Canonical representation of order 2 transient Markov and rational arrival processes☆

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Abstract

Similar to other processes that are modulated by background Markov chains the matrix representation of a transient Markov arrival process is not unique and the use of a convenient unique canonical form is essential for practical computations.

The paper presents a set of 5 Markovian forms which provide a unique and minimal representation for all members of the TMAP(2) class. In the course of the derivation we also show the identity of the TMAP(2) and the TRAP(2) classes.

Keywords: Transient Markov arrival process, canonical form, Transient rational arrival process.

1. Introduction

With the evolution of the complexity of stochastic models a new modelling paradigm arises recently on the field of point processes. Instead of defining processes of individual events, processes of a finite series of events are used when they better describe the occurrence of events. Such point processes have gained attention in various fields, e.g., demography [1], epidemiology [2], risk processes [3], port consumption modelling of web requests [4]. The following example demonstrates this modelling paradigm. *Assume that a single user browses web pages on the internet. A single click on a website initiates a process of downloading several embedded objects. To model the overall download process of objects it is a natural approach to separate the user's activity from the behaviour of the*

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download process of embedded objects associated with a single website. The process of user clicks can be any general (e.g. stationary) point process generating infinite number of events (clicks) as time tends to infinity and the process generated by a single click is a transient process which terminates after the required (random) number of objects are downloaded. This paper is about the transient process initiated by a single click in this example.

The stochastic description of terminating point processes is a current modelling challenge. One analytically convenient model of such processes is the transient Markov arrival process (TMAP) [5]. As suggested by its name, the transient Markov arrival process is an extension of the Markov arrival process (MAP) which terminates after a finite number of events. Several properties of TMAPs are inherited from the ones of MAPs [6]. Based on [6] representation transformation procedures and functions computing various properties of TMAPs (e.q., correlation parameter, extinction distribution) have been implemented in the BuTools 1.0 program package [7]. In this paper we focus on a special property of TMAPs, their canonical representation.

One of the basic properties of MAPs and TMAPs is the non-uniqueness of their matrix representation (set of vectors and matrices which define the process as detailed below). As a result there is a need for a conveniently defined unique representation of TMAPs in order to decide whether two TMAPs given by their matrix representations are identical. Such a representation is referred to as canonical from. There is freedom for defining a convenient canonical form. In this respect a convenient representation is minimal (the number of non-zero matrix elements is equal to the number of parameters that define the model) and Markovian (the process has a nice stochastic interpretation thanks to the background continuous time Markov chain (CTMC) which modulates the arrivals).

Apart from the theoretical benefits, the most important practical application of canonical forms is in the fitting of experimental data with Markov modulated stochastic models, e.g. in [8]. The complexity and the numerical properties of the fitting methods benefit from the minimal parameter representation and the known (non-negative) boundaries of the parameters of the canonical forms.

Canonical forms are not easy to define in general. Up to now, canonical forms are provided only for order 2 point process models (which are governed by a 2-state background CTMC). For stationary order 2 MAPs canonical forms were presented in [9] and for non-stationary order 2 MAPs in [10, 11]. The rising popularity of TMAPs initiated research for efficient fitting methods of TMAP models. In this paper we present a canonical representation of the order 2 TMAP class (TMAP(2)), which is characterized by 7 parameters, in contrast to the 4 parameters of the stationary MAP(2) and the 5 parameters of the non-stationary MAP(2) classes. The main approach of this paper is similar to the one in [9] and in [11], but due to the increased number of parameters it is far more complex. For example, we need 5 forms to cover the TMAP(2) class, while 2 forms covers the stationary and non-stationary order 2 MAPs. The provided proofs are built on hard to follow intricate details without particular methodological novelty. The importance of the results lie in the conclusion, which has to be accurately proved. From a practical point of view this proof is necessary for trusting the canonical representation function of the TMAP(2) class which we plan to integrate into the BuTools package [7].

In the course of the derivation of the canonical representation of TMAP(2)s we also show that the transient point processes defined by (non-Markovian) order 2 vectors and matrices, referred to as order 2 transient rational arrival processes (TRAP(2)s), are indeed identical with TMAP(2). This result is a generalization of the related results for stationary MAP(2) in [9] and for non-stationary MAP(2) in [11].

Due to the high complexity of the problem (or at least the high number of different cases to handle) the detailed derivation is quite long. To respect space limitation we neglect some elements of the derivations which are somewhat similar to other presented elements.

Restriction: In this paper we focus our attention on the non-zero measure open subsets of TRAP(2). The investigation of the borders of such subsets is possible, but requires the evaluation of an enormous number of cases, and is neglected here. Some direct consequences of this restriction are that we present strict inequalities all along the paper, we do not discuss the identity of different degenerate canonical forms, and exclude the case of identical eigenvalues in (7). An intuitive explanation of this limitation can be given in Figure 2, where we provide a division of the plane, but do not discuss the set membership of the border lines between the neighbouring sets.

The rest of the paper is organized as follows. Section 2 introduces the TRAP(2) and the TMAP(2) classes and their basic properties. Section 3 provides direct and iterative constraints on the behaviour of the TRAP(2) class. The main theorem that the TRAP(2) class can be described by 5 Markovian forms is presented and proved in Section 4. The uniqueness of the canonical representation comes from the non-overlapping behaviour of the 5 forms, which is proved in Section 5.

2. Background

In this section we provide the theoretical background that will be built upon in the rest of the paper.

Let $\mathcal{X}(t)$ be a point process on $\mathbb{R}^+$ with joint probability density function (joint pdf) of inter-event times $f(x_0, x_1, \dots, x_k)$ for $k = 0, 1, 2, \dots$.

Definition 1. *$\mathcal{X}(t)$ is called a stationary rational arrival process if there exists a finite $(\boldsymbol{H_0}, \boldsymbol{H_1})$ square matrix pair such that $(\boldsymbol{H_0} + \boldsymbol{H_1})\mathbb{1} = \mathbf{0}$ (where $\mathbb{1}$ and $\mathbf{0}$ are the column vectors of ones and zeros, respectively, with appropriate size), the*

$$\pi(-\boldsymbol{H_0})^{-1}\boldsymbol{H_1} = \pi, \quad \pi\mathbb{1} = \mathbb{1}, \tag{1}$$

system of linear equations has a unique solution for π, and for $\forall x_0, \dots, x_k \geq 0, k \geq 0$ the joint pdf of the process is

$$f(x_0, x_1, \dots, x_k) = \pi e^{\boldsymbol{H_0} x_0}\boldsymbol{H_1} e^{\boldsymbol{H_0} x_1}\boldsymbol{H_1} \dots e^{\boldsymbol{H_0} x_k}\boldsymbol{H_1}\mathbb{1}. \tag{2}$$

In this case we say that $\mathcal{X}(t)$ is a stationary rational arrival process (RAP) with representation $(\boldsymbol{H_0}, \boldsymbol{H_1})$, or shortly, RAP$(\boldsymbol{H_0}, \boldsymbol{H_1})$.

