Canonical form of order 2 non-stationary Markov arrival processes*

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Abstract. Canonical forms of Markovian distributions and processes provide an efficient way of describing these structures by eliminating the redundancy of general representations. Canonical forms of order 2 stationary Markov arrival processes (MAPs) have already been established for both continuous and discrete time. In this paper we present canonical form of continuous time non-stationary MAPs of order 2. We also investigate the relation of the order 2 stationary Markov arrival processes and rational arrival processes. It turns out that the irreducibility of the underlying Markov chain has important qualitative consequences.

1 Introduction

Markov chain based stochastic models, and, among them, phase type distributions (PHs) and Markov arrival processes (MAPs), are used in a wide array of fields from healthcare [6] to risk theory [2] and, most notably, queueing theory [12]. One of the main benefits of using Markovian structures in queueing models is that they enable the application of the matrix analytic methodology [9], which provides a powerful tool for analysing these systems. When trying to model a real life system these Markovian structures have to be constructed by fitting to empirical data. Several fitting methods have been produced for this purpose. Some use special structures or heuristic fitting methods, e.g. [7], while other methods apply general optimisation techniques such as expectation maximisation [1], or a mixture of these two. Using special structures reduces the flexibility of the stochastic model, while using the general structure the efficiency diminishes due to the redundancy in the standard description of the respective stochastic models. This issue can be eliminated by the usage canonical forms. The canonical form of a Markovian distribution or process is its unique representation that is defined by a minimal number of parameters. This means that every distribution or process has to have a one-to-one correspondence with a canonical form description. In the past years canonical forms for several Markovian structures have been devised. Canonical forms have been established for order 2 phase type

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distributions (PHs) [13, 11] and stationary Markov arrival processes (MAPs) [3, 10], and order 3 phase type distributions [8, 13] in both continuous and discrete time.

The non-Markovian generalizations of these Markov chain based models, matrix exponential distributions and rational arrival processes, can be efficiently used for overcoming some limitations of the Markovian models. For example Markovian models with low coefficients of variation can be represented far more efficiently with non-Markovian generalizations [4]. In the analysis of these stochastic models it is an important to determine whether the Markovian and the non-Markovian class of the same order has the same flexibility or not. In the former case there is no need for the investigation of more complex non-Markovian models. For stationary Markov arrival processes (MAPs) it has been proved that the order 2 Markocian class and the order 2 non-Markovian class are identical [3, 10] (both in case of continuous time and discrete time models).

In this paper we focus on the non-stationary, continuous time, order 2 Markov arrival processes and rational arrival processes and investigate their canonical representation and the relation of those set of processes. We found that in some cases the results available for order 2 stationary MAPs similarly extends to order 2 non-stationary MAPs, but in other cases completely different qualitative properties occur.

The rest of the paper is organized as follows. The next section presents the necessary background of Markov arrival processes and rational arrival processes. Section 3 summarizes the existing results for stationary arrival processes of order 2. Section 4 and 5 present the new results, the canonical representation of non-stationary Markov arrival processes of order 2 and the relation of the Markovian and the non-Markovian sets of processes. The paper is concluded in Section 6.

2 Theoretical background

In this section we present the definitions and some basic characteristics of stationary and non-stationary Markov arrival processes and their non-Markovian generalizations.

Let $\mathcal{X}(t)$ be a point process on $\mathbb{R}^+$ with joint probability density function (joint pdf) of inter-event times $f(x_0, x_1, \ldots, x_k)$ for $k = 1, 2, \ldots$.

Definition 1. *$\mathcal{X}(t)$ is called a stationary rational arrival process if there exists a finite $(\boldsymbol{H_0}, \boldsymbol{H_1})$ square matrix pair such that $(\boldsymbol{H_0} + \boldsymbol{H_1})\mathbb{1} = \mathbf{0}$ (where $\mathbb{1}$ and $\mathbf{0}$ are the column vectors of ones and zeros, respectively, with appropriate size),*

$$\pi(-\boldsymbol{H_0})^{-1}\boldsymbol{H_1} = \pi, \quad \pi\mathbb{1} = \mathbb{1} \ , \tag{1}$$

has a unique solution, and for $\forall k \geq 0, \ x_0, \ldots, x_k$ its joint pdf is

$$f(x_0, x_1, \ldots, x_k) = \pi e^{\boldsymbol{H_0} x_0} \boldsymbol{H_1} e^{\boldsymbol{H_0} x_1} \boldsymbol{H_1} \ldots e^{\boldsymbol{H_0} x_k} \boldsymbol{H_1} \mathbb{1}. \tag{2}$$

In this case we say that $\mathcal{X}(t)$ is a stationary rational arrival process (RAP) with representation $(\boldsymbol{H_0}, \boldsymbol{H_1})$, or shortly, RAP$(\boldsymbol{H_0}, \boldsymbol{H_1})$.

Definition 2. *If $\mathcal{X}(t)$ is a stationary RAP($\boldsymbol{H_0}, \boldsymbol{H_1}$), where $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$ have the following properties:*

- $\boldsymbol{H_1}$ *has only non-negative elements*
- $\boldsymbol{H_0}_{ii} < 0$, $\boldsymbol{H_0}_{ij} \geq 0$ *for* $i \neq j$, $\boldsymbol{H_0}\mathbb{1} \leq 0$,

then we say that $\mathcal{X}(t)$ is a stationary Markov arrival process (MAP) with representation ($\boldsymbol{H_0}, \boldsymbol{H_1}$), or shortly, MAP($\boldsymbol{H_0}, \boldsymbol{H_1}$).

The pairs of matrices whose elements satisfy the sign properties of Definition 2 are referred to as Markovian.

The importance of the MAP class comes from the associated stochastic interpretation. Every MAP representation can be mapped to a continuous time Markov chain (referred to as the background Markov chain) with generator $\boldsymbol{H} = \boldsymbol{H_0} + \boldsymbol{H_1}$ where $\boldsymbol{H_1}$ contains transition rates with arrivals and $\boldsymbol{H_0}$ contains transition rates without arrivals and the Markov chain starts from initial distribution π which is the stationary probability vector embedded at arrivals. In such a Markov chain (2) is the joint pdf of the inter-arrival times. We note here that an arbitrary ($\boldsymbol{H_0}, \boldsymbol{H_1}$) square matrix pair satisfying (1) does not necessarily define a valid RAP as (2) may still give negative values for some $x_0, \ldots, x_k$. If an ($\boldsymbol{H_0}, \boldsymbol{H_1}$) matrix pair fulfils the additional sign constraints of MAPs in Definition 2, however, then (2) is guaranteed to be positive for arbitrary $x_0, \ldots, x_k$ as can be seen from the mapping to Markov chains. One of the major advantages of MAPs to RAPs is exactly this difference.

