Canonical representation of discrete order 2 MAP and RAP

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Abstract. Matrix-geometric distributions (MG) and discrete (time) rational arrival processes (DRAP) are natural extensions of discrete phase-type distributions (DPH) and discrete Markov arrival processes (DMAP) respectively. However, the exact relation of the Markovian classes and their non-Markovian counterparts and the boundaries of these classes are not known yet. It has been shown that for the order two case the MG and DPH classes are equivalent. In this paper we prove that the equivalence holds for the order two DMAPs and DRAPs as well. We prove this equivalence by introducing a Markovian canonical form for order two DRAPs and by showing, that this canonical form can indeed be used to describe the whole order two DRAP class.

Keywords: discrete Markov arrival process, discrete rational arrival process, canonical representation

1 Introduction

Stochastic models with underlying Markov chains have been widely used since the introduction of matrix analytic methods [1], which allow efficient numerical analysis of such stochastic models. Relaxing the limitations of stochastic processes with underlying Markov chains, non-Markovian generalizations of these processes, matrix exponential distributions (ME) [2] and continuous rational arrival processes (CRAP) [3], have been introduced. More recently it has turned out that these non-Markovian generalizations inherit the applicability of the efficient numerical procedures for their analysis [4]. Due to the nice computational properties, parameter estimation (fitting) and moments matching of CMAP and CRAP processes have gained significant attention [5, 6]. The order two models (the lowest order non-trivial models) allow explicit analytical treatment. For order two continuous processes the canonical representation and the moments matching were investigated in [7]. It was shown that order two CMAP $\equiv$ order two CRAP. In this paper we investigate the discrete counterparts of these processes and introduce a canonical representation for the order two DRAP class, we prove that the order two DMAP $\equiv$ order two DRAP relation also holds, and we present explicit moments and correlation matching formulas.

The rest of the paper is organized as follows. In Section 2 we survey the necessary definitions and essential properties of existing Markov chain driven stochastic processes and their non-Markovian generalizations. Unfortunately, we need to introduce a lot of concepts in this section, which makes it rather dense. The next section focuses on the special properties of the order 2 class of these processes. The main result of the paper, the canonical representation of order 2 DMAP and DRAP processes, is presented in Section 4. Finally, explicit moments and correlation matching formulas are provided in Section 5.

2 Markov chain driven point processes in discrete and continuous time and their non-Markovian generalizations

The following subsections summarize the main properties of simple stochastic models with a background discrete state Markov chain and their non-Markovian generalizations. If the background chain is a discrete time Markov chain we obtain discrete (time) stochastic models and if it is a continuous time Markov chain we obtain continuous (time) stochastic models. The main focus of the paper is on the discrete models, but some results are related to their continuous counterparts and that is why we introduce both of them.

2.1 Discrete Phase type and matrix geometric distributions

The following stochastic models define discrete distributions on the positive integers.

Definition 1. *Let $\mathcal{X}$ be a discrete random variable on $\mathbb{N}^+$ with probability mass function (pmf)*

$$P_{\mathcal{X}}(i) = Pr(\mathcal{X} = i) = \alpha \boldsymbol{A}^{i-1}(\mathbb{1} - \boldsymbol{A}\mathbb{1}) \quad \forall i \in \mathbb{N}^+, \tag{1}$$

where α is a row vector of size n, $\boldsymbol{A}$ is a square matrix of size $n \times n$, and $\mathbb{1}$ is the column vector of ones of size n. If the pmf has this matrix geometric form, then we say that $\mathcal{X}$ is matrix geometrically distributed with representation $\alpha, \boldsymbol{A}$, or shortly, $MG(\alpha, \boldsymbol{A})$ distributed.

In this and the subsequent models the scalar quantity is obtained as a product of a row vector, a given number of square matrices and a column vector. In the sequel we refer to the row vector as initial vector and to the column vector as closing vector. It is an important consequence of Definition 1 that α and $\boldsymbol{A}$ have to be such that (1) is non-negative.

Definition 2. *If $\mathcal{X}$ is an $MG(\alpha, \boldsymbol{A})$ distributed random variable, where α and $\boldsymbol{A}$ have the following properties:*

- $\alpha_i \geq 0$,

– $A_{ij} \geq 0$, $\boldsymbol{A}\mathbb{1} \leq \mathbb{1}$,

then we say that $\mathcal{X}$ is discrete phase type distributed with representation $\alpha, \boldsymbol{A}$, or shortly, DPH($\alpha, \boldsymbol{A}$) distributed.

The vector-matrix representations satisfying the conditions of Definition 2 are called Markovian.

In this paper we focus on distributions on the positive integers, consequently, $\alpha\mathbb{1} = 1$. The cumulative density function (cdf), the moment generating function, and the factorial moments of $\mathcal{X}$ are

$$F_{\mathcal{X}}(i) = Pr(\mathcal{X} \leq i) = 1 - \alpha \boldsymbol{A}^i \mathbb{1}, \tag{2}$$

$$\mathcal{F}_{\mathcal{X}}(z) = E(z^{\mathcal{X}}) = z\alpha(\boldsymbol{I} - z\boldsymbol{A})^{-1}(\mathbb{1} - \boldsymbol{A}\mathbb{1}), \tag{3}$$

$$f_n = E(\mathcal{X}(\mathcal{X}-1)\ldots(\mathcal{X}-n+1)) = \frac{d^n}{dz^n}\mathcal{F}_{\mathcal{X}}(z)|_{z=1} = n!\alpha(\boldsymbol{I}-\boldsymbol{A})^{-n}\boldsymbol{A}^{n-1}\mathbb{1}. \tag{4}$$

2.2 Discrete Markov arrival process and discrete rational arrival process

Let $\mathcal{X}(t)$ be a point process on $\mathbb{N}^+$ with joint probability mass function of inter-event times $P_{\mathcal{X}}(x_0, x_1, \ldots, x_k)$ for $k = 1, 2, \ldots$ and $x_0, \ldots, x_k \in \mathbb{N}^+$.