Definition 2. *If $\mathcal{X}(t)$ is a stationary RAP$(\boldsymbol{H_0}, \boldsymbol{H_1})$, where*

- $\boldsymbol{H_1} \geq 0$ *(element-wise),*
- $\boldsymbol{H_0}_{ii} < 0$, $\boldsymbol{H_0}_{ij} \geq 0$ *for* $i \neq j$ *and* $\boldsymbol{H_0}\mathbb{1} \leq 0$,

then we say that $\mathcal{X}(t)$ is a stationary Markov arrival process (MAP) with representation $(\boldsymbol{H_0}, \boldsymbol{H_1})$, or shortly, MAP$(\boldsymbol{H_0}, \boldsymbol{H_1})$.

The importance of the MAP class comes from the associated stochastic interpretation. Every MAP representation can be mapped to a continuous time Markov chain with generator $\boldsymbol{H} = \boldsymbol{H_0} + \boldsymbol{H_1}$ where $\boldsymbol{H_1}$ contains transition rates with arrivals and $\boldsymbol{H_0}$ contains transition rates without arrivals and the Markov chain starts from initial distribution π, which is the stationary probability vector embedded at arrivals. In such a Markov chain (2) is the joint pdf of the inter-arrival times. We note here that an arbitrary $(\boldsymbol{H_0}, \boldsymbol{H_1})$ square matrix pair satisfying (1) does not necessarily define a valid RAP as (2) may still give negative values for some $x_0, \ldots, x_k \geq 0$. If an $(\boldsymbol{H_0}, \boldsymbol{H_1})$ matrix pair fulfils the additional sign constraints of MAPs in Definition 2, however, then (2) is guaranteed to be positive for arbitrary $x_0, \ldots, x_k > 0$ as can be seen from the mapping to Markov chains. One of the major advantages of MAPs compared to RAPs is that the non-negativity of (2) is guaranteed.

RAPs (MAPs) have infinite different representations (as it is demonstrated below for their non-stationary counterparts), i.e., matrix pair sets that give the same $f(x_0, x_1, \ldots, x_k)$ joint probability density function. The different representations might have different sizes [12], where the size refers to the dimension of the square matrices $\boldsymbol{H_0}, \boldsymbol{H_1}$. The size of the smallest among those representations is referred to as the order of the RAP (MAP). The class of order n RAPs (MAPs) is denoted by RAP(n) (MAP(n)). From Definition 1 and 2 it follows that MAP(n)⊆RAP(n).

Stationary RAPs are processes with an infinite number of arrivals. In many cases, however, we need to describe systems with a transient behaviour, where the process generates a finite series of arrivals and terminates after that. To model these situations transient RAPs (TRAPs) are defined.

Definition 3. *$\mathcal{X}(t)$ is called a transient rational arrival process if there exists a $(\pi, \boldsymbol{H_0}, \boldsymbol{H_1})$ (vector, matrix, matrix) tuple with elements of finite size such that $\pi\mathbb{1} = 1$, $(\boldsymbol{H_0} + \boldsymbol{H_1})\mathbb{1} \leq \boldsymbol{0}$,*

$$f^C(t_1, \ldots, t_k) = \pi e^{\boldsymbol{H_0} t_1} \boldsymbol{H_1} e^{\boldsymbol{H_0} t_2} \boldsymbol{H_1} \ldots e^{\boldsymbol{H_0} t_k} \boldsymbol{H_1} \mathbb{1} \geq 0 \tag{3}$$

and

$$f^T(t_1, \ldots, t_k) = \pi e^{\boldsymbol{H_0} t_1} \boldsymbol{H_1} e^{\boldsymbol{H_0} t_2} \boldsymbol{H_1} \ldots e^{\boldsymbol{H_0} t_k} \boldsymbol{\eta} \geq 0 \tag{4}$$

for $\forall k \in \mathbb{N}^+$ and $\forall \{t_1, \ldots, t_k\} \in {\mathbb{R}^+}^k$, where $\boldsymbol{\eta} = -(\boldsymbol{H_0} + \boldsymbol{H_1})\mathbb{1}$. Function $f^C(t_1, \ldots, t_k)$ is the joint probability density of inter-event times $\{t_1, \ldots, t_k\}$, *where all the events are arrivals and the arrival process continues with either a new arrival or termination after some additional t_{k+1} time, and $f^T(t_1,\ldots,t_k)$ is the joint probability density of inter-event times $\{t_1,\ldots,t_{k-1},t_k\}$, where the first $k-1$ events are arrivals and the last event is the termination of the process.*

Definition 4. *If $\mathcal{X}(t)$ is a transient RAP$(\pi,\boldsymbol{H_0},\boldsymbol{H_1})$, where*

- $\pi \geq 0$,
- $\boldsymbol{H_1} \geq 0$,
- $\boldsymbol{H_0}_{ii} < 0$, $\boldsymbol{H_0}_{ij} \geq 0$ *for* $i \neq j$, *and* $(\boldsymbol{H_0}+\boldsymbol{H_1})\mathbb{1} \leq 0$,

then we say that $\mathcal{X}(t)$ is a transient Markov arrival process (TMAP) with representation $(\pi,\boldsymbol{H_0},\boldsymbol{H_1})$, or shortly, TMAP$(\pi,\boldsymbol{H_0},\boldsymbol{H_1})$.

Just like their stationary counterparts, TRAPs (TMAPs) have infinite different representations, i.e., $(\hat{\pi},\hat{\boldsymbol{H}}_{\boldsymbol{0}},\hat{\boldsymbol{H}}_{\boldsymbol{1}})$ tuples that give the same $f^C(x_0,x_1,\ldots,x_k)$ and $f^T(x_0,x_1,\ldots,x_k)$ joint probability density functions. One way to get a different representation of a TRAP$(\pi,\boldsymbol{H_0},\boldsymbol{H_1})$ with the same size is the application of the similarity transformation

$$\hat{\pi} = \pi\boldsymbol{T}, \quad \hat{\boldsymbol{H}}_{\boldsymbol{0}} = \boldsymbol{T}^{-1}\boldsymbol{H_0}\boldsymbol{T}, \quad \hat{\boldsymbol{H}}_{\boldsymbol{1}} = \boldsymbol{T}^{-1}\boldsymbol{H_1}\boldsymbol{T}, \tag{5}$$

where $\boldsymbol{T}$ is an arbitrary non-singular transformation matrix with $\boldsymbol{T}\mathbb{1} = \mathbb{1}$. The transformed representation gives the same joint pdfs, as