RAPs (MAPs) have infinite different representations (as it is demonstrated below for the non-stationary case), i.e. matrix pair sets that give the same $f(x_0, x_1, \ldots, x_k)$ joint probability density function. The different representations might have different sizes [5]. The size of the smallest among those representations is referred to as the order of the RAP (MAP). The class of order n RAPs (MAPs) is denoted by RAP(n) (MAP(n)). From Definition 1 and 2 it follows that MAP(n)⊆RAP(n).

In Definition 1 and 2 the initial vector in (2), π, has to fulfil (1). That is why π is also referred to as the embedded stationary vector. By relaxing this constraint we obtain the class of non-stationary RAPs and MAPs.

Definition 3. *$\mathcal{X}(t)$ is called a non-stationary rational arrival process if there exists a finite ($\pi_0, \boldsymbol{H_0}, \boldsymbol{H_1}$) initial vector and square matrix pair triple such that*

$$f(x_0, x_1, \ldots, x_k) = \pi_0 e^{\boldsymbol{H_0} x_0} \boldsymbol{H_1} e^{\boldsymbol{H_0} x_1} \boldsymbol{H_1} \ldots e^{\boldsymbol{H_0} x_k} \boldsymbol{H_1} \mathbb{1}. \qquad (3)$$

In this case we say that $\mathcal{X}(t)$ is a non-stationary rational arrival process (NRAP) with representation ($\pi_0, \boldsymbol{H_0}, \boldsymbol{H_1}$), or shortly, NRAP($\pi_0, \boldsymbol{H_0}, \boldsymbol{H_1}$).

Definition 4. *If $\mathcal{X}(t)$ is a non-stationary RAP($\pi_0, \boldsymbol{H_0}, \boldsymbol{H_1}$), where*

- π_0 *has only non-negative elements*
- $\boldsymbol{H_1}$ *has only non-negative elements*
- $\boldsymbol{H_0}_{ii} < 0$, $\boldsymbol{H_0}_{ij} \geq 0$ *for* $i \neq j$, $\boldsymbol{H_0}\mathbb{1} \leq 0$,

then we say that $\mathcal{X}(t)$ is a non-stationary Markov arrival process (NMAP) with representation $(\pi_0, \boldsymbol{H_0}, \boldsymbol{H_1})$, or shortly, NMAP$(\pi_0, \boldsymbol{H_0}, \boldsymbol{H_1})$.

Similar to the stationary case every NMAP representation can be mapped to a continuous time Markov chain with generator $\boldsymbol{H} = \boldsymbol{H_0} + \boldsymbol{H_1}$ where the initial distribution is π_0, and every NRAP (NMAP) has infinite different representations, i.e. $(\pi_0, \boldsymbol{H_0}, \boldsymbol{H_1})$ sets that give the same $f(x_0, x_1, \ldots, x_k)$ joint probability density function. One way to get a different representation of an NRAP$(\pi_0, \boldsymbol{H_0}, \boldsymbol{H_1})$ with the same size is the application of the similarity transformation

$$\pi_0' = \pi_0 \boldsymbol{T}, \quad \boldsymbol{H_0'} = \boldsymbol{T}^{-1}\boldsymbol{H_0}\boldsymbol{T}, \quad \boldsymbol{H_1'} = \boldsymbol{T}^{-1}\boldsymbol{H_1}\boldsymbol{T}, \tag{4}$$

where $\boldsymbol{T}$ is a non-singular transformation matrix with $\boldsymbol{T}\mathbb{1} = \mathbb{1}$. The transformed representation gives the same joint pdf as

$$\begin{aligned} f(x_0, x_1, \ldots, x_k) &= \pi e^{\boldsymbol{H_0} x_0} \boldsymbol{H_1} e^{\boldsymbol{H_0} x_1} \boldsymbol{H_1} \ldots e^{\boldsymbol{H_0} x_k} \boldsymbol{H_1} \mathbb{1} = \\ &= \pi \boldsymbol{T}\boldsymbol{T}^{-1} e^{\boldsymbol{H_0} x_0} \boldsymbol{T}\boldsymbol{T}^{-1} \boldsymbol{H_1} \boldsymbol{T}\boldsymbol{T}^{-1} e^{\boldsymbol{H_0} x_1} \boldsymbol{T}\boldsymbol{T}^{-1} \boldsymbol{H_1} \boldsymbol{T} \ldots \boldsymbol{T}^{-1} e^{\boldsymbol{H_0} x_k} \boldsymbol{T}\boldsymbol{T}^{-1} \mathbb{1} = \\ &= \pi e^{\boldsymbol{H_0'} x_0} \boldsymbol{H_1'} e^{\boldsymbol{H_0'} x_1} \boldsymbol{H_1'} \ldots e^{\boldsymbol{H_0'} x_k} \boldsymbol{H_1'} \mathbb{1} = f(x_0, x_1, \ldots, x_k), \end{aligned} \tag{5}$$

where we used that $\boldsymbol{T}^{-1}\mathbb{1} = \mathbb{1}$ (from $\boldsymbol{T}\mathbb{1} = \mathbb{1}$).

The order of NRAPs and NMAPs is defined similarly as for RAPs and MAPs. The class of order n NRAPs (NMAPs) is denoted by NRAP(n) (NMAP(n)). From Definition 3 and 4 it follows that NMAP(n)⊆NRAP(n).

2.1 Order classification of NMAP(2)

The order of NMAPs and the order of their stationary counterparts are not necessary identical. This problem is associated with the connectivity classification of the background Markov chain. In case of an irreducible Markov chain the set of states with nonzero transient ($t > 0$) probabilities and the set of states with nonzero stationary probabilities are identical, but it is not the case for Markov chains with a transient subset of states. Furthermore there are Markov chains with more than one communicating classes whose transient behavior depends on the initial distribution. Definition 2 excludes the presence of more than one communicating classes (by requiring the unique solution of the linear system) and the presence of transient states do not require special treatment due to the associated zero stationary probability. But neither of these properties apply for NMAPs (and NRAPs) in Definition 4 (and 3). In case of transient processes we need to consider transient states and multiple communicating classes. Fortunately for order two processes there are only a limited number of cases:

a) one communicating set of two states,
b) two communicating sets, each with one state,
c) one transient state and a communicating set of one state.

Case a) is referred to as order 2 NMAP with order 2 stationary behavior and is discussed in Section 4.1 and 5.1. Case b) represents a Poisson process (order 1 MAP) because the superposition of two independent Poisson processes is a Poisson process. Case c), referred to as order 2 NMAP with order 1 stationary behavior, is discussed in Section 4.2 and 5.2.

3 Previous results for MAP(2) and RAP(2) processes

Before discussing the canonical form of NMAP(2) we summarize the results on the canonical structure of MAP(2) from [3] as these will provide the basis for the subsequent argumentation.