Definition 3. *$\mathcal{X}(t)$ is called a rational arrival process if there exists a finite $(\boldsymbol{H_0}, \boldsymbol{H_1})$ square matrix pair such that $(\boldsymbol{H_0} + \boldsymbol{H_1})\mathbb{1} = \mathbb{1}$,*

$$\underline{\pi}(\boldsymbol{I} - \boldsymbol{H_0})^{-1}\boldsymbol{H_1} = \underline{\pi}, \quad \underline{\pi}\mathbb{1} = \mathbb{1} \tag{5}$$

has a unique solution and

$$P_{\mathcal{X}(t)}(x_0, x_1, \ldots, x_k) = \underline{\pi}\boldsymbol{H_0}^{x_0-1}\boldsymbol{H_1}\boldsymbol{H_0}^{x_1-1}\boldsymbol{H_1}\ldots\boldsymbol{H_0}^{x_k-1}\boldsymbol{H_1}\mathbb{1}, \tag{6}$$

In this case we say that $\mathcal{X}(t)$ is a discrete rational arrival process with representation $(\boldsymbol{H_0}, \boldsymbol{H_1})$, or shortly, DRAP$(\boldsymbol{H_0}, \boldsymbol{H_1})$.

The size of the $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$ matrices is also referred to as the order of the associated process. An important consequence of Definition 3 is that $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$ have to be such that (6) is always non-negative.

Definition 4. *If $\mathcal{X}(t)$ is a DRAP$(\boldsymbol{H_0}, \boldsymbol{H_1})$, where $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$ are non-negative, we say that $\mathcal{X}(t)$ is a Discrete Markov arrival process with representation $(\boldsymbol{H_0}, \boldsymbol{H_1})$, or shortly, DMAP$(\boldsymbol{H_0}, \boldsymbol{H_1})$.*

The matrix pairs satisfying the conditions of Definition 4 are called Markovian and the matrix pairs violating Definition 4 are called non-Markovian.

Definition 5. *The correlation parameter, γ, of a DRAP$(\boldsymbol{H_0}, \boldsymbol{H_1})$ is the eigenvalue of $(\boldsymbol{I} - \boldsymbol{H_0})^{-1}\boldsymbol{H_1}$ with the second largest absolute value.*

One of the eigenvalues of $(\boldsymbol{I}-\boldsymbol{H_0})^{-1}\boldsymbol{H_1}$ is 1, because $(\boldsymbol{H_0}+\boldsymbol{H_1})\mathbb{1} = \mathbb{1}$, and the other eigenvalues are on the unit disk. If γ is real, it is between -1 and 1. This parameter is especially important in case of order 2 DRAPs, as their ρ_k lag-k autocorrelation coefficient can be given as $\rho_k = \gamma^k c_0$, where c_0 depends only on the stationary inter-arrival time distribution of the process.

In general, a DMAP has infinitely many different Markovian and non-Markovian representations (matrix pairs, that fulfill (6)). One way to get a different representation of a DMAP$(\boldsymbol{D_0}, \boldsymbol{D_1})$ with the same size is the application of the similarity transformation

$$\boldsymbol{H_0} = \boldsymbol{T}^{-1}\boldsymbol{D_0}\boldsymbol{T}, \quad \boldsymbol{H_1} = \boldsymbol{T}^{-1}\boldsymbol{D_1}\boldsymbol{T}, \tag{7}$$

where $\boldsymbol{T}$ is an arbitrary non-singular matrix for which $\boldsymbol{T}\mathbb{1} = \mathbb{1}$. The (stationary) marginal distribution of the inter-event time of DRAP$(\boldsymbol{H_0}, \boldsymbol{H_1})$ is MG$(\pi, \boldsymbol{H_0})$, where π is the unique solution of (5).

2.3 Continuous Phase type and matrix exponential distributions

The continuous counterparts of the above introduced models are defined as follows.

Definition 6. *Let $\mathcal{X}$ be a continuous random variable with support on $\mathbb{R}^+$ and cumulative distribution function (cdf)*

$$F_X(x) = Pr(\mathcal{X} < x) = 1 - \alpha e^{\boldsymbol{A}x}\mathbb{1},$$

where α is a row vector of size n, $\boldsymbol{A}$ is a square matrix of size $n \times n$, and $\mathbb{1}$ is the column vector of ones of size n. In this case, we say that $\mathcal{X}$ is matrix exponentially distributed with representation $\alpha, \boldsymbol{A}$, or shortly, ME$(\alpha, \boldsymbol{A})$ distributed.

Definition 7. *If $\mathcal{X}$ is an ME$(\alpha, \boldsymbol{A})$ distributed random variable, where α and $\boldsymbol{A}$ have the following properties:*

- $\alpha_i \geq 0$, $\alpha\mathbb{1} = 1$ *(there is no probability mass at $x = 0$),*
- $A_{ii} < 0$, $A_{ij} \geq 0$ *for $i \neq j$,* $\boldsymbol{A}\mathbb{1} \leq 0$,

we say that $\mathcal{X}$ is phase type distributed with representation $\alpha, \boldsymbol{A}$, or shortly, CPH$(\alpha, \boldsymbol{A})$ distributed.

The vector-matrix representations satisfying the conditions of Definition 7 are called Markovian.

