. I.e.,

$$f(t) = e^{pt}(|a_c| + \Re(a_c)\cos(p_c t) + \Im(a_c)\sin(p_c t))$$

where $a_c = |a_c|e^{i\phi}$ is the complex coefficient of the complex pole. $f(t)$ can be rewritten as

$$\begin{aligned} f(t) &= e^{pt} \; (|a_c| + \Re(a_c)\cos(p_c t) + \Im(a_c)\sin(p_c t)) \\ &= e^{pt} \; |a_c|(1 + \cos(\phi)\cos(p_c t) + \sin(\phi)\sin(p_c t)) \\ &= e^{pt} \; |a_c|(1 + \cos(p_c t - \phi)) \\ &= e^{pt} \; 2|a_c|\cos^2\left(\frac{p_c t - \phi}{2}\right). \end{aligned}$$

At $p = 1$, $|a_c| = 1$ (where the density is not normalized) the minimal coefficient of variation is calculated by numerical optimization. The optimum is reached at $\phi = 3.47863$, $p_c = 1.03593$ and it is $mincv(3) = 0.200902$.

2.2 n is odd

The general form of the (non-normalized) minimal solution is

$$f(t) = e^t \prod_{i=1}^{k} \cos^2(\omega t - \phi_i),$$

where $k = (n-1)/2$. There are $k + 1 = (n+1)/2$ parameters to optimize in this function, ω and $\phi_1, \ldots, \phi_k$.

Note, that the non-negativity of $f(t)$ is ensured by the general form.

The table below contains the results of numerical minimization of the coefficient of variation based on this function for various n values.

2.3 $n = 4$

The minimal coefficient of variation is obtained with a pair of complex poles ($p^c = p \pm i p_c$) and two real ones whose decay are p (identical with the real part of the complex one) and $p_1 (> p)$. In this case

$$f(t) = b_1 e^{p_1 t} + e^{pt}(b + |a| + bt + \Re(a)\cos(p_c t) + \Im(a)\sin(p_c t))$$

The resulted the minimal cv is such that $f(0) = 0$, and the function osculate with the x axes at a single point t_1, i.e., $f(t_1) = 0$ and $f'(t_1) = 0$. This three conditions eliminates 3 unknown from the expression. Optimizing for the remaining 3 parameters we obtain the minimum given in the table below.

2.4 n is even

The general form of the minimal solution is

$$f(t) = ce^{\alpha t} + e^t \; (b + \prod_{i=1}^{k} \cos^2(\omega t - \phi_i))$$

where $n = 2k + 2$. In this expression there are $k + 4$ parameters. The minimal density function is such that $f(0) = 0$, and the function osculate with the x axes at a single point t_1, i.e., $f(t_1) = 0$ and $f'(t_1) = 0$. Applying these conjectures we can eliminate two more parameters (three equations with one unknown t_1) and we remain with $k + 2$ parameters to optimize.

For $n > 4$ we cannot optimize the function (provided its non-negativity) yet.

Poles	MinVar	1/MinVar	MinVar	1/MinVar
	real poles		complex poles	
3	0.276583	3.61556	0.200902	4.97756
4	0.19333	5.17251	0.149808	6.6752
5	0.138453	7.22266	0.0812643	12.3055
6	0.108623	9.20619	x	x
7	0.0861277	11.6107	0.04288	23.3209
8	0.0717026	13.9465	x	x
9	0.0600486	16.6532	0.0261569	38.2309
10	0.0518365	19.2914	x	x
11	0.0449173	22.2632	0.017494	57.1625
12	0.0397335	25.1677	x	x
13	0.0352403	28.3766	0.0124696	80.1951
14	0.031726	31.5199	x	x
15	0.0286172	34.944	0.00931281	107.379

Table 1: Minimal relative variance