For MAP(2)s one of the eigenvalues of matrix $(-\boldsymbol{H_0})^{-1}\boldsymbol{H_1}$ is 1, since $\pi(-\boldsymbol{H_0})^{-1}\boldsymbol{H_1} = \pi$. The other eigenvalue is denoted by γ, for which we have $-1 \leq \gamma < 1$. Based on the sign of γ the following canonical forms can be applied.

Theorem 1. *[3] If the γ parameter of the order 2 RAP($\boldsymbol{H_0}, \boldsymbol{H_1}$) is*

– *non-negative, then it can be represented in the following Markovian canonical form*

$$\boldsymbol{D_0} = \begin{bmatrix} -\lambda_1 & (1-a)\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} a\lambda_1 & 0 \\ (1-b)\lambda_2 & b\lambda_2 \end{bmatrix} .$$

where $0 < \lambda_1 \leq \lambda_2$, $0 < a, b < 1$, $b \geq a\frac{\lambda_1}{\lambda_2}$, $\gamma = ab$, and the associated embedded stationary vector is $\boldsymbol{\pi} = \left[\,\frac{1-b}{1-ab}\ \frac{b-ab}{1-ab}\,\right]$,

– *negative, then it can be represented in the following Markovian canonical form*

$$\boldsymbol{D_0} = \begin{bmatrix} -\lambda_1 & (1-a)\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} 0 & a\lambda_1 \\ b\lambda_2 & (1-b)\lambda_2 \end{bmatrix} ,$$

where $0 < \lambda_1 \leq \lambda_2$, $0 \leq a \leq 1$, $0 < b \leq 1$, $b \geq a\frac{\lambda_1}{\lambda_2}$, $\gamma = -ab$ and the associated embedded stationary vector is $\boldsymbol{\pi} = \left[\,\frac{b}{1+ab}\ 1 - \frac{b}{1+ab}\,\right]$.

Theorem 2. *[3] For the MAP(2) and RAP(2) sets of point processes we have*

$$MAP(2) \equiv RAP(2).$$

The aim of this paper is to verify the existence of Theorem 1 and 2 for non-stationary processes, NMAP(2) and NRAP(2).

4 Canonical form of order 2 NMAP

In this section we present the canonical form of NMAP(2) and prove that such canonical form is Markovian for any valid NMAP(2).

4.1 Canonical form of NMAP(2) with order 2 stationary behavior

Theorem 3. *An order 2 NMAP($\pi_0, \boldsymbol{H_0}, \boldsymbol{H_1}$) with order 2 stationary behavior can be represented in the ($\delta, \boldsymbol{D_0}, \boldsymbol{D_1}$)=($\pi_0\boldsymbol{T}, \boldsymbol{T}^{-1}\boldsymbol{H_0}\boldsymbol{T}, \boldsymbol{T}^{-1}\boldsymbol{H_1}\boldsymbol{T}$) canonical form, where $\boldsymbol{T}$ is the transformation matrix which transforms $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$ to the MAP(2) canonical form ($\boldsymbol{D_0}, \boldsymbol{D_1}$)=($\boldsymbol{T}^{-1}\boldsymbol{H_0}\boldsymbol{T}, \boldsymbol{T}^{-1}\boldsymbol{H_1}\boldsymbol{T}$).*

This theorem simply means that the canonical form that was used for MAP(2)s can be used for NMAP(2)s with a natural extension to the initial vector.

Proof. The sign properties of the elements of $\boldsymbol{D_0}$ and $\boldsymbol{D_1}$ defined in Definition 2 are trivially satisfied because of Theorem 1, thus the ($\delta, \boldsymbol{D_0}, \boldsymbol{D_1}$) representation is Markovian if and only if the elements of δ are non-negative, which can be formally described as $\delta\boldsymbol{e}_i \geq 0$ for $i = 1, 2$, where $\delta = \pi_0\boldsymbol{T}$ and $\boldsymbol{e}_i$ is the ith unit column vector (whose elements equal to zero except the ith one which is one). As π_0 is non-negative, δ will also be non-negative if the elements of $\boldsymbol{T}$ are non-negative. (This is a sufficient, but not a necessary condition.) In the following we show that $\boldsymbol{T}$ is indeed non-negative for any initial Markovian ($\pi_0, \boldsymbol{H_0}, \boldsymbol{H_1}$) representation.

Because every Markovian NMAP(2) representation can be obtained from the canonical forms using similarity transformations, the previous statement can be reversed to get the following equivalent: If ($\delta, \boldsymbol{D_0}, \boldsymbol{D_1}$) is an arbitrary NMAP(2) in canonical form, then its similarity transform

$$(\pi_0, \boldsymbol{H_0}, \boldsymbol{H_1}) = (\delta\boldsymbol{T}^{-1}, \boldsymbol{T}\boldsymbol{D_0}\boldsymbol{T}^{-1}, \boldsymbol{T}\boldsymbol{D_1}\boldsymbol{T}^{-1}) \tag{6}$$

is Markovian only if $\boldsymbol{T}$ is non-negative. In other words we "reverse similarity transform" the canonical form (note that here the transformation matrix is $\boldsymbol{T}^{-1}$ while in (4) it was $\boldsymbol{T}$) and examine what could the original representation be and prove that for every possible original representation that satisfies the MAP representation constraints in Definition 2, matrix $\boldsymbol{T}$ is non-negative. In the following we prove this last version of the theorem.

We have different canonical forms for negative and non-negative γ that we have to examine separately. Let us first consider NMAPs with non-negative γ. In this case the matrices of the canonical form are

$$\boldsymbol{D_0} = \begin{bmatrix} -\lambda_1 & (1-a)\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} a\lambda_1 & 0 \\ (1-b)\lambda_2 & b\lambda_2 \end{bmatrix}.$$

Let

$$\boldsymbol{T} = \begin{bmatrix} 1-t_1 & t_1 \\ t_2 & 1-t_2 \end{bmatrix}. \tag{7}$$

From (6) we get

$$\boldsymbol{H_0} = \begin{bmatrix} -\frac{(1-t_1)(1-at_2)\lambda_1 - t_1 t_2 \lambda_2}{1-t_1-t_2} & \frac{(1-t_1)((1-a(1-t_1))\lambda_1 - t_1\lambda_2)}{1-t_1-t_2} \\ \frac{t_2((1-t_2)\lambda_2 + at_2\lambda_1 - \lambda_1)}{1-t_1-t_2} & -\frac{(1-t_1)(1-t_2)\lambda_2 - (1-a-at_1)t_2\lambda_1}{1-t_1-t_2} \end{bmatrix}$$
$$\boldsymbol{H_1} = \begin{bmatrix} \frac{a(1-t_1)(1-t_2)\lambda_1 + t_1\lambda_2(1-b-t_2)}{1-t_1-t_2} & \frac{t_1(\lambda_2(b-t_1) - a\lambda_1(1-t_1))}{1-t_1-t_2} \\ \frac{(1-t_2)(a\lambda_1 t_2 + \lambda_2(1-b-t_2))}{1-t_1-t_2} & \frac{\lambda_2(1-t_2)(b-t_1) - a\lambda_1 t_1 t_2}{1-t_1-t_2} \end{bmatrix}. \tag{8}$$