The probability density function (pdf), the Laplace transform, and the moments of $\mathcal{X}$ are

$$f_{\mathcal{X}}(x) = -\alpha e^{\boldsymbol{A}x}\boldsymbol{A}\mathbb{1}, \tag{8}$$

$$f^*_{\mathcal{X}}(s) = E(e^{-s\mathcal{X}}) = -\alpha(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{A}\mathbb{1}, \tag{9}$$

$$\mu_n = E(\mathcal{X}^n) = n!\alpha(-\boldsymbol{A})^{-n}\mathbb{1}. \tag{10}$$

2.4 Continuous Markov arrival process and continuous rational arrival process

Let $\mathcal{X}(t)$ be a point process on $\mathbb{R}^+$ with joint probability density function of inter-event times $f(x_0, x_1, \ldots, x_k)$ for $k = 1, 2, \ldots$.

Definition 8. *$\mathcal{X}(t)$ is called a rational arrival process if there exists a finite $(\boldsymbol{H_0}, \boldsymbol{H_1})$ square matrix pair such that $(\boldsymbol{H_0} + \boldsymbol{H_1})\mathbb{1} = 0$,*

$$\underline{\pi}(-\boldsymbol{H_0})^{-1}\boldsymbol{H_1} = \underline{\pi}, \quad \underline{\pi}\mathbb{1} = \mathbb{1} \ , \tag{11}$$

has a unique solution, and

$$f(x_0, x_1, \ldots, x_k) = \underline{\pi} e^{\boldsymbol{H_0} x_0} \boldsymbol{H_1} e^{\boldsymbol{H_0} x_1} \boldsymbol{H_1} \ldots e^{\boldsymbol{H_0} x_k} \boldsymbol{H_1} \mathbb{1}. \tag{12}$$

In this case we say that $\mathcal{X}(t)$ is a rational arrival process with representation $(\boldsymbol{H_0}, \boldsymbol{H_1})$, or shortly, RAP$(\boldsymbol{H_0}, \boldsymbol{H_1})$.

Definition 9. *If $\mathcal{X}(t)$ is a RAP$(\boldsymbol{H_0}, \boldsymbol{H_1})$, where $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$ have the following properties:*

- $\boldsymbol{H_1}_{ij} \geq 0$,
- $\boldsymbol{H_0}_{ii} < 0$, $\boldsymbol{H_0}_{ij} \geq 0$ *for* $i \neq j$, $\boldsymbol{H_0}\mathbb{1} \leq 0$,

we say that $\mathcal{X}(t)$ is a Markov arrival process with representation $(\boldsymbol{H_0}, \boldsymbol{H_1})$, or shortly, MAP$(\boldsymbol{H_0}, \boldsymbol{H_1})$.

Similar to the discrete case, the representations satisfying the conditions of Definition 9 are called Markovian and similarity transformations generate different representations of the same process.

3 Some properties of order 2 DPH and MG distributions

In this section we summarize some recent results concerning the canonical representation of order 2 DPH and MG distributions (DPH(2) and MG(2), respectively) from [8], which are going to be utilized in the subsequent sections. Matrix $\boldsymbol{A}$ of an order 2 MG distribution has two (not necessarily distinct) real eigenvalues, out of which at least one is positive. The cases when both eigenvalues of $\boldsymbol{A}$ are positive can always be represented with an acyclic Markovian canonical representation, whose properties are studied in [9]. The cases when one of the eigenvalues is negative can always be represented with a cyclic Markovian canonical representation as it is summarized below.

Theorem 1. *[8] The pmf of an MG(2) distribution has one of the following two forms*

- *different eigenvalues:*

$$p_i = a_1 s_1^{i-1} + a_2 s_2^{i-1}, \tag{13}$$

  where s_1, s_2 are real, $0 < s_1 < 1$, $s_1 > |s_2|$, $a_2 = (1 - s_2)\left(1 - \frac{a_1}{1-s_1}\right)$ and a_1 is such that

- *if* $s_2 > 0$, *then* $0 \le a_1 \le \frac{(1-s_1)(1-s_2)}{s_1-s_2}$ *and*
- *if* $s_2 < 0$, *then* $\frac{(1-s_1)(1-s_2)s_2}{(1-s_2)s_2-(1-s_1)s_1} \le a_1 \le \frac{(1-s_1)(1-s_2)}{s_1-s_2}$,

– *identical eigenvalues:*

$$p_i = (a_1(i-1) + a_2)s^{i-1}, \tag{14}$$

where s *is real* $0 < s < 1$, *and* a_1, a_2 *are such that* $0 < a_1 \le \frac{(1-s)^2}{s}$ *and* $a_2 = \frac{(1-s)^2 - a_1 s}{1-s}$.

Theorem 2. *[9] If* $\mathcal{X}$ *is* $MG(2)$ *distributed with two distinct positive eigenvalues* $(0 < s_2 < s_1 < 1)$*, it can be represented as DPH*$(\alpha, \boldsymbol{A})$*, where*

$$\alpha = \left[\frac{a_1(s_1-s_2)}{(s_1-1)(s_2-1)}, \frac{a_1+a_2}{1-s_2}\right], \qquad \boldsymbol{A} = \begin{bmatrix} s_1 & 1-s_1 \\ 0 & s_2 \end{bmatrix}.$$

Theorem 3. *[8] If* $\mathcal{X}$ *is* $MG(2)$ *distributed with a dominant positive and a negative eigenvalue* $(s_2 < 0 < s_1 < 1$ *and* $s_1 + s_2 > 0)$*, it can be represented as DPH*$(\alpha, \boldsymbol{A})$*, where*

$$\alpha = \left[\frac{a_1 s_1 + a_2 s_2}{(s_1-1)(s_2-1)}, \frac{(a_1+a_2)(1-s_1-s_2)}{(s_1-1)(s_2-1)}\right], \qquad \boldsymbol{A} = \begin{bmatrix} 1-\beta_1 & \beta_1 \\ \beta_2 & 0 \end{bmatrix},$$

$\beta_1 = 1 - s_1 - s_2$ *and* $\beta_2 = \frac{s_1 s_2}{s_1+s_2-1}$.