$$\begin{aligned}
&\hat{f}^C(t_1,\ldots,t_k) = \pi' e^{\hat{\boldsymbol{H}}_{\boldsymbol{0}} t_0}\hat{\boldsymbol{H}}_{\boldsymbol{1}}\ldots e^{\hat{\boldsymbol{H}}_{\boldsymbol{0}} t_k}\hat{\boldsymbol{H}}_{\boldsymbol{1}}\mathbb{1} = \\
&= \pi\boldsymbol{T}e^{\boldsymbol{T}^{-1}\boldsymbol{H_0}\boldsymbol{T}t_1}\boldsymbol{T}^{-1}\boldsymbol{H_1}\boldsymbol{T}\ldots e^{\boldsymbol{T}^{-1}\boldsymbol{H_0}\boldsymbol{T}t_k}\boldsymbol{T}^{-1}\boldsymbol{H_1}\boldsymbol{T}\mathbb{1} = \\
&= \pi\boldsymbol{T}\boldsymbol{T}^{-1}e^{\boldsymbol{H_0}t_1}\boldsymbol{T}\boldsymbol{T}^{-1}\boldsymbol{H_1}\boldsymbol{T}\ldots\boldsymbol{T}^{-1}e^{\boldsymbol{H_0}t_k}\boldsymbol{T}\boldsymbol{T}^{-1}\boldsymbol{H_1}\boldsymbol{T}^{-1}\boldsymbol{T}\mathbb{1} = \\
&= \pi e^{\boldsymbol{H_0}t_1}\boldsymbol{H_1}\ldots e^{\boldsymbol{H_0}t_k}\boldsymbol{H_1}\mathbb{1} = f^C(t_1,\ldots,t_k),
\end{aligned} \tag{6}$$

where we used that $\boldsymbol{T}\mathbb{1} = \mathbb{1}$. The equality between $\hat{f}^T(t_1,\ldots,t_k)$ and $f^T(t_1,\ldots,t_k)$ can be shown in a similar fashion.

The order of TRAPs and TMAPs is defined similarly as that of RAPs and MAPs. The class of order n TRAPs (TMAPs) is denoted by TRAP(n) (TMAP(n)). From Definition 3 and 4 it follows that TMAP$(n)\subseteq$TRAP(n).

3. Constraints of second order TRAPs

In this section we discuss the boundaries of the TRAP(2) class in preparation of proving that TRAP(2)≡TMAP(2) (i.e., the classes of order 2 TRAPs and TMAPs are equivalent). We will establish a set of constraints, but we will not show in this section that these are tight bounds. In the next section, we show that TMAP(2) completely fills the space defined by these constraints. Because TMAP(2) is a subclass of TRAP(2) this also shows that further constraints are not necessary, that is, the set of constraints provides tight boundaries.

We derive these constraints for the parameters of TRAP(2) using two different approaches. First, we obtain boundaries by examining the $f^C(t)$ and $f^T(t)$ probability density functions for $t = 0$ and $t \to \infty$ and the change a single arrival makes on the π phase distribution vector. We will call these direct parameter constraints. Then we apply a methodology very similar to what was developed for order 2 stationary MAPs in [9]. This methodology is based on the evolution of the phase distribution vector right after arrivals, when a long series of very fast ($t_i = 0, \forall i = 1, \dots, k$) or very slow ($t_i \to \infty, \forall i = 1, \dots, k$) arrivals happens. The resulting boundaries will be called iterative parameter constraints.

In order to investigate the limits of the TRAP(2) class we consider the following representation with diagonal $\boldsymbol{H_0}$

$$\pi = [\pi_0, 1-\pi_0], \boldsymbol{H_0} = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}, \boldsymbol{H_1} = \begin{bmatrix} ab\lambda_1 & a(1-b)\lambda_1 \\ c(1-d)\lambda_2 & cd\lambda_2 \end{bmatrix}, \boldsymbol{\eta} = \begin{bmatrix} (1-a)\lambda_1 \\ (1-c)\lambda_2 \end{bmatrix}, \tag{7}$$

where $\lambda_1 \leq \lambda_2$. Under this constraint the representation is unique, since it is based on the spectral decomposition of the $\boldsymbol{H_0}$ with ordered eigenvalues.

From [9, Theorem 1] we have that all the parameters of the diagonal representation have to be real and λ_1, λ_2 have to be positive. We define $\alpha = \lambda_1/\lambda_2$. Because λ_1 and λ_2 are both positive,

$$0 < \alpha < 1 \tag{8}$$

has to hold (where the second strict inequality comes from **Restriction**). Multiplying λ_1 and λ_2 by an arbitrary positive r parameter corresponds to speeding up the process by factor r, consequently it does not affect whether the representation describes a valid process. Thus, during the analysis of the limits of the TRAP(2) set we assume that $\lambda_2 = 1$ and $\lambda_1 = \alpha$. This assumption makes the analysis simpler, but it does not violate its validity. In the following we explore the area defined by the remaining 6 parameters of the above representation that define a valid TRAP using the direct and the iterative approaches.

3.1. Direct parameter constraints

In this section we use the $f^C(t)$ and $f^T(t)$ functions (functions (3) and (4) with $k = 1$ and $t_1 = t$, i.e., the probability density functions of the first event) to obtain parameter constraints. These can be rewritten using the diagonal TRAP(2) representation in (7) as

$$f^C(t) = \pi e^{\boldsymbol{H_0} t} \boldsymbol{H_1} \mathbb{1} = \pi_0 a\alpha e^{-t\alpha} + (1-\pi_0)ce^{-t}, \tag{9}$$

$$f^T(t) = \pi e^{\boldsymbol{H_0} t} \boldsymbol{\eta} = \pi_0(1-a)\alpha e^{\alpha t} + (1-\pi_0)(1-c)e^{-t}, \tag{10}$$

$$f(t) = f^C(t) + f^T(t) = \pi_0 \alpha e^{-\alpha t} + (1-\pi_0)e^{-t}, \tag{11}$$

where $f(t)$ is the probability density function of the time of the first event.

Before looking at the direct parameter constraints, similar to [9] we define $\boldsymbol{v}(x,t) = [u(x,t), 1-u(x,t)]$ such that

$$\boldsymbol{v}(x,t) = [u(x,t), 1-u(x,t)] = \frac{[x, 1-x]e^{\boldsymbol{H_0}t}\boldsymbol{H_1}}{[x, 1-x]e^{\boldsymbol{H_0}t}\boldsymbol{H_1}\mathbb{1}}. \tag{12}$$

The $\boldsymbol{v}(x,t)$ vector describes the effect of an inter-arrival period of length t on the phase distribution assuming that the process continues.

