On the Minimal Coefficient of Variation of Matrix Exponential Distributions

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1 ME with complex poles

The density of a ME distribution has the following general form

\[ f(t) = \sum_{i=1}^{N_r} g_i^{n_i}(t) e^{p_i t} + \sum_{i=1}^{N_c/2} a_i^c e^{p_i^c t} + \bar{a}_i^c e^{\bar{p}_i^c t} \]

where \( N_r \) is the number of different real poles and \( N_c \) is the number of complex poles of \( f^*(s) \), \( n_i \) is the multiplicity of the real eigenvalue \( p_i \), \( g_i^{p_i^{n_i}}(t) \) is a real polynomial of order \( n_i - 1 \).

The complex poles can also be represented as

\[ \sum_{i=1}^{N_c/2} a_i^c e^{p_i^c t} + \bar{a}_i^c e^{\bar{p}_i^c t} = \sum_{i=1}^{N_c/2} 2|a_i^{p_i^c}| e^{-\Re(p_i^c) t} (\Re(a_i^{p_i^c}) \cos(\Im(p_i^c) t) + \Im(a_i^{p_i^c}) \sin(\Im(p_i^c) t)) \]

where \( a_i^{p_i^c} = |a_i^{p_i^c}| e^{i\phi} \).

In the rest we only check the non-negativity of \( f(t) \) and do not normalize and scale it. The coefficient of variation is computed as \( cv = \frac{m_i}{m_0 m_1} - 1 \), where \( m_i = \int_0^\infty t^i f(t) dt \) for \( i = 0, 1, 2 \).

2 Conjectures

2.1 \( n = 3 \)

The minimal coefficient of variation is obtained when there is a pair of complex poles \( (p_i^c = p \pm ip_c) \) and a real one whose decay is identical with the complex one \( (p) \). The coefficient of the real pole is such that the density function hits the x axes infinitely many times. I.e.,

\[ f(t) = e^{pt} (|a_c| + \Re(a_c) \cos(p_c t) + \Im(a_c) \sin(p_c t)) \]
where \( a_c = |a_c|e^{i\phi} \) is the complex coefficient of the complex pole. \( f(t) \) can be rewritten as

\[
\begin{align*}
  f(t) &= e^{pt} (|a_c| + \Re(a_c) \cos(p_c t) + \Im(a_c) \sin(p_c t)) \\
  &= e^{pt} |a_c|(1 + \cos(\phi) \cos(p_c t) + \sin(\phi) \sin(p_c t)) \\
  &= e^{pt} |a_c|(1 + \cos(p_c t - \phi)) \\
  &= e^{pt} 2|a_c| \cos^2 \left(\frac{p_c t - \phi}{2}\right).
\end{align*}
\]

At \( p = 1 \), \( |a_c| = 1 \) (where the density is not normalized) the minimal coefficient of variation is calculated by numerical optimization. The optimum is reached at \( \phi = 3.47863, p_c = 1.03593 \) and it is \( \text{mincv}(3) = 0.200902 \).

2.2 \( n \) is odd

The general form of the (non-normalized) minimal solution is

\[
f(t) = e^t \prod_{i=1}^{k} \cos^2(\omega t - \phi_i),
\]

where \( k = (n - 1)/2 \). There are \( k + 1 = (n + 1)/2 \) parameters to optimize in this function, \( \omega \) and \( \phi_1, \ldots, \phi_k \).

Note, that the non-negativity of \( f(t) \) is ensured by the general form.

The table below contains the results of numerical minimization of the coefficient of variation based on this function for various \( n \) values.

2.3 \( n = 4 \)

The minimal coefficient of variation is obtained with a pair of complex poles \((p^c = p \pm ip_c)\) and two real ones whose decay are \( p \) (identical with the real part of the complex one) and \( p_1 (> p) \). In this case

\[
f(t) = b_1 e^{p_1 t} + e^{pt} (b + |a| + bt + \Re(a) \cos(p_c t) + \Im(a) \sin(p_c t))
\]

The resulted the minimal \( \text{cv} \) is such that \( f(0) = 0 \), and the function osculate with the x axes at a single point \( t_1 \), i.e., \( f(t_1) = 0 \) and \( f'(t_1) = 0 \). This three conditions eliminates 3 unknown from the expression. Optimizing for the remaining 3 parameters we obtain the minimum given in the table below.

2.4 \( n \) is even

The general form of the minimal solution is

\[
f(t) = e^{c_1 t} + e^{t} \left(b + \prod_{i=1}^{k} \cos^2(\omega t - \phi_i)\right)
\]

where \( n = 2k + 2 \). In this expression there are \( k + 4 \) parameters. The minimal density function is such that \( f(0) = 0 \), and the function osculate with the x axes at a single point \( t_1 \), i.e., \( f(t_1) = 0 \) and \( f'(t_1) = 0 \). Applying these conjectures we can eliminate two more parameters (three equations with one unknown \( t_1 \)) and we remain with \( k + 2 \) parameters to optimize.

For \( n > 4 \) we cannot optimize the function (provided its non-negativity) yet.
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<th>Poles</th>
<th>MinVar real</th>
<th>1/MinVar real poles</th>
<th>MinVar complex poles</th>
<th>1/MinVar complex poles</th>
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<td>(x)</td>
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<tr>
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</table>

Table 1: Minimal relative variance