We have to prove that if the off-diagonal elements of $\boldsymbol{H_0}$ and the elements of $\boldsymbol{H_1}$ are non-negative, then $0 \le t_1 \le 1$ and $0 \le t_2 \le 1$. By using the restrictions on a, b in Theorem 1 we can derive constraints from the elements of $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$. From the $(1,2)$ element of $\boldsymbol{H_0}$ we obtain

$$t_1 + t_2 < 1 \;\&\&\; \left(t_1 > 1 \;||\; t_1 < \frac{(1-a)\lambda_1}{\lambda_2 - a\lambda_1}\right) \tag{9a}$$

or

$$t_1 + t_2 > 1 \;\&\&\; t_1 > 1 \;\&\&\; t_1 > \frac{(1-a)\lambda_1}{\lambda_2 - a\lambda_1} \tag{9b}$$

From the $(2,1)$ element of $\boldsymbol{H_0}$ we have

$$t_1 < \frac{(1-a)\lambda_1}{\lambda_2 - a\lambda_1} \;\&\&\; t_1 + t_2 > 1 \;||\; \left(0 < t_2 \;\&\&\; t_2 < \frac{\lambda_2 - \lambda_1}{\lambda_2 - a\lambda_1}\right) \tag{10a}$$

or

$$t_1 > \frac{(1-a)\lambda_1}{\lambda_2 - a\lambda_1} \;\&\&\; (t_1 < 1) \;\&\&\; \left(t_2 > \frac{\lambda_2 - \lambda_1}{\lambda_2 - a\lambda_1} \;||\; (0 < t_2 \;\&\&\; t_1 + t_2 \le 1)\right) \tag{10b}$$

Combining the constraints for the two elements we get

$$t_1 < \frac{\lambda_1(1-a)}{\lambda_2 - \lambda_1} \;\&\&\; 0 < t_2 < \frac{\lambda_2 - \lambda_1}{\lambda_2 - a\lambda_1} \tag{11a}$$

or

$$\frac{\lambda_1(1-a)}{\lambda_2 - a\lambda_1} < t_1 < 1 \;\&\&\; t_2 > \frac{\lambda_2 - \lambda_1}{\lambda_2 - a\lambda_1} \tag{11b}$$

The first case corresponds to $t_1 + t_2 < 1$, while the second to $t_1 + t_2 > 1$. From the $(1,1)$ element of $\boldsymbol{H_1}$ we have

$$t_1 + t_2 < 1 \;\&\&\; 0 < t_1 < \frac{b\lambda_2 - a\lambda_1}{\lambda_2 - a\lambda_1} \tag{12a}$$

or

$$t_1 + t_2 > 1 \;\&\&\; \left(t_1 > \frac{b\lambda_2 - a\lambda_1}{\lambda_2 - a\lambda_1} \;||\; t_1 < 0\right) \tag{12b}$$

From (11a) and (12a) we get that $t_1 > 0$ and $t_2 > 0$ for the $t_1 + t_2 < 1$ case, from which $0 \leq t_1, t_2 \leq 1$ if $t_1 + t_2 < 1$. It remains to show the same for the $t_1 + t_2 > 1$ case. From (11b) we have that $0 \leq t_1 \leq 1$ and $0 \leq t_2$ thus we only have to prove that $t_2 \leq 1$ also holds. From the $(2,1)$ element of $\boldsymbol{H_1}$ we get

$$t_1 < 0 \;\&\&\; \left(t_2 < \frac{(1-b)\lambda_2}{\lambda_2 - a\lambda_1} \;||\; (t_1 + t_2 < 1 \;\&\&\; t_2 > 1)\right) \tag{13a}$$

or

$$t_1 > 0 \;\&\&\; t_1 < \frac{b\lambda_2 - a\lambda_1}{\lambda_2 - a\lambda_1} \;\&\&\; \left(t_2 < \frac{(1-b)\lambda_2}{\lambda_2 - \lambda_1} \;||\; (t_1 + t_2 > 1 \;\&\&\; t_2 < 1)\right) \tag{13b}$$

or

$$t_1 > \frac{b\lambda_2 - a\lambda_1}{\lambda_2 - a\lambda_1} \;\&\&\; \left(t_1 + t_2 < 1 \;||\; \left(t_2 < 1 \;\&\&\; t_2 > \frac{(1-b)\lambda_2}{\lambda_2 - a\lambda_1}\right)\right) \tag{13c}$$

In the (13c) subcase $t_2 \leq 1$ is explicitly stated if $t_1 + t_2 > 1$. For the other two subcases $t_2 \leq 1$ holds if $\frac{(1-b)\lambda_2}{\lambda_2 - a\lambda_1} \leq 1$. But this is true because from Theorem 1 we know that $b \geq a\frac{\lambda_1}{\lambda_2}$, consequently we can use the transformation

$$t_2 < \frac{(1-b)\lambda_2}{\lambda_2 - a\lambda_1} \leq \frac{\left(1 - a\frac{\lambda_1}{\lambda_2}\right)\lambda_2}{\lambda_2 - a\lambda_1} = 1. \tag{14}$$

Substituting this into (13a) and (13b) we obtain that $t_2 < 1$ if $t_1 + t_2 > 1$ in both subcases, thus $0 \leq t_1, t_2 \leq 1$ is proven for $\gamma \geq 0$, which means that the proposed canonical form is valid for $\gamma \geq 0$.