That is, if the initial phase distribution is $\pi = [x, 1-x]$, then, after an inter-arrival period of length t, the phase distribution is $\boldsymbol{v}(x,t) = [u(x,t), 1-u(x,t)]$, where, according to (7)

$$u(x,t) = \frac{[x, 1-x]e^{\boldsymbol{H_0}t}\boldsymbol{H_1}\boldsymbol{e_1}}{[x, 1-x]e^{\boldsymbol{H_0}t}\boldsymbol{H_1}\mathbb{1}} = \frac{xab\alpha e^{-\alpha t} + (1-x)c(1-d)e^{-t}}{xa\alpha e^{-\alpha t} + (1-x)ce^{-t}}, \tag{13}$$

and $\boldsymbol{e_1}$ is the column vector whose only non-zero element is the first element which is one. The denominator of this fraction is $f^C(t)$ (assuming $\pi = [x, 1-x]$), therefore it has to be non-negative if $[x, 1-x]$ is a valid phase distribution. Based on the meaning of $\boldsymbol{v}(x,t)$ it is clear that if TRAP($[x, 1-x], \boldsymbol{H_0}, \boldsymbol{H_1}$) is a valid TRAP, then TRAP($\boldsymbol{v}(x,t), \boldsymbol{H_0}, \boldsymbol{H_1}$) has to be a valid TRAP as well (i.e. the same constraints have to hold for both phase distributions). We will use this property in the following.

The $f^C(t), f^T(t), f(t)$ functions in (9)-(11) have to be non-negative for $\forall t \geq 0$, which results in the following constraints.

Lemma 1. *For a TRAP(2) the elements of the diagonal representation in* (7) *satisfy*

$$\pi_0 > 0, \tag{14}$$

$$0 < a < 1. \tag{15}$$

Proof. Function $f(t)$ has to be non-negative for large t values, that is $f(t) = \pi_0\alpha e^{-\alpha t} + (1-\pi_0)e^{-t} \overset{t\to\infty}{\approx} \pi_0\alpha e^{-\alpha t}$ has to be non-negative, therefore due to (8) we obtain (14).

Function $f^C(t)$ has to be non-negative for large t, i.e., $f^C(t) = \pi_0 a\alpha e^{-\alpha t} - (1-\pi_0)ce^{-t} \overset{t\to\infty}{\approx} \pi_0 a\alpha e^{-\alpha t} > 0$ from which (using (8) and (14)) we have $a > 0$. Similarly, function $f^T(t)$ has to be non-negative for large t, i.e., $f^T(t) = \pi_0(1-a)\alpha e^{-\alpha t} - (1-c)(1-\pi_0)e^{-t} \overset{t\to\infty}{\approx} \pi_0(1-a)\alpha e^{-\alpha t} > 0$ from which $a < 1$. $\square$

Lemma 2. *The* π_0 *element of* (7) *satisfies*

$$\frac{c}{c-a\alpha} < \ \pi_0 < \frac{1-c}{1-c-\alpha+a\alpha}, \qquad \text{if } c<0, \tag{16a}$$

$$0 < \ \pi_0 < \frac{1-c}{1-c-\alpha+a\alpha}, \qquad \text{if } 0<c<a, \tag{16b}$$

$$0 < \ \pi_0 < \frac{c}{c-a\alpha}, \qquad \text{if } a<c<1, \tag{16c}$$

$$\frac{1-c}{1-c-\alpha+a\alpha} < \ \pi_0 < \frac{c}{c-a\alpha}, \qquad \text{if } c>1. \tag{16d}$$

Proof. $f^C(0) = \pi_0\alpha a + (1-\pi_0)c$ and $f^T(0) = \pi_0\alpha(1-a) + (1-\pi_0)(1-c)$ are non-negative. Rearranging these to π_0 and using (8), (14), (15) gives the lemma. A more detailed derivation can be found in [13]. □

Lemma 3. *The* b *parameter of* (7) *satisfies the same constraints as* π_0 *in* (16a)-(16d)*.*

Proof. According to (13), after a long inter-arrival period the first element of the phase distribution changes to $\lim_{t\to\infty} u(\pi_0, t) = b$, which means that the initial phase distribution after this very late arrival, $[b, 1-b]$, has to fulfil the same constraints as $[\pi_0, 1-\pi_0]$. □

3.2. Iterative parameter constraints

Now we move on to the constraints that can be derived from the examination of a series of arrivals.

Lemma 4. *If* $u(x,0) < b$ *then*

$$u(x,0) < u(x,t) < b$$

and if $u(x,0) > b$ *then*

$$b < u(x,t) < u(x,0).$$

Proof. Function $u(x,t)$ is a monotone function of t for $\forall x \in \{0, \frac{1}{1-\alpha}\}$, because

$$\frac{\partial u(x,t)}{\partial t} = -\frac{ac(1-b-d)\alpha(1-\alpha)x(1-x)e^{(1+\alpha)t}}{\left(a\alpha x e^t + c(1-x)e^{\alpha t}\right)^2},$$

and the sign of this expression does not change with t, therefore $u(x,t)$ is either monotonically increasing or decreasing in t and $\lim_{t\to\infty} u(x,t) = b$. □

In the following we focus on $u(x,0)$, therefore we abuse notation and simply write $u(x)$ instead of $u(x,0)$. Function

$$u(x) = \frac{xab\alpha + (1-x)c(1-d)}{xa\alpha + (1-x)c} = \frac{ab\alpha - c(1-d)}{a\alpha - c} + \frac{\frac{ac\alpha(1-b-d)}{(a\alpha-c)^2}}{x - \frac{c}{c-a\alpha}} \tag{17}$$

is a hyperbola with vertical asymptote

$$v = \frac{c}{c - a\alpha}, \tag{18}$$

and horizontal asymptote

$$h = \frac{ab\alpha - c(1-d)}{a\alpha - c}. \tag{19}$$

The denominator of $u(x)$ changes sign if $xa\alpha + (1-x)c = 0$, that is, in $x = v$. Due to the fact that the denominator is $f^C(0)$, which is non-negative, we only need to consider one side of this hyperbola where $xa\alpha + (1-x)c > 0$. The right side of the hyperbola opens upwards if the factor of x if the numerator of the second term is positive, that is, if $\frac{ac\alpha(1-b-d)}{(a\alpha-c)^2} > 0$, which holds if $c(1-b-d) > 0$ (because $\alpha > 0$ and $a > 0$ hold true). Otherwise the right side opens downwards (c.f. Fig. 1).