Theorem 4. *[9] If* $\mathcal{X}$ *is* $MG(2)$ *distributed with two identical eigenvalues* $(0 < s = s_2 = s_1 < 1)$*, it can be represented as DPH*$(\alpha, \boldsymbol{A})$*, where*

$$\alpha = \left[\frac{a_1 s}{(1-s)^2}, \frac{a_2}{1-s}\right], \qquad \boldsymbol{A} = \begin{bmatrix} s & 1-s \\ 0 & s \end{bmatrix}.$$

There are several interesting consequences of Theorem 1 – 4. First of all

$$\text{DPH}(2) \equiv \text{MG}(2),$$

that is all MG(2) can be represented with a Markovian vector-matrix pair. Further more

$$\text{ADPH}(2) \equiv \text{MG}(2) \text{ with positive eigenvalues},$$

where ADPH(2) denotes the subclass of DPH(2) with acyclic matrix $\boldsymbol{A}$.

The canonical representation of the stochastic models introduced in Section 2 is a convenient Markovian representation that takes Cumani's acyclic canonical form [10] if possible and contains the maximal number of zero elements. In some cases these principles completely define the canonical representation, while additional criteria are applied in other cases. The representations in Theorem 2 – 4 are recommended as canonical representations in [8, 9].

The ADPH(2) canonical forms (Theorem 2 and 4) have an interesting relation with the Cumani's canonical form of CPH distributions. If MG$(\gamma, \boldsymbol{G})$ is a MG(2) with positive eigenvalues then vector γ and matrix $\boldsymbol{G} - \boldsymbol{I}$ define a ME(2) distribution, ME($\gamma, \boldsymbol{G}-\boldsymbol{I}$). Let PH($\delta, \boldsymbol{D}$) be the Cumani's acyclic canonical form of ME($\gamma, \boldsymbol{G}-\boldsymbol{I}$), which always exists [9]. Vector δ and matrix $\boldsymbol{D}+\boldsymbol{I}$ define the canonical representation of MG($\gamma, \boldsymbol{G}$) according to Theorem 2 or 4. That is

$$\begin{aligned} \mathrm{MG}(\gamma, \boldsymbol{G}) &\overset{D\rightrightarrows C}{\Rightarrow} \mathrm{ME}(\gamma, \boldsymbol{G}-\boldsymbol{I}) \equiv \mathrm{CPH}(\underbrace{\gamma\boldsymbol{T}}_{\delta}, \underbrace{\boldsymbol{T}^{-1}(\boldsymbol{G}-\boldsymbol{I})\boldsymbol{T}}_{\boldsymbol{D}}) \\ &\overset{C\rightrightarrows D}{\Rightarrow} \mathrm{DPH}(\gamma\boldsymbol{T}, \boldsymbol{T}^{-1}(\boldsymbol{G}-\boldsymbol{I})\boldsymbol{T}+\boldsymbol{I}) \equiv \mathrm{DPH}(\gamma\boldsymbol{T}, \boldsymbol{T}^{-1}\boldsymbol{G}\boldsymbol{T}), \quad (15) \end{aligned}$$

where the eigenvalues of $\boldsymbol{G}$ and $\boldsymbol{T}^{-1}\boldsymbol{G}\boldsymbol{T}$ are between in $(0,1)$ and the eigenvalues of $\boldsymbol{D}$ are in $(-1,0)$. Note that the similarity transformation $\boldsymbol{T}^{-1}\boldsymbol{G}\boldsymbol{T}$ maintains the eigenvalue structure of $\boldsymbol{G}$.

4 Canonical representation of DRAP(2) processes

The main goal of this paper is to define Markovian canonical forms for order 2 DRAP processes.

The DRAP(2) processes are defined by 4 parameters [11], e.g. the first 3 factorial moments of the stationary inter-arrival time distribution, f_1, f_2, f_3, and the correlation parameter, γ. $\boldsymbol{D_0}$ and $\boldsymbol{D_1}$ of size 2×2 has a total of 8 elements (free parameters). The $(\boldsymbol{D_0}+\boldsymbol{D_1})\mathbb{1} = \mathbb{1}$ constraint reduces the number of free parameters to 6. If additionally, two elements of the representation are set to 0 then the obtained (canonical) representation characterizes the process exactly with 4 parameters.

4.1 Canonical forms of CMAP(2)

The last paragraph of the previous section discusses the relation of the discrete and continuous distributions. We are going to utilize a similar relation between DMAP(2) and CMAP(2). To this end we summarize the canonical representation of CMAP(2) from [7].

Theorem 5. *[7] If the correlation parameter of the order 2 CRAP($\boldsymbol{H_0}, \boldsymbol{H_1}$) is*

- *non-negative, then it can be represented in the following Markovian canonical form*

$$\boldsymbol{D_0} = \begin{bmatrix} -\lambda_1 & (1-a)\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} a\lambda_1 & 0 \\ (1-b)\lambda_2 & b\lambda_2 \end{bmatrix}.$$

  where $0 < \lambda_1 \le \lambda_2$, $0 \le a, b \le 1$, $min\{a, b\} \ne 1$, $\gamma = ab$ and the associated embedded stationary vector is $\boldsymbol{\pi} = \left[\, \frac{1-b}{1-ab} \; \frac{b-ab}{1-ab} \,\right]$,
- *negative, then it can be represented in the following Markovian canonical form*

$$\boldsymbol{D_0} = \begin{bmatrix} -\lambda_1 & (1-a)\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} 0 & a\lambda_1 \\ b\lambda_2 & (1-b)\lambda_2 \end{bmatrix},$$

  where $0 < \lambda_1 \le \lambda_2$, $0 \le a \le 1$, $0 < b \le 1$, $\gamma = -ab$ and the associated embedded stationary vector is $\boldsymbol{\pi} = \left[\, \frac{b}{1+ab} \; 1 - \frac{b}{1+ab} \,\right]$.