Let us now consider NMAPs with $\gamma < 0$. The matrices of the canonical form are

$$\boldsymbol{D_0} = \begin{bmatrix} -\lambda_1 & (1-a)\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} 0 & a\lambda_1 \\ b\lambda_2 & (1-b)\lambda_2 \end{bmatrix}.$$

Using the (6) similarity transformation we get

$$\begin{aligned} \boldsymbol{H_0} &= \begin{bmatrix} -\frac{(1-t_1)(1-at_2)\lambda_1 - t_1 t_2 \lambda_2}{1-t_1-t_2} & \frac{(1-t_1)((1-a(1-t_1))\lambda_1 - t_1\lambda_2)}{1-t_1-t_2} \\ \frac{t_2(1-t_2)\lambda_2 + at_2\lambda_1 - \lambda_1}{1-t_1-t_2} & -\frac{(1-t_1)(1-t_2)\lambda_2 - (1-a+at_1)t_2\lambda_1}{1-t_1-t_2} \end{bmatrix} \\ \boldsymbol{H_1} &= \begin{bmatrix} \frac{t_1(1-b-t_2)\lambda_2 - a(1-t_1)t_2\lambda_1}{1-t_1-t_2} & \frac{a(1-t_1)^2\lambda_1 + t_1(b-t_1)\lambda_2}{1-t_1-t_2} \\ \frac{(1-t_2)(1-b-t_2)\lambda_2 - at_2^2\lambda_1}{1-t_1-t_2} & \frac{a(1-t_1)t_2\lambda_1 + (b-t_1)(1-t_2)\lambda_2}{1-t_1-t_2} \end{bmatrix}. \end{aligned} \tag{15}$$

We apply the same approach as before, i.e., we prove that if the respective elements of $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$ are non-negative then $0 \leq t_1, t_2 \leq 1$ has to hold. The $\boldsymbol{H_0}$ matrix is the same as for $\gamma \geq 0$, therefore we get (11) again. In the following we use substitutions

$$e = \frac{b\lambda_2 - 2a\lambda_1}{\lambda_2 - a\lambda_1}, \quad f = \frac{(2-b)\lambda_2}{\lambda_2 - a\lambda_1}, \quad g = \frac{\sqrt{\lambda_2(4a(1-b) + b^2\lambda_2)}}{\lambda_2 - a\lambda_1}$$

From the $(1,2)$ element of $\boldsymbol{H_1}$

$$t_1 + t_2 > 1 \ \&\& \ t_1 > \frac{1}{2}(e+g) \tag{16a}$$

or

$$t_1 + t_2 > 1 \ \&\& \ t_1 < \frac{1}{2}(e-g) \tag{16b}$$

or

$$t_1 + t_2 < 1 \ \&\& \ t_1 < \frac{1}{2}(e+g) \ \&\& \ t_1 > \frac{1}{2}(e-g). \tag{16c}$$

from the constraints on the $(2,1)$ element of $\boldsymbol{H_1}$ we get

$$t_1 + t_2 < 1 \ \&\& \ t_2 < \frac{1}{2}(f-g) \ \&\& \ t1 < \frac{1}{2}(e+g) \tag{17a}$$

or

$$t_1 + t_2 < 1 \ \&\& \ t1 > \frac{1}{2}(e+g) \tag{17b}$$

or

$$t_1 + t_2 < 1 \ \&\& \ t_2 < \frac{1}{2}(f+g) \ \&\& \ t1 < \frac{1}{2}(e-g) \tag{17c}$$

or

$$t_1 + t_2 > 1 \ \&\& \ t_2 < \frac{1}{2}(f+g) \ \&\& \ \frac{1}{2}(e-g) < t1 < \frac{1}{2}(e+g) \tag{17d}$$

or

$$t_1 + t_2 > 1 \ \&\& \ \frac{1}{2}(f-g) < t_2 < \frac{1}{2}(f+g) \ \&\& \ t1 > \frac{1}{2}(e-g) \tag{17e}$$

If (16a) then only the (17e) subcase is possible. None of the constraints in (17) allow (16b) to be true, while is (16c) only possible for the (17a) subcase. Summarising these we get

$$t_1 + t_2 > 1 \ \&\& \ \frac{1}{2}(f-g) < t_2 < \frac{1}{2}(f+g) \ \&\& \ t1 > \frac{1}{2}(e+g) \tag{18a}$$

or

$$t_1 + t_2 < 1 \ \&\& \ \frac{1}{2}(e-g) < t_1 < \frac{1}{2}(e+g) \ \&\& \ t2 < \frac{1}{2}(f-g) \tag{18b}$$

Using the further substitution

$$h = \frac{(1-b)t_1\lambda_2}{t_1\lambda_2 + a(1-t_1)\lambda_1}$$

from the constraints on the $(1,1)$ element of $\boldsymbol{H_1}$ we obtain

$$t_1 + t_2 < 1 \;\&\&\; t_1 < \frac{1}{2}(e - g) \;\&\&\; t_2 > h \tag{19a}$$

or

$$t_1 + t_2 < 1 \;\&\&\; t_1 > \frac{1}{2}(e + g) \tag{19b}$$

or

$$t_1 + t_2 > 1 \;\&\&\; \frac{1}{2}(e - g) < t_1 < -\frac{a\lambda_1}{\lambda_2 - a\lambda_1} \;\&\&\; t_2 < h \tag{19c}$$

or

$$t_1 + t_2 > 1 \;\&\&\; -\frac{a\lambda_1}{\lambda_2 - a\lambda_1} < t_1 < \frac{1}{2}(e + g) \tag{19d}$$

or

$$-\frac{a\lambda_1}{\lambda_2 - a\lambda_1} < t_1 < \frac{1}{2}(e + g) \;\&\&\; t_2 < h \tag{19e}$$

or

$$t_1 > \frac{1}{2}(e + g) \;\&\&\; t_2 > h \tag{19f}$$

We combine these with the previously obtained expressions for the elements of $\boldsymbol{H_0}$ and $\boldsymbol{H_1}$. Let us examine first the $t_1+t_2 < 1$ case. As (18b) has to hold, only the (19e) subcase is possible. Because of the constraints on the elements of $\boldsymbol{H_0}$, for $t_1 + t_2 < 1$ (11a) has to hold, i.e., $t_1 < \frac{\lambda_1(1-a)}{\lambda_2-\lambda_1} \leq 1$ and $0 \leq t_2 \leq \frac{\lambda_2-\lambda_1}{\lambda_2-a\lambda_1} \leq 1$ are true, thus we only have to show that $0 \leq t_1$ is also true. From (19e) we have that

$$t_2 = h = \frac{(1-b)t_1\lambda_2}{t_1\lambda_2 + a(1-t_1)\lambda_1}. \tag{20}$$

From Definition 1 we know that $\lambda_2 \geq \frac{a}{b}\lambda_1$ has to hold. The expression is monotonically increasing in t_1 for $t_1 > -\frac{a\lambda_1}{\lambda_2-a\lambda_1}$ (which is the other constraint in (19e)) and non-negative only if $t_1 \geq 0$. But we know from (11a), that $t_2 \geq 0$ has to hold, thus, for t_2 to have a valid range $t_1 \geq 0$ also has to be true. This proves that $0 \leq t_1, t_2 \leq 1$ if $t_1 + t_2 < 1$.