The effect of a very fast ($t \to 0$) arrival on the phase distribution $[x, 1-x]$ corresponds to one $x \mapsto u(x)$ iteration on $u(x)$, therefore after a long series of very fast arrivals the phase distribution either converges to a fixed point (where $u(x) = x$) or diverges (because $u(x)$ is a hyperbola). Due to (16a)-(16d) the latter case is not allowed for a TRAP(2) (the allowed range of phase distributions is restricted). The fixed points of $u(x)$ (the solutions of $u(x) = x$) are

$$x_{1,2} = \frac{cd + ab\alpha - 2c \pm \sqrt{(2c - cd - ab\alpha)^2 - 4c(1-d)(c - a\alpha)}}{2(a\alpha - c)}. \tag{20}$$

These solutions have to be real, which means that

$$(2c - cd - ab\alpha)^2 - 4c(1-d)(c - a\alpha) \geq 0 \tag{21}$$

has to hold. Because $u(x)$ is a hyperbola, one of the fixed points ($x_i, i = \{1,2\}$) is stable, the other one is unstable. The x_i fixed point is stable if $|\frac{\partial u(x)}{\partial x}|_{x=x_i} < 1$. If $v > h$, then the stable fixed point is on the left side of the hyperbola, and if $v < h$, then the stable fixed point is on the right side of the hyperbola (c.f. Fig. 1). If the right side opens upwards, the stable and unstable fixed points are on the opposite sides of the hyperbola, otherwise they are on the same side. In the following we denote the stable fixed point by x_s and the unstable fixed point by x_u.

Lemma 5. *In addition to* (16a)-(16d), π_0 *is bounded by the following iterative parameter constraints*

$$\begin{array}{ll} 0 < \pi_0 < v, & \textit{if } c(1-b-d) > 0 \textit{ and } v > h, \\ v < \pi_0 < \frac{1}{1-\alpha}, & \textit{if } c(1-b-d) > 0 \textit{ and } v < h, \\ 0 < \pi_0 < x_u, & \textit{if } c(1-b-d) < 0 \textit{ and } v > h, \\ x_u < \pi_0 < \frac{1}{1-\alpha}, & \textit{if } c(1-b-d) < 0 \textit{ and } v < h. \end{array} \tag{22}$$

Proof. Considering that the stable fix point is the one which is closer to the horizontal asymptote (where the absolute value of the derivative is less than one) we have the following four possibilities:

- If $c(1-b-d) > 0$ and $v > h$, then the right side of the hyperbola opens upwards, the stable fixed point is on the left side and the unstable fixed point is on the right side (Fig. 1a).
- If $c(1-b-d) > 0$ and $v < h$, then the right side of the hyperbola opens upwards, the stable fixed point is on the right side and the unstable fixed point is on the left side (Fig. 1b).
- If $c(1-b-d) < 0$ and $v > h$, then the left side of the hyperbola opens upwards, the stable fixed point is the most left and the unstable fixed point is between the stable fix point and the vertical asymptote (Fig. 1c).
- If $c(1-b-d) < 0$ and $v < h$, then the left side of the hyperbola opens upwards, the stable fixed point is the most right and the unstable fixed point is between the stable fix point and the vertical asymptote (Fig. 1d).

We define the attractive basin of the stable fixed point (x_s) as the interval from which the iterative application of $x \mapsto u(x)$ converges to x_s (e.g., the attractive basin is (v, ∞) in Fig. 1a and Fig. 1b, $(-\infty, x_u)$ in Fig. 1c, and it is (x_u, ∞) in Fig. 1d). The first element of the phase distribution (π_0) has to be in the attractive basin of the stable fixed point. This does not hold, when the unstable fixed point (x_u) or the vertical asymptote (v) is between x_s and π_0. Combining these constraints and (14) we get the statement of the lemma. □

Lemma 6. *The stable fixed point, x_s, satisfies the same constraints as π_0 in* (16a)-(16d).

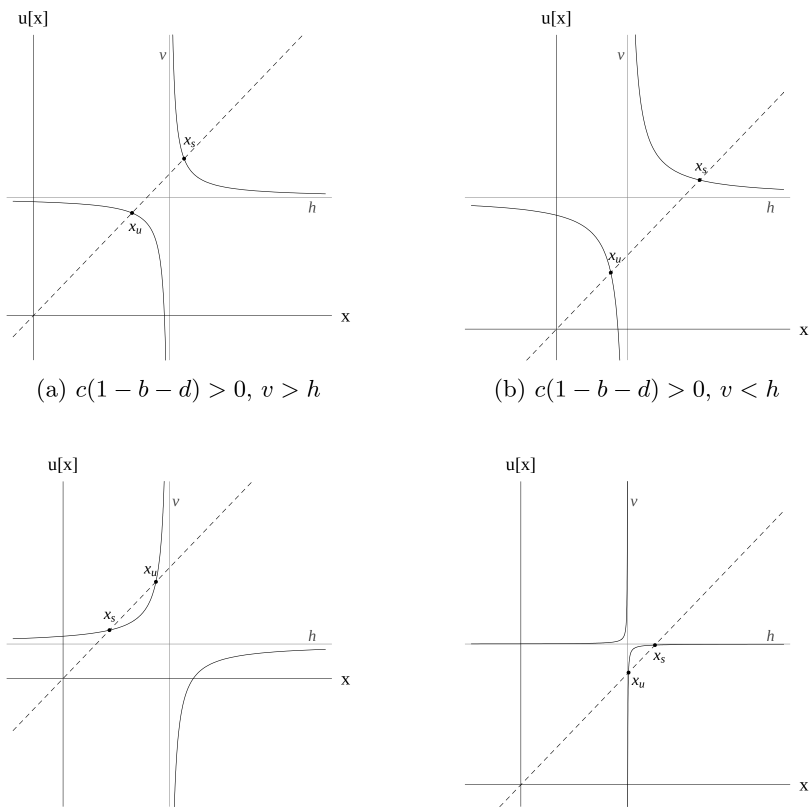
Proof. x_s can be reached from an arbitrary valid $[\pi_0, 1-\pi_0]$ starting vector with a sequence of $t \to 0$ arrivals, consequently $[x_s, 1-x_s]$ has to be a legal initial distribution for the TRAP(2) as well. □

According to (7), the matrix describing the phase distribution changes due to an arrival is

$$\boldsymbol{P} = \int_{t=0}^{\infty} e^{\boldsymbol{H_0} t} \boldsymbol{H_1} dt = (-\boldsymbol{H_0})^{-1} \boldsymbol{H_1} = \begin{bmatrix} ab & a(1-b) \\ c(1-d) & cd \end{bmatrix}.$$

Its eigenvalues are $\mu_{1,2} = \frac{1}{2}\left(ab + cd \pm \sqrt{(ab+cd)^2 + 4(ac - abc - acd)}\right)$ and the product of the eigenvalues is $\mu_1 \mu_2 = -ac(1-b-d)$. The larger eigenvalue has to be positive ($\mu_1 > 0$) and dominant ($|\mu_1| > |\mu_2|$) for a TRAP(2), otherwise an element of the phase distribution would become negative after a series of arrivals. We define $\gamma = \mu_2$, which is a kind of correlation parameter of the TRAP(2) process. Because $\mu_1 > 0$, the sign of γ is identical with the sign of $-ac(1-b-d)$ (where $a > 0$ according to (15)). As a result the cases in (22) and in Figure 1 are indeed classified according to the sign of γ and c.

Appendix A summarizes the feasible TMAP(2) regions satisfying the direct and the iterative parameter constraints for different values of γ and c.