4.2 Canonical forms of DMAP(2) with positive eigenvalues

Theorem 6. *If the eigenvalues of $\boldsymbol{H_0}$ are positive and the correlation parameter of the order 2 DRAP($\boldsymbol{H_0}, \boldsymbol{H_1}$) is*

- *non-negative, then it can be represented in the following Markovian canonical form*

$$\boldsymbol{D_0} = \begin{bmatrix} 1-\lambda_1 & (1-a)\lambda_1 \\ 0 & 1-\lambda_2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} a\lambda_1 & 0 \\ (1-b)\lambda_2 & b\lambda_2 \end{bmatrix}. \tag{16}$$

  where $0 < \lambda_1 \leq \lambda_2$, $0 \leq a, b < 1$, $\gamma = ab$ and the associated embedded stationary vector is $\boldsymbol{\pi} = \left[\frac{1-b}{1-ab} \; \frac{b-ab}{1-ab} \right]$,
- *negative, then it can be represented in the following Markovian canonical form*

$$\boldsymbol{D_0} = \begin{bmatrix} 1-\lambda_1 & (1-a)\lambda_1 \\ 0 & 1-\lambda_2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} 0 & a\lambda_1 \\ b\lambda_2 & (1-b)\lambda_2 \end{bmatrix}, \tag{17}$$

  where $0 < \lambda_1 \leq \lambda_2$, $s_1 = 1-\lambda_1$, $s_2 = 1-\lambda_2$, $0 \leq a \leq 1$, $0 < b \leq 1$, $\gamma = -ab$ and the associated embedded stationary vector is $\boldsymbol{\pi} = \left[\frac{b}{1+ab} \; 1 - \frac{b}{1+ab} \right]$.

Proof. Practically the same approach is applied here as in (15). The detailed proof of the theorem follows the same pattern as the proof of Theorem 5 in [7] which we omit here because we focus on the proof of Theorem 7, the related theorem with negative eigenvalues.

4.3 Canonical forms of DMAP(2) with a negative eigenvalue

Theorem 7. *If one eigenvalue of $\boldsymbol{H_0}$ is negative and the γ correlation parameter of the order 2 DRAP($\boldsymbol{H_0}, \boldsymbol{H_1}$) is*

- *non-negative, then it can be represented in the following Markovian canonical form*

$$\boldsymbol{D_0} = \begin{bmatrix} 1-\beta_1 & a\beta_1 \\ \frac{1}{a}\beta_2 & 0 \end{bmatrix}, \boldsymbol{D_1} = \begin{bmatrix} (1-a)\beta_1 & 0 \\ (1-\frac{1}{a}\beta_2)b & (1-\frac{1}{a}\beta_2)(1-b) \end{bmatrix}, \tag{18}$$

- *negative, then it can be represented in the following Markovian canonical form*

$$\boldsymbol{D_0} = \begin{bmatrix} 1-\beta_1 & a\beta_1 \\ \frac{1}{a}\beta_2 & 0 \end{bmatrix}, \boldsymbol{D_1} = \begin{bmatrix} 0 & (1-a)\beta_1 \\ (1-\frac{1}{a}\beta_2)b & (1-\frac{1}{a}\beta_2)(1-b) \end{bmatrix}, \tag{19}$$

where the eigenvalues are such that $s_2 < 0 < s_1 < 1$, $s_1 + s_2 > 0$, the relation of the parameters and the eigenvalues is $\beta_1 = 1 - s_1 - s_2$, $\beta_2 = \frac{s_1 s_2}{s_1+s_2-1}$, $0 \leq b < 1$ and $\beta_2 \leq a \leq \min\left(1, b\frac{1-s_2}{1-s_1}\right)$ in case of $\gamma \geq 0$ or $\beta_2 \leq a \leq 1$ in case of $\gamma < 0$,

The correlation parameter and the first coordinate of the embedded stationary probability vectors (the unique solution of (5))

– of (18) are

$$\gamma = (1-a)(1-b)\left(1 + \frac{1-a}{a}\frac{s_1 s_2}{1 - s_1 - s_2 + s_1 s_2}\right), \tag{20}$$

$$\pi_1 = \frac{1 - \frac{1}{1-a}\gamma}{1-\gamma}, \tag{21}$$

– of (19) are

$$\gamma = -(1-a)b\left(1 + \frac{1-a}{a}\frac{s_1 s_2}{1 - s_1 - s_2 + s_1 s_2}\right), \tag{22}$$

$$\pi_1 = 1 - \frac{1 + \frac{a}{1-a}\gamma}{1-\gamma}. \tag{23}$$

We prove the theorem by considering the full flexibility of the DRAP(2) class with a negative eigenvalue and showing that the canonical forms of Theorem 7 cover this whole set of processes. To this end we first investigate the flexibility of the DRAP(2) class.

Constraints of the DRAP(2) class We investigate the flexibility of the DRAP(2) class based on the following representation

$$\boldsymbol{H_0} = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}, \boldsymbol{H_1} = \begin{bmatrix} a_1 + (1-a_1-s_1)\gamma & (1-a_1-s_1)(1-\gamma) \\ \frac{a_1(1-s_2)(1-\gamma)}{1-s_1} & \frac{(1-s_2)(1-a_1-s_1+a_1\gamma)}{1-s_1} \end{bmatrix}, \tag{24}$$

where s_1 is the positive, s_2 is the negative eigenvalue, γ is the correlation parameter and a_1 is the parameter that characterizes the stationary inter-arrival distribution together with the eigenvalues according to (13). With this representation the first coordinate of the embedded stationary vector is $\pi_1 = \frac{a_1}{1-s_1}$.