Finally, from (11b) we have that $0 \leq \frac{\lambda_2(1-a)}{\lambda_2-\lambda_1} \leq t_1 \leq 1$ and $0 \leq \frac{\lambda_2-\lambda_1}{\lambda_2-a\lambda_1} \leq t_2$, thus we only have to show that $t_2 \leq 1$. First we note that from (19) only the (19f) subcase is possible due to the $t_1 > \frac{1}{2}(e+g)$ condition in (18). We use the constraints on the $(2,2)$ element of $\boldsymbol{H_1}$ with substitutions

$$i = \frac{b\lambda_2 - a\lambda_1}{\lambda_2 - \lambda_1}, \quad j = \frac{(b-t_1)\lambda_2}{(b-t_1)\lambda_2 - a(1-t_1)\lambda_1}$$

to get

$$t_1 + t_2 < 1 \;\&\&\; t_1 < \frac{1}{2}(e-g) \tag{21a}$$

or

$$t_1 + t_2 < 1 \;\&\&\; \frac{1}{2}(e-g) < t_1 < i \tag{21b}$$

or

$$t_1 + t_2 < 1 \;\&\&\; i < t_1 < \frac{1}{2}(e+g) \;\&\&\; t2 > j \tag{21c}$$

or

$$t_1 < \frac{1}{2}(e-g) \;\&\&\; t_2 < j \tag{21d}$$

or

$$\frac{1}{2}(e-g) < t_1 < i \;\&\&\; t_2 > j \tag{21e}$$

or

$$t_1 + t_2 > 1 \;\&\&\; t_1 > \frac{1}{2}(e+g) \;\&\&\; t_2 < j \tag{21f}$$

For the (21d)-(21f) subcases $t_1 + t_2 > 1$ holds, but because of the constraint $t_1 > \frac{1}{2}(e+g)$ in (18a), subcase (21d) and (21e) are not possible. For the last remaining (21f) subcase the last needed constraint $t_2 \leq 1$ has to hold, because

$$t_2 < j = \frac{(b-t_1)\lambda_2}{(b-t_1)\lambda_2 - a(1-t_1)\lambda_1} \leq 1$$

as j is monotonously increasing between $t_1 = b$ and $t_1 = 1$ and $j = 1$ if $t_1 = 1$, but we know from (11) that $t_1 \leq 1$, thus $t_2 \leq 1$ also holds.

4.2 Canonical form of NMAP(2) with order 1 stationary behavior

In this section we present the canonical form of NMAP(2) with order 1 stationary behavior. Recall that this is only possible, if its background Markov chain consists of a transient state and a single absorbing state. If we number the states such that state 1 is the transient state then every such NMAP(2) has the following general form

$$\pi_0 = [p, 1-p], \quad \boldsymbol{H_0} = \begin{bmatrix} -\lambda_1 & a\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{H_1} = \begin{bmatrix} b\lambda_1 & (1-a-b)\lambda_1 \\ 0 & \lambda_2 \end{bmatrix}. \tag{22}$$

Because state 1 is a transient state, transitions from state 1 are not restricted in any way, but transition from state 2 to state 1 is not possible. Note that the Markovian constraints (i.e. $\lambda_1, \lambda_2 > 0,\ 0 \leq a, b \leq 1,\ a \leq 1-b$) still have to hold, but λ_1 can be both smaller and larger than λ_2. The canonical form of such an NMAP(2) can be described by parameters λ_1, λ_2 and b as stated in the following theorem.

Theorem 4. *Let us consider an order 2 NMAP with order 1 stationary behavior described by Markovian representation*

$$\pi_0 = [p, 1-p], \quad \boldsymbol{H_0} = \begin{bmatrix} -\lambda_1 & a\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{H_1} = \begin{bmatrix} b\lambda_1 & (1-a-b)\lambda_1 \\ 0 & \lambda_2 \end{bmatrix}. \tag{23}$$

This NMAP can be represented in one of the following two canonical forms

– *if* $\lambda_2 \geq (1-a)\lambda_1$*, then it can be represented as*

$$\delta = \left[p\frac{\lambda_2 - (1-a)\lambda_1}{\lambda_2 - b\lambda_1}, 1 - \frac{\lambda_2 - (1-a)\lambda_1}{\lambda_2 - b\lambda_1}\right]$$
$$\boldsymbol{D_0} = \begin{bmatrix} -\lambda_1 & (1-b)\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} b\lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

  where $0 < \lambda_1, \lambda_2$ *and* $0 \leq b \leq 1$

– *if* $\lambda_2 < (1-a)\lambda_1$*, then it can be represented as*

$$\delta = \left[p\frac{\lambda_2 - (1-a)\lambda_1}{\lambda_2 - \lambda_1}, 1 - \frac{\lambda_2 - (1-a)\lambda_1}{\lambda_2 - \lambda_1}\right]$$
$$\boldsymbol{D_0} = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} b\lambda_1 & (1-b)\lambda_1 \\ 0 & \lambda_2 \end{bmatrix}$$

  where $0 < \lambda_1, \lambda_2$ *and* $0 \leq b \leq 1$

Note that the λ_1, λ_2 and b parameters are the same as in the general representation in (22).

Proof. If we similarity transform the general representation in (22), as long as the new representation is Markovian, its $\boldsymbol{D_0}$ matrix has to have a zero off-diagonal element, otherwise the background Markov chain of the new representation would have a single communicating class of two states, which is not possible. Because of this, if we fix that state 1 is the transient state then the $(2,1)$ element of $\boldsymbol{D_0}$ is zero, and the $(1,1)$ and $(2,2)$ elements are the eigenvalues of $\boldsymbol{D_0}$, λ_1 and λ_2, that is, these three elements are fixed. Let us consider a similarity transformation with transformation matrix $\boldsymbol{T}$ such that

$$\boldsymbol{D_0} = \boldsymbol{T}^{-1}\boldsymbol{H_0}\boldsymbol{T} = \begin{bmatrix} -\lambda_1 & x\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \tag{24}$$

where $\boldsymbol{H_0}$ is in the general form (22). Than

$$\boldsymbol{T} = \begin{bmatrix} \frac{\lambda_2-(1-a)\lambda_1}{\lambda_2-(1-x)\lambda_1} & \frac{(a-x)\lambda_1}{\lambda_2-(1-x)\lambda_1} \\ 0 & 1 \end{bmatrix}. \tag{25}$$

The previous expression shows that there is only one degree of freedom (represented by x) in the similarity transformation for the order 1 stationary behavior. Applying this similarity transformation to $\boldsymbol{H_1}$ we get

$$\boldsymbol{D_1} = \boldsymbol{T}^{-1}\boldsymbol{H_1}\boldsymbol{T} = \begin{bmatrix} b\lambda_1 & (1-b-x)\lambda_1 \\ 0 & \lambda_2 \end{bmatrix}, \tag{26}$$

from which $\boldsymbol{D_0}$ and $\boldsymbol{D_1}$ are Markovian for $0 \le x \le 1-b$. When we apply the transformation to π_0, we get

$$\delta = \pi_0 \boldsymbol{T} = \left[p\frac{\lambda_2 - (1-a)\lambda_1}{\lambda_2 - (1-x)\lambda_1}, 1 - p\frac{\lambda_2 - (1-a)\lambda_1}{\lambda_2 - (1-x)\lambda_1} \right]. \tag{27}$$