(a) $c(1-b-d) > 0$, $v > h$

(b) $c(1-b-d) > 0$, $v < h$

(c) $c(1-b-d) < 0$ and $v > h$

(d) $c(1-b-d) < 0$ and $v < h$

Figure 1: The possible cases of the $u(x)$ hyperbola

4. Equivalence of TRAP(2) and TMAP(2)

In this section we prove the equivalence of TMAP(2) and TRAP(2). We present a constructive proof, that is, we define a set of (Markovian) TMAP(2) representations and show that every TRAP(2) can be transformed to one of these Markovian forms.

Theorem 1. *Every order 2 transient rational arrival process can be represented with one of the following Markovian forms*

- *Form 1:* $\gamma > 0$

$$\boldsymbol{D_0^{(1)}} = \begin{bmatrix} -\lambda_1 & \hat{c}\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}, \ \boldsymbol{D_1^{(1)}} = \begin{bmatrix} \hat{a}\lambda_1 & 0 \\ \hat{d}\lambda_2 & \hat{b}\lambda_2 \end{bmatrix}, \ \boldsymbol{\delta^{(1)}} = \begin{bmatrix} (1-\hat{a}-\hat{c})\lambda_1 \\ (1-\hat{b}-\hat{d})\lambda_2 \end{bmatrix},$$

where $0 < \lambda_1 < \lambda_2$, $\hat{b} > \hat{a}\frac{\lambda_1}{\lambda_2}$,

- *Form 2:* $\gamma > 0$

$$\boldsymbol{D_0^{(2)}} = \begin{bmatrix} -\bar{\lambda}_1 & \bar{c}\bar{\lambda}_1 \\ (1-\bar{b}-\bar{d})\bar{\lambda}_2 & -\bar{\lambda}_2 \end{bmatrix},\ \boldsymbol{D_1^{(2)}} = \begin{bmatrix} \bar{a}\bar{\lambda}_1 & 0 \\ \bar{d}\bar{\lambda}_2 & \bar{b}\bar{\lambda}_2 \end{bmatrix},\ \boldsymbol{\delta^{(2)}} = \begin{bmatrix} (1-\bar{a}-\bar{c})\bar{\lambda}_1 \\ 0 \end{bmatrix},$$

where $0 < \bar{\lambda}_1, \bar{\lambda}_2,\ \bar{b} > \bar{a}\frac{\bar{\lambda_1}}{\bar{\lambda_2}}$*,*

- *Form 3:* $\gamma < 0$

$$\boldsymbol{D_0^{(3)}} = \begin{bmatrix} -\lambda_1 & \check{c}\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix},\ \boldsymbol{D_1^{(3)}} = \begin{bmatrix} 0 & \check{a}\lambda_1 \\ \check{d}\lambda_2 & \check{b}\lambda_2 \end{bmatrix},\ \boldsymbol{\delta^{(3)}} = \begin{bmatrix} (1-\check{a}-\check{c})\lambda_1 \\ (1-\check{b}-\check{d})\lambda_2 \end{bmatrix},$$

where $0 < \lambda_1 < \lambda_2$*,*

- *Form 4:* $\gamma < 0$

$$\boldsymbol{D_0^{(4)}} = \begin{bmatrix} -\tilde{\lambda}_1 & \tilde{c}\tilde{\lambda}_1 \\ (1-\tilde{b}-\tilde{d})\lambda_2 & -\tilde{\lambda}_2 \end{bmatrix},\ \boldsymbol{D_1^{(4)}} = \begin{bmatrix} 0 & \tilde{a}\tilde{\lambda}_1 \\ \tilde{d}\tilde{\lambda}_2 & \tilde{b}\tilde{\lambda}_2 \end{bmatrix},\ \boldsymbol{\delta^{(4)}} = \begin{bmatrix} (1-\tilde{a}-\tilde{c})\tilde{\lambda}_1 \\ 0 \end{bmatrix},$$

where $0 < \tilde{\lambda}_1 < \tilde{\lambda}_2$*,*

- *Form 5: no restriction on the sign of* γ

$$\boldsymbol{D_0^{(5)}} = \begin{bmatrix} -\lambda_1 & \dot{c}\lambda_1 \\ 0 & -\lambda_2 \end{bmatrix},\ \boldsymbol{D_1^{(5)}} = \begin{bmatrix} \dot{a}\lambda_1 & (1-\dot{a}-\dot{c})\lambda_1 \\ \dot{d}\lambda_2 & \dot{b}\lambda_2 \end{bmatrix},\ \boldsymbol{\delta^{(5)}} = \begin{bmatrix} 0 \\ (1-\dot{b}-\dot{d})\lambda_2 \end{bmatrix},$$

where $0 < \lambda_1 < \lambda_2$*,*

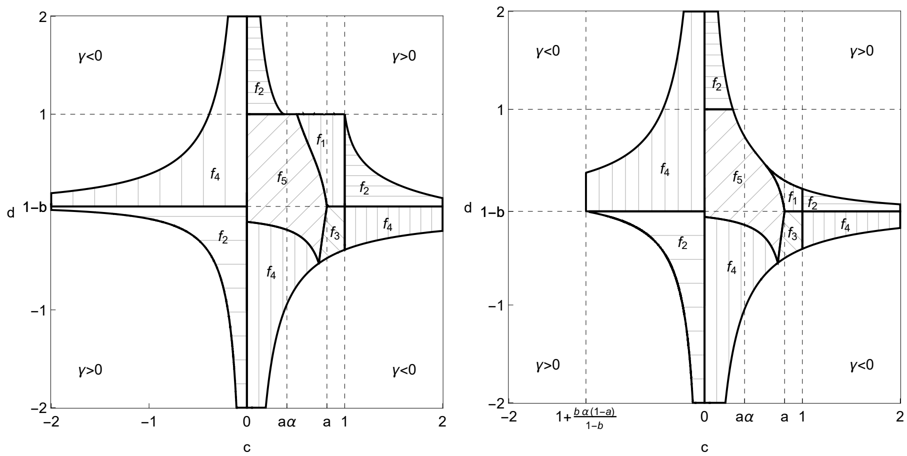
and all the respective λ_1, λ_2 *parameters are positive and all the respective* $a, b, c, d, 1-a-c, 1-b-d$ *parameters are between* 0 *and* 1*.*

In the proof of the theorem we consider the diagonal TRAP(2) representation from (7) assuming $\lambda_1 = \alpha$, $\lambda_2 = 1$, that is

$$\pi = [\pi_0, 1-\pi_0],\ \boldsymbol{H_0} = \begin{bmatrix} -\alpha & 0 \\ 0 & -1 \end{bmatrix}, \boldsymbol{H_1} = \begin{bmatrix} ab\alpha & a(1-b)\alpha \\ c(1-d) & cd \end{bmatrix},\ \boldsymbol{\eta} = \begin{bmatrix} (1-a)\alpha \\ 1-c \end{bmatrix}. \tag{23}$$

Figure 2 illustrates the areas that can be represented by the different Markovian representations of the theorem as a function of c and d for given values of α, a, b. In the figure f_i denotes the area covered by form i. Intuitively, Theorem 1 states that the outer boundaries of the covered area in Figure 2 coincide with the boundaries defined in the previous section for TRAP(2) and Theorem 2 in the next section states that the areas covered by different TMAP(2) forms do not overlap.