For a given pair of eigenvalues, $s_1 > 0$ and $s_2 < 0$, Theorem 1 defines the limits of a_1. According to these limits the first coordinate of any embedded vector of DRAP($\boldsymbol{H_0}, \boldsymbol{H_1}$) should be bounded by

$$\frac{(1-s_2)s_2}{(1-s_2)s_2 - (1-s_1)s_1} \leq x \leq \frac{(1-s_2)(1-s_2)}{s_1 - s_2}. \tag{25}$$

Function $U_n(x)$ describes the effect of an n long inter-arrival period on the first coordinate of the embedded vector.

$$U_n(x) = \frac{(x, 1-x)\boldsymbol{H_0}^{n-1}\boldsymbol{H_1}}{(x, 1-x)\boldsymbol{H_0}^{n-1}\boldsymbol{H_1}\mathbb{1}}(1,0)^T. \tag{26}$$

If the embedded vector is $(x, 1-x)$ at an arrival instance and the next inter-arrival is n time unit long, the embedded vector is going to be $(U_n(x), 1-U_n(x))$ at the next arrival instance. In case of DMAPs the embedded vector represents the probability distribution of the background Markov chain at arrivals, but in case of DRAPs it does not have any probabilistic interpretations. $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$ has to be such that starting from the stationary embedded vector π for any series of inter-arrival times the first coordinate of the embedded vector satisfy (25). Based on this property we define simple constraints.

- *long series of* 1 *time unit long inter-arrivals:*
  $U_1(x) = x$ has to have a real solution between the bounds in (25), because if $U_1(x)$ would be larger (smaller) than x between the bounds then a series of one time unit long inter-arrivals would increase (decrease) the first coordinate above the upper (below the lower) limit (cf. Figure 1). This constraint results in
$$\gamma \leq \frac{(\sqrt{c_1} - \sqrt{c_2})^2}{(c_3 - a_1 s_2)^2}. \tag{27}$$
- *a long series of* 1 *time unit long inter-arrivals, then a* 2 *time unit long inter-arrival:*
  If $\gamma > 0$, then $U_1(x)$ is a shifted negative hyperbolic function which increases monotonously between the bounds in (25). If $U_1(x) = x$ has two solutions, w_1, w_2 ($w_1 < w_2$), then w_1 is stable and w_2 is unstable, which means that starting from $x < w_1$ or $w_1 < x < w_2$ and having a long series of 1 time unit long inter-arrivals the first coordinate converges to w_1, while starting from $x > w_2$ and having a long series of 1 time unit long inter-arrivals the first coordinate diverges. Consequently a long series of 1 time unit long inter-arrivals and a 2 time unit long inter-arrival keep the first coordinate between the bounds if $U_2(w_1) \leq w_2$ holds. This constraint results in
$$\gamma \leq \frac{s_1 s_2 c_2 - c_1(1 - s_1 - s_2) - \sqrt{s_1 s_2 c_1 c_2 (s_1 + s_2)^2}}{c_4 c_5}. \tag{28}$$
- *long series of* 2 *time unit long inter-arrivals:*
  Similar to the first constraint $U_2(x) = x$ has to have a real solution which results in
$$\gamma \geq \frac{\sqrt{s_1 s_2 c_2} + \sqrt{c_6})^2}{{c_4}^2}. \tag{29}$$
- *a long series of* 1 *time unit long inter-arrivals:*
  If $\gamma < 0$ then $U_1(x)$ is a shifted hyperbolic function which decreases monotonously between the bounds in (25). $U_1(x) = x$ has to have a stable real solution (w_1) between the bounds in (25), which holds if $\frac{d}{dx} U_1(x)|_{x=w_1} > -1$ (cf. Figure 2) (in case of a long series of 1 time unit long inter-arrivals the first coordinate converge to w_1). This constraint results in
$$\gamma \geq \frac{s_2(1 - a_1 - s_1) + a_1 s_1}{(c_3 - a_1 s_1)^2}. \tag{30}$$

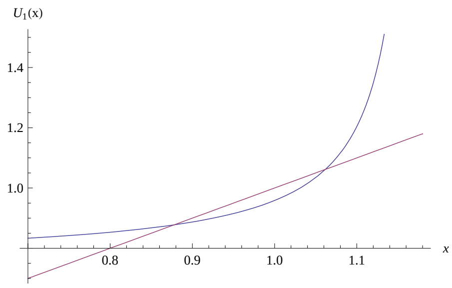

Fig. 1. The $U_1(x)$ function when $s_1 = 0.8, s_2 = -0.3, a_1 = 0.19, \gamma = 0.17$.

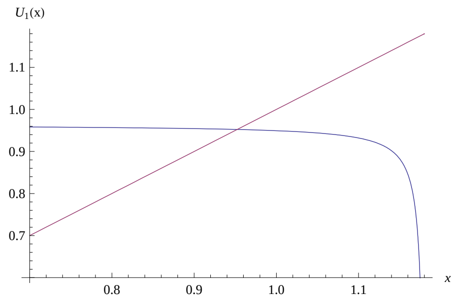

Fig. 2. The $U_1(x)$ function when $s_1 = 0.8, s_2 = -0.3, a_1 = 0.19, \gamma = -0.012$.