It remaind to show that $0 < \delta e_1 < 1$. If $\lambda_2 > (1-a)\lambda_1$, then $p\lambda_2 - (1-a)\lambda_1$, the numerator of δe_1 is positive, thus the denominator also has to be positive for δe_1 to be non-negative. If $x > a$, then the denominator of δe_1 is positive and greater than $\lambda_2 - (1-a)\lambda_1$, from which $0 \le \delta e_1 < p$. In this case the canonical form takes the largest feasible value $x = 1-b$. If $\lambda_2 > (1-a)\lambda_1$, then the numerator of δe_1 is negative, thus the denominator has to be negative as well. If $x < a$ this will be true and $0 < \delta e_1 < p$ will hold. In this case the canonical form takes the smallest feasible value with $x = 0$. Substituting these values into the formulas of $\pi_0 \boldsymbol{T}$, $\boldsymbol{T}^{-1}\boldsymbol{H_0}\boldsymbol{T}$, $\boldsymbol{T}^{-1}\boldsymbol{H_1}\boldsymbol{T}$ provides the theorem.

5 Relation of the NMAP(2) and NRAP(2) classes

5.1 Equivalence of the NMAP(2) and NRAP(2) classes with order 2 stationary behavior

Theorem 5. *For the NMAP(2) and NRAP(2) with order 2 stationary behavior sets of point processes we have*

$$\textit{NMAP(2)} \equiv \textit{NRAP(2)}.$$

That is, every NRAP(2) process has a Markovian representation of size 2.

Proof. The proof follows a similar pattern as the one that proves the equivalence between MAP(2) and RAP(2) in [3], therefore we reiterate some of the main points from there. The kth inter-arrival time in an NRAP has joint probability density function

$$f(X_k = x_k | X_0 = x_0, X_1 = x_1, \dots, X_{k-1} = x_{k-1}) = \\ = \frac{\pi_0 e^{\boldsymbol{H_0}x_0}\boldsymbol{H_1} e^{\boldsymbol{H_0}x_1}\boldsymbol{H_1} \dots e^{\boldsymbol{H_0}x_{k-1}}\boldsymbol{H_1}}{\pi_0 e^{\boldsymbol{H_0}x_0}\boldsymbol{H_1} e^{\boldsymbol{H_0}x_1}\boldsymbol{H_1} \dots e^{\boldsymbol{H_0}x_{k-1}}\boldsymbol{H_1}\mathbb{1}} e^{\boldsymbol{H_0}x_k}\boldsymbol{H_1}\mathbb{1}. \tag{28}$$

The X_k random variable has to have a valid distribution for $\forall k \ge 0$ and $\forall x_0, \dots, x_k \ge 0$. Let $\pi_k(x_0, x_1, \dots, x_{k-1})$ be the initial vector before the kth inter-arrival, X_k. It is given by

$$\pi_k(x_0, x_1, \dots, x_{k-1}) = \frac{\pi_0 e^{\boldsymbol{H_0}x_0}\boldsymbol{H_1} e^{\boldsymbol{H_0}x_1}\boldsymbol{H_1} \dots e^{\boldsymbol{H_0}x_{k-1}}\boldsymbol{H_1}}{\pi_0 e^{\boldsymbol{H_0}x_0}\boldsymbol{H_1} e^{\boldsymbol{H_0}x_1}\boldsymbol{H_1} \dots e^{\boldsymbol{H_0}x_{k-1}}\boldsymbol{H_1}\mathbb{1}}. \tag{29}$$

If $(\pi_0, \boldsymbol{H_0}, \boldsymbol{H_1})$ in the previous expression is an NRAP(2) representation in NMAP(2) canonical form, then the first element of vector $\pi_k(x_0, x_1, \dots, x_{k-1})$ has to be in the range of

$$0 \le \pi_k(x_0, x_1, \dots, x_{k-1})\boldsymbol{e_1} \le \frac{1}{1 - a\frac{\lambda_1}{\lambda_2}}, \tag{30}$$

otherwise the joint pdf in (28) is not strictly non-negative (see [3] for more details). To prove the equivalence of NRAP(2) and NMAP(2) we show that if the $\boldsymbol{H_0}$, $\boldsymbol{H_1}$ matrices of an NRAP(2) are in canonical form, then its initial vector is Markovian (non-negative) as well. Let $u(x,t)$ be the first element of the initial vector after an inter-arrival time of length t if the initial vector after the previous arrival was $[x, 1-x]$. Then $u(x,t)$ can be expressed as

$$u(x,t) = \frac{[x, 1-x]e^{\boldsymbol{H_0}t}\boldsymbol{H_1}\boldsymbol{e}_1}{[x, 1-x]e^{\boldsymbol{H_0}t}\boldsymbol{H_1}\mathbb{1}}. \tag{31}$$

From (30) it is clear that $\delta\boldsymbol{e}_1 \geq 0$ holds regardless of the value of γ. We have to show that $\delta\boldsymbol{e}_1 \leq 1$ ($\delta\boldsymbol{e}_2 \geq 0$) is also true. We will assume a series of arrivals with negligibly small inter-arrival time ($t \to 0$ and consequently $e^{\boldsymbol{H_0}t} \to \boldsymbol{I}$) and prove that for x to satisfy the constraints in (30) $\delta\boldsymbol{e}_1 \leq 1$ has to hold. First we examine the $\gamma \geq 0$ case. From (31) after using the respective canonical form in Definition 1 and simplifying the expression we get that

$$u(x,0) = \frac{[x, 1-x]\boldsymbol{H_1}\boldsymbol{e}_1}{[x, 1-x]\boldsymbol{H_1}\mathbb{1}} = \frac{ax\lambda_1 + (1-x)(1-b)\lambda_2}{ax\lambda_1 + (1-x)\lambda_2}. \tag{32}$$

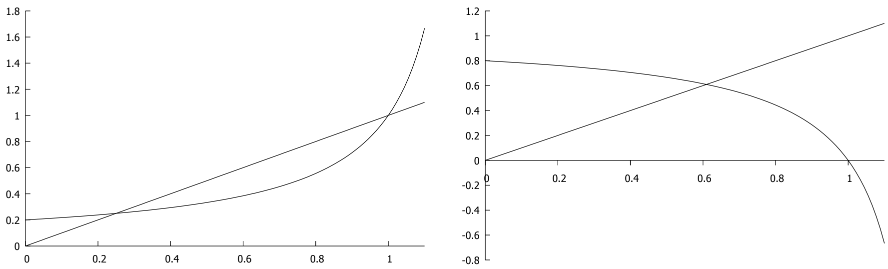

Fig. 1. Behavior of $u(x,0)$ for positive (left) and negative (correlation) for $\lambda_1 = 1$, $\lambda_2 = 2$, $a = 0.4$, $b = 0.8$