Figure 2 also indicates that the shape of the valid region has two different types behaviour depending on the value of b (below and above 1), which suggests that the proof splits into different cases.



(a) $\alpha = 1/2,\ a = 9/11,\ b = 66/70 < 1$ (b) $\alpha = 1/2,\ a = 9/11,\ b = 73/70 > 1$

Figure 2: The valid range of c and d for the proposed canonical forms

Proof. The proof is organized as follows. We consider some of the constraints for TRAP(2) and show that under these constraints a TRAP(2) can be represented by one of the Markovian froms in Theorem 1. Because TMAP(2)⊆TRAP(2) this also proves that the chosen constraints provide the tight boundaries of TRAP(2).

More precisely, in the previous section we obtained several constraints of TRAP(2) for different regions of the parameters each of which require detailed derivation of involved formulas. Here we do not present all of those cases, but restrict the attention to the cases when $\gamma > 0,\ a\alpha < c < a$, which are provided in (A.2) – (A.5). As illustrated in Figure 2 this area can be covered using Form 1 and Form 5.

First we look at the constraints of the Markovian representations of Form 5. To transform the diagonal TRAP(2) form in (23) to Form 5 we have to apply the similarity transformation shown in (5) to the diagonal form. Using $(\boldsymbol{D_0^{(5)}})_{2,1} = 0$, $((\boldsymbol{D_0^{(5)}} + \boldsymbol{D_1^{(5)}})\mathbb{1})_1 = 0$ and $(\boldsymbol{D_0^{(5)}})_{1,1} > (\boldsymbol{D_0^{(5)}})_{2,2}$ we get that $\boldsymbol{T^{(5)}}$ is the solution of the following system of constrained equations

$$(\boldsymbol{T^{(5)}}\boldsymbol{H_0}\boldsymbol{T^{(5)}}^{-1})_{2,1} = 0, \quad (\boldsymbol{T^{(5)}}(\boldsymbol{H_0} + \boldsymbol{H_1})\mathbb{1})_1 = 0$$
$$(\boldsymbol{T^{(5)}}\boldsymbol{H_0}\boldsymbol{T^{(5)}}^{-1})_{1,1} > (\boldsymbol{T^{(5)}}\boldsymbol{H_0}\boldsymbol{T^{(5)}}^{-1})_{2,2},$$

Solving this for $\boldsymbol{T^{(5)}}$ and substituting into (5) we get that Form 5 can be written as

$$\pi^{(5)} = \left[\frac{1-c-\alpha+a\alpha}{1-c}\pi_0, 1 - \frac{1-c-\alpha+a\alpha}{1-c}\pi_0\right],$$

$$\boldsymbol{\delta}^{(\mathbf{5})} = \begin{bmatrix} 0 \\ 1-c \end{bmatrix}, \quad \boldsymbol{D}_{\mathbf{0}}^{(\mathbf{5})} = \begin{bmatrix} -\alpha & \frac{\alpha(1-a)(1-\alpha)}{1-c-\alpha+a\alpha} \\ 0 & -1 \end{bmatrix},$$

$$\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{5})} = \begin{bmatrix} \frac{a\alpha[(b(1-c)+c(1-d)]-c\alpha(1-d)}{1-c} & w_1 \\ \frac{c(1-d)(1-c-\alpha+a\alpha)}{1-c} & \frac{c[\alpha(1-a)+d(1-c-\alpha+a\alpha)]}{1-c} \end{bmatrix},$$

where $w_1 = \alpha - (\boldsymbol{D}_{\mathbf{0}}^{(\mathbf{5})})_{1,2} - (\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{5})})_{1,1}$. The $\pi^{(5)}$ vector is non-negative if $0 < \pi_1^{(5)} < 1$. For $a\alpha < c < a$ the fraction in $\pi^{(5)}$ is positive, because

$$1 - c - \alpha + a\alpha > 1 - a - \alpha(1-a) = (1-\alpha)(1-a) > 0. \tag{24}$$

Rearranging $0 < \pi_1^{(5)} < 1$ to π_0 we get

$$0 < \pi_0 < \frac{1-c}{1-c-\alpha+a\alpha}. \tag{25}$$

The $(1,1)$, $(2,1)$ and $(2,2)$ elements of the $\boldsymbol{D}_{\mathbf{0}}^{(\mathbf{5})}$ matrix are trivially Markovian and the $(1,2)$ element is also non-negative because its denominator is non-negative from (24).

For the $(1,1)$ element of the $\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{5})}$, we have that $(\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{5})})_{1,1}|_{d=1-b} = \frac{b(a-c)\alpha}{1-c}$, which is zero at $b = 0$ and positive for $a\alpha < c < a$, $b > 0$. The partial derivatives of $(\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{5})})_{1,1}$ with respect to b and d are

$$\frac{\partial(\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{5})})_{1,1}}{\partial b} = a\alpha \quad \text{and} \quad \frac{\partial(\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{5})})_{1,1}}{\partial d} = \frac{c\alpha(1-a)}{1-c}, \tag{26}$$

which are both positive, that is, $(\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{5})})_{1,1}$ increases in both b and d, therefore it is positive if

$$d > 1 - b, \; b > 0. \tag{27}$$

The partial derivative of $(\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{5})})_{1,2}$ with respect to d is

$$\frac{\partial(\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{5})})_{1,2}}{\partial d} = -\frac{c\alpha(1-a)}{1-c}, \tag{28}$$

which is negative and $(\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{5})})_{1,2} = 0$ at

$$d = \frac{a(1-b)(1-c)^2 - (1-a)\alpha(c(1-a) - ab(1-c))}{c(1-a)(1-c-\alpha+a\alpha)},$$

therefore $(\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{5})})_{1,2}$ is positive if

$$d < \frac{a(1-b)(1-c)^2 - (1-a)\alpha(c(1-a) - ab(1-c))}{c(1-a)(1-c-\alpha+a\alpha)}. \tag{29}$$

Element $(\boldsymbol{D_1^{(5)}})_{2,1}$ is positive if

$$d < 1, \tag{30}$$

because $1-c>0$ and $1-c-\alpha+a\alpha>0$ (see (24)). From (29) and (30) we get that

$$d < \min\left(1,\ \frac{a(1-b)(1-c)^2-(1-a)\alpha(c(1-a)-ab(1-c))}{c(1-a)(1-c+\alpha-a\alpha)}\right). \tag{31}$$

The $(\boldsymbol{D_1^{(5)}})_{2,2}$ element is positive if its numerator is positive, which is true if $d > \frac{\alpha(1-a)}{1-c-\alpha+a\alpha}$. Using $d>1-b$ this can be rewritten as

$$b < \frac{1-c}{1-c-a+a\alpha}. \tag{32}$$

Finally, the non-negativity of $\boldsymbol{\delta^{(5)}}$ is straightforward.