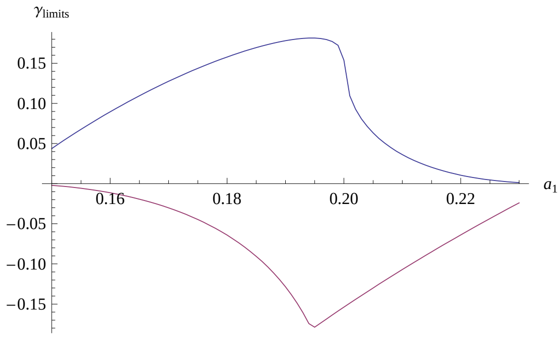

Fig. 3. The upper and lower γ limits as a function of a_1 when $s_1 = 0.8, s_2 = -0.3$

In the above expressions the auxiliary variables are

$$
\begin{aligned}
c_1 &= -a_1(s_1 - s_2)^2(1 - a_1 - s_1), \\
c_2 &= (1 - s_1)^3(1 - s_2), \\
c_3 &= 1 - s_1(2 - a_1 - s_1), \\
c_4 &= s_1(1 - s1)(1 - a_1 - s_1) + a_1 s_2(1 - s_2), \\
c_5 &= (a_1(s_1 - s_2) + s_2(1 - s_1)^2), \\
c_6 &= -a_1(1 - a_1 - s_1)(s_1(1 - s_1) - s_2(1 - s_2))^2.
\end{aligned} \tag{31}
$$

We summarize the results of this subsection in the following theorem.

Theorem 8. *For DRAP($\boldsymbol{H_0}, \boldsymbol{H_1}$) defined in* (24) *with* $0 < s_1 < 1$, $-s_1 < s_2 < 0$ *and* $a1$ *satisfying Theorem 1 the correlation parameter satisfies the inequalities* (27) - (30).

Theorem 8 defines only some bounds of the set of DRAP(2) processes, but the subsequent analysis of the canonical DMAP(2) proves that these bounds are tight.

Constraints of the set of canonical DMAP(2) processes Having the bounds of the DRAP(2) class from Theorem 8 we are ready to prove Theorem 7.

Proof. (Theorem 7) First we need to relate the variables of the canonical representation with the parameters used for characterizing the DMAP(2) processes. The relation of β_1, β_2 with s_1, s_2 is

$$s_{1,2} = \frac{1}{2}(1 - \beta_1 \pm \sqrt{(1-\beta_1)^2 + 4\beta_1\beta_2}) \tag{32}$$

The relation of s_1, s_2, a_1, γ with a and b can be obtained from (20) and (21) for the first canonical form and form (22) and (23) for the second canonical form.

If $\gamma > 0$, then

$$a = \frac{g_1 + \sqrt{g_1^2 - g_2}}{2e_1}, \quad b = 1 - \frac{a\gamma(1 - s_1 - s_2 + s_1 s_2)}{(1-a)(a(1 - s_1 - s_2) + s_1 s_2)}, \tag{33}$$

where

$$\begin{aligned} e_1 &= (1-s_1)(1-s_1-s_2)^2, \\ e_2 &= (1-s_1-s_2)(a_1(s_1-s_2)(1-\gamma) - s_1(1-s_1)), \\ e_3 &= \gamma(1-s_1)^2, \\ g_1 &= e_1 + e_2 - e_3(1-s_1-s_2), \\ g_2 &= 4e_1(e_2 + e_3 s_1) \end{aligned} \tag{34}$$

and if $\gamma < 0$, then

$$a = \frac{g_3 - \sqrt{g_3^2 + g_4}}{g_5}, \quad b = 1 - \frac{a\gamma(1 - s_1 - s_2 - s_1 s_2)}{(1-a)(a(1 - s_1 - s_2) + s_1 s_2)}, \tag{35}$$

where

$$\begin{aligned} e_6 &= a_1(s1 - s2)(1-\gamma), \\ e_7 &= (1-s_1)(s_2(1-\gamma) - (1-s_1-s_2)\gamma), \\ e_8 &= (1-s_1-s_2)(1-s_1)s_2, \\ g_3 &= -(1-s_1-s_2)e_6 + e_7 s_1 - e_8, \\ g_4 &= 4(e_6 + e_7)e_8 s_1, \\ g_5 &= -2(1-s_1-s_2)(e_6+e_7). \end{aligned} \tag{36}$$

Based on these relations the constraints of the canonical DMAP(2) processes can be obtained using the fact that all the elements of $\boldsymbol{D_0}$ and $\boldsymbol{D_1}$ have to be non-negative real numbers. That is a is real, $\beta_2 \le a \le 1$ and $0 \le b \le 1$. a is real when the expression under the square root sign in (33) for $\gamma > 0$ and in (33) for $\gamma < 0$ is non-negative. All together these constrains result in 5 inequalities for $\gamma > 0$ and 5 for $\gamma < 0$. Out of these the following ones are relevant.

- Case $\gamma > 0$:
  - a is real when $g_1^2 - g_2 \geq 0$ which translates to (27),
  - the inequality $b \leq 1$ translates to (28),
- Case $\gamma < 0$:
  - a is real when $g_3^2 + g_4 \geq 0$ which translates to (29),
  - the inequality $b \leq 0$ translates to (30).

We neglect the details of the other derivations here.

5 Explicit moments and correlation matching with the canonical forms

One of the most important applications of the introduced canonical forms is the moments and correlation matching of DMAP(2) processes. Using the different canonical forms ((16) - (19)) we can obtain analytical formulas for their 4 characterizing parameters the first 3 factorial moments (f_1, f_2, f_3) and the correlation parameter (γ). Obviously, the different canonical forms result in different equations.