This function is a hyperbola that has one or two fix points (points where $u(x,0) = x$, see Figure 1 for illustration). These are $\frac{(1-b)\lambda_2}{\lambda_2 - a\lambda_1}$ and 1. (There is only one fix point in $x = 1$ if $a = b = 0$ or $\lambda_2 = \frac{a}{b}\lambda_1$.) Because $\lambda_2 \geq \frac{a}{b}\lambda_1$, we know that $x = 1$ is the higher fix point. The first element of δ cannot be higher than this value, because for $1 < x < \frac{1}{1-a\frac{\lambda_1}{\lambda_2}}$ we have $u(x,0) > x$, which means that the first coordinate of the initial vector would increase after every arrival and would finally go above the upper limit of $x = \frac{1}{1-a\frac{\lambda_1}{\lambda_2}}$ in (30) (this value is the vertical asymptote of the hyperbola). This means that $0 \leq \delta\boldsymbol{e}_1 \leq 1$ has to hold for $\gamma \geq 0$.

Now let us investigate the $\gamma < 0$ case. As before we examine the $u(x,0)$ function and substitute the canonical form from Definition 1 corresponding to $\gamma < 0$. Doing so we get

$$u(x,0) = \frac{[x, 1-x]\boldsymbol{H_1}\boldsymbol{e}_1}{[x, 1-x]\boldsymbol{H_1}\mathbb{1}} = \frac{b\lambda_2(1-x)}{(1-x)\lambda_2 + ax\lambda_1}. \tag{33}$$

Again from (30) we know that $\delta\boldsymbol{e}_1 < \frac{1}{1-a\frac{\lambda_1}{\lambda_2}}$, and we have to prove that $\delta\boldsymbol{e}_1 < 1$ The numerator of the expression becomes negative for $x > 1$, while the denominator is negative for $1 < x < \frac{1}{1-a\frac{\lambda_1}{\lambda_2}}$, thus for $\delta\boldsymbol{e}_1 > 1$ the first coordinate of the initial vector would become negative, which is not allowed according to (30). Thus $0 \le \delta\boldsymbol{e}_1 \le 1$ has to hold for $\gamma < 0$ as well.

To summarize, we showed that $0 < \delta\boldsymbol{e}_1 \le 1$ has to hold for any NRAP(2) transformed to the canonical NMAP(2) representation thus we proved that any NRAP(2) can be transformed to a Markovian canonical form, thus NRAP(2)≡NMAP(2).

5.2 Relation of the NMAP(2) and NRAP(2) classes with order 1 stationary behavior

Theorem 6. *The NMAP(2) sets of point processes with order 1 stationary behavior is a valid subset of the NRAP(2) sets of point processes with order 1 stationary behavior. That is*

$$\textit{NMAP(2)} \subset \textit{NRAP(2)}.$$

That is, there exist NRAP(2) process which does not have a Markovian representation of size 2.

Proof. The following NRAP(2) process which does not have a Markovian representation of size 2

$$\pi_0 = [2, -1]\,, \quad \boldsymbol{D_0} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \boldsymbol{D_1} = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 2 \end{bmatrix}.$$

6 Conclusion

In this paper we proposed a canonical form for NMAP(2)s and proved that this canonical form is Markovian for every NMAP(2). We also investigated the relation of the classes of NMAP(2) and NRAP(2) processes and we found that non-irreducible background Markov chains causes unexpected qualitative behaviors. In the course of this work we got informed of a similar effort [14] with partially similar goals. [14] considers only the canonical form of NMAP(2) with order 2 stationary behavior and it obtains the same conclusion as Theorem 3, but with a different approach.

References

1. Asmussen, S., Nerman, O., Olsson, M.: Fitting phase-type distributions via the EM algorithm. Scandinavian Journal of Statistics pp. 419–441 (1996)
2. Bladt, M.: A review on phase-type distributions and their use in risk theory. ASTIN Bulletin 35, 145–161 (5 2005), http://journals.cambridge.org/article_S0515036100014100
3. Bodrog, L., Heindl, A., Horváth, G., Telek, M.: A Markovian canonical form of second-order matrix-exponential processes. European Journal of Operation Research 190, 459–477 (2008)
4. Buchholz, P., Horvath, A., Telek, M.: Stochastic Petri nets with low variation matrix exponentially distributed firing time. International Journal of Performability Engineering 7, 441–454 (2011), special issue on Performance and Dependability Modeling of Dynamic Systems
5. Buchholz, P., Telek, M.: On minimal representation of rational arrival processes. Annals of Operations Research 202, 35–58 (2013), dOI 10.1007/s10479-011-1001-5
6. Fackrell, M.: Modelling healthcare systems with phase-type distributions. Health Care Management Science 12(1), 11–26 (2009), http://dx.doi.org/10.1007/s10729-008-9070-y
7. Horváth, A., Telek, M.: A Markovian point process exhibiting multifractal behaviour and its application to traffic modeling. In: Proceedings of 4th International Conference on Matrix-Analytic Methods in Stochastic models. pp. 183–208 (2002)
8. Horváth, G., Telek, M.: A canonical representation of order 3 phase type distributions. Formal Methods and Stochastic Models for Performance Evaluation pp. 48–62 (2007)
9. Latouche, G., Ramaswami, V.: Introduction to Matrix Analytic Methods in Stochastic Modeling. ASA-SIAM Series on Statistics and Applied Probability, Society for Industrial and Applied Mathematics (1999), https://books.google.com/books?id=J31a3P69K8IC
10. Mészáros, A., Telek, M.: Canonical representation of discrete order 2 MAP and RAP. In: Balsamo, M., Knottenbelt, W., Marin, A. (eds.) Computer Performance Engineering, Lecture Notes in Computer Science, vol. 8168, pp. 89–103. Springer Berlin Heidelberg (2013), http://dx.doi.org/10.1007/978-3-642-40725-3_8
11. Mitchell, K., van de Liefvoort, A.: Approximation models of feed-forward G/G/1/N queueing networks with correlated arrivals. Performance Evaluation 51(2), 137–152 (2003)
12. Neuts, M.F.: Models based on the Markovian arrival process. IEICE Transactions on Communications 75(12), 1255–1265 (1992)
13. Papp, J., Telek, M.: Canonical representation of discrete phase type distributions of order 2 and 3. In: In Proc. of UK Performance Evaluation Workshop, UKPEW 2013 (2013)
14. Rodriguez, J., Lillo, R.E., Ramirez-Cobo, P.: A canonical form for the non-stationary two-state Markovian arrival process (2015), submitted for publication