The moments and correlation matching requires the inverse of the computation of these parameters, that is the appropriate canonical form and its parameters have to be found for a given f_1, f_2, f_3 and γ. Unfortunately, based on f_1, f_2, f_3 it is not obvious how to decide if the eigenvalues are positive or one of them is negative and consequently, it is not trivial to decide which canonical form needs to be used. However, for any given set of f_1, f_2, f_3 and γ parameters at most one canonical form gives a Markovian representation. In the following we present methods to obtain the different canonical DMAP(2) from f_1, f_2, f_3 and γ. These methods consist of two steps. The first step is the calculation of the representation of the stationary inter-arrival time, i.e., α and $\boldsymbol{A}$ of Theorem 2 and 3 using the first three factorial moments, the second step is the computation of the parameters associated with γ.

Transformation to DMAP(2) canonical form with positive eigenvalues
As in the previous section we will first consider the DMAP(2) canonical form with positive eigenvalues ((16) and (17)). In this case the first step is based on Table 3 in [9]. the s_1 and s_2 elements of matrix $\boldsymbol{A}$ and vector α can be calculated as

$$\alpha = [p, 1-p]\,,\ p = \frac{-z(h_3 - 6f_1h_1) + \sqrt{h_4}}{zh_3 + \sqrt{h_4}},$$
$$s_1 = 1 - \frac{h_3 - z\sqrt{h_4}}{h_2},\ s_2 = 1 - \frac{h_3 + z\sqrt{h_4}}{h_2},$$

where

$$h_1 = 2{f_1}^2 - 2f_1 - f_2,\ h_2 = 3{f_2}^2 - 2f_1f_3,$$
$$h_3 = 3f_1f_2 - 6(f_1 + f_2 - f_1^2) - f_3,\ h_4 = h_3^2 - 6h_1h_2,\ z = \frac{h_2}{|h_2|}.$$

The second step is the calculation of a, b of Theorem 6. If $\gamma = 0$, then $a = 1,\ b = 0$. If $\gamma > 0$, then a and b can be computed using

$$a = \frac{d_1 - \sqrt{d_2}}{2(1-s_1)}, \quad b = \frac{d_1 + \sqrt{d_2}}{2(1-s_2)},$$

with

$$d_1 = 1 - s_2 - p(1-s_2)(1-\gamma) + (1-s_1)\gamma, \quad d_2 = d_1^2 - 4(1-s_1)(1-s_2)\gamma.$$

If $\gamma \le 0$, then

$$a = \frac{-\gamma(1-s_2)}{p(1-s_2)(1-\gamma) - \gamma(1-s_1)}, \ b = \frac{p(1-s_2)(1-\gamma) - \gamma(1-s_1)}{1-s_2}.$$

Transformation to canonical form with a negative eigenvalue For the DMAP(2) canonical form with a negative eigenvalue the $\beta_1,\ \beta_2$ parameters and the α vector can be calculated using

$$\begin{aligned}
\beta_1 &= \frac{12f_1^2 - 3f_2(4+f_2) - 2f_3 + 2f_1(-6+3f_2+f_3)}{(3f_2^2 - 2f_1f_3)} \\
\beta_2 &= \frac{-3f_2(2-2f_1+f_2) + 2(-1+f_1)f_3}{12f_1^2 - 3f_2(4+f_2) - 2f_3 + 2f_1(-6+3f_2+f_3)} \\
p &= \frac{\beta_1 - f_1\beta_1 + \beta_2 + f_1\beta_1\beta_2}{-1+\beta_2}, \ \alpha = [p, 1-p]\,.
\end{aligned}$$

From β_1 and β_2 the eigenvalues s_1 and s_2 are obtained by (32). In the second step a, b of Theorem 7 are calculated. If $\gamma = 0$ then $a = 1,\ b = 0$ stands again. Otherwise

$$\begin{aligned}
a &= \frac{k_1 + \sqrt{k_1^2 - k_2}}{2\beta_1}, \ b = 1 - \frac{a\gamma(1-\beta_2)}{(1-a)(a-\beta_2)}, \quad \text{if } \gamma > 0, \\
a &= \frac{k_3 + \sqrt{k_3^2 + 4\beta_2k_4}}{2k_4}, \ b = -\frac{a\gamma(1-\beta_2)}{(1-a)(a-\beta_2)}, \quad \text{if } \gamma < 0,
\end{aligned}$$

where

$$\begin{aligned}
&k_1 = (1-\gamma)(p+\beta_1+\beta_2-p\beta_2) - 1 + \beta_1, \quad k_2 = 4\beta_1(k_1 - \beta_1 + \gamma - \beta_2\gamma), \\
&k_3 = (1-\gamma)(-p(1-\beta_2) - 2\beta_2) - \gamma(1-\beta_1), \ k_4 = k_3 + \beta_2 + \gamma - \beta_2\gamma.
\end{aligned}$$

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6 Conclusions

We have investigated the properties of order 2 DMAP and DRAP processes and found that some of their properties are identical with the ones of order 2 CMAP and CRAP, specifically the subset of order 2 DMAP and DRAP processes with positive eigenvalues can be mapped to the class of order 2 CMAP and CRAP, while the subset of order 2 DMAP and DRAP processes with one negative eigenvalue differs from the order 2 CMAP and CRAP and requires a different treatment. We showed that the whole set of order 2 DMAP and DRAP cannot be represented with acyclic Markovian $\boldsymbol{D_0}$ matrix, which was the case with order 2 CMAP and CRAP, but allowing cyclic representations as well the whole order 2 DRAP class can be represented with Markovian matrices.

We proposed a minimal (contains exactly 4 parameters) Markovian canonical representation of order 2 DMAPs and DRAPs. This canonical representation can be used efficiently for fitting, because the limits of the parameters are known a priori. Additionally, we presented simple explicit procedures for moments and correlation matching of canonical DMAP(2)s.

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