Let us look at the (31) constraint one more time. Using notation $w_2 = \frac{a(1-b)(1-c)^2-(1-a)\alpha(c(1-a)-ab(1-c))}{c(1-a)(1-c-\alpha+a\alpha)}$ we have that $w_2>1$ exactly if $b<\frac{a-c}{1-c-\alpha+a\alpha}$, because $w_2|_{b=\frac{a-c}{1-c-\alpha+a\alpha}}=1$ and $\frac{\partial w_2}{\partial b} = -\frac{a(1-c)}{c(1-a)}<0$, thus from (25), (27), (31), and (32) we get that Form 5 is Markovian for $a\alpha<c<a$ if

$$\begin{aligned}
0 < \pi_0 &< \frac{1-c}{1-c-\alpha+a\alpha},\\
0 < b &< \frac{1-c}{1-c-\alpha+a\alpha},\\
1-b < d &< \begin{cases} 1, & \text{if } b<\frac{a-c}{1-c-\alpha+a\alpha},\\ \frac{a(1-b)(1-c)^2-(1-a)\alpha(c(1-a)-ab(1-c))}{c(1-a)(1-c-\alpha+a\alpha)}, & \text{if } b>\frac{a-c}{1-c-\alpha+a\alpha}.\end{cases}
\end{aligned} \tag{33}$$

Now we examine the boundaries of the Markovian representations of Form 1. To transform the diagonal TRAP(2) form in (23) to Form 1 we have to apply a similarity transformation with $\boldsymbol{T^{(1)}}$ to the diagonal form, where $\boldsymbol{T^{(1)}}$ is the solution of the following set of constrained equations

$$\begin{gathered}
(\boldsymbol{T^{(1)}H_0T^{(1)}}^{-1})_{2,1}=0,\ (\boldsymbol{T^{(1)}H_1T^{(1)}}^{-1})_{1,2}=0,\\
(\boldsymbol{T^{(1)}H_0T^{(1)}}^{-1})_{1,1} > (\boldsymbol{TD_0T}^{-1})_{2,2},\\
(\boldsymbol{T^{(1)}H_1T^{(1)}}^{-1})_{1,1} > (\boldsymbol{T^{(1)}D_1^{(1)}T^{(1)}}^{-1})_{2,2}.
\end{gathered}$$

Solving this for $\boldsymbol{T}^{(\mathbf{1})}$ we get

$$
\begin{aligned}
\pi^{(1)} &= \left[\frac{2(c-a\alpha)\pi_0}{2c-cd-ab\alpha+w_3}, \frac{2(c-a\alpha)\pi_0}{2c-cd-ab\alpha+w_3}\right], \\
\boldsymbol{\delta}^{(\mathbf{1})} &= \begin{bmatrix} \frac{2(c-a)+(1-c-\alpha+a\alpha)(cd+ab\alpha-w_3)}{2(c-a\alpha)} \\ 1-c \end{bmatrix}, \\
\boldsymbol{D}_{\mathbf{0}}^{(\mathbf{1})} &= \begin{bmatrix} -\alpha & \frac{(1-\alpha)(2a\alpha-cd-ab\alpha+w_3)}{2(c-a\alpha)} \\ 0 & -1 \end{bmatrix}, \\
\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{1})} &= \begin{bmatrix} \frac{cd+ab\alpha-w_3}{2} & 0 \\ \frac{2c(1-d)(c-a\alpha)}{2c-cd-ab\alpha+w_3} & \frac{cd+ab\alpha+w_3}{2} \end{bmatrix},
\end{aligned}
$$

where $w_3 = \sqrt{(2c-cd-ab\alpha)^2 - 4c(1-d)(c-a\alpha)}$. From (21) we know that the quantity below the square root is non-negative for a TRAP(2). The roots of w_3 are

$$
d_{1,2} = \frac{a\alpha(2-b) \pm 2\sqrt{a\alpha(b-1)(c-a\alpha)}}{c}. \tag{34}
$$

If $b < 1$, then the $d_{1,2}$ roots are complex, therefore the quadratic function never changes signs and is always positive (and the elements of the representation are real). If $b > 1$, then $d_{1,2}$ is real, thus the quadratic function is negative and the representation has some complex elements if $d_1 < d < d_2$, and every element is real if

$$
d < d_1 = \frac{a\alpha(2-b) - 2\sqrt{a\alpha(1-b)(c-a\alpha)}}{c}. \tag{35}
$$

This constraint is identical to the corresponding TRAP(2) constraint (A.5), therefore the $d > d_2$ case does not have to be considered.

The $(1,1)$, $(2,1)$, and $(2,2)$ elements of $\boldsymbol{D}_{\mathbf{0}}^{(\mathbf{1})}$ are trivially Markovian. The denominator of the $(1,2)$ element is positive for $a\alpha < c < a$ and the numerator is positive as well because of the following. If $2a\alpha > cd + ab\alpha$ then $(\boldsymbol{D}_{\mathbf{0}}^{(\mathbf{1})})_{1,2}$ is trivially positive. Let us now look at the $2a\alpha < cd + ab\alpha$ case. If $b > 1$, then $2a\alpha - cd + ab\alpha < 0$ can be rearranged to d as $d > \frac{2a\alpha - ab\alpha}{c}$, which is higher than the upper boundary for TRAP(2) in (A.5), therefore does not need to be considered. If $b < 1$, then $(\boldsymbol{D}_{\mathbf{0}}^{(\mathbf{1})})_{1,2} < 0$ only if $-cd + 2a\alpha - ab\alpha < 0$, thus we need

$$
-|2a\alpha - cd - ab\alpha| + \sqrt{(2c-cd-ab\alpha)^2 - 4c(1-d)(c-a\alpha)} < 0. \tag{36}
$$

Relations $-|w_4| + \sqrt{w_5} < 0$ and $-w_4^2 + w_5 < 0$ are equivalent, thus (36) is also equivalent with

$$
\begin{aligned}
-(2a\alpha - cd - ab\alpha)^2 + (2c-cd-ab\alpha)^2 - 4c(1-d)(c-a\alpha) &= \\
= 4a\alpha(1-b)(c-a\alpha) < 0, \quad (37)
\end{aligned}
$$

which cannot hold if $b < 1$. From the above it follows, that $(\boldsymbol{D_0^{(1)}})_{1,2}$ cannot be negative for any value of b.

The $(\boldsymbol{D_1^{(1)}})_{1,1}$ element equals to 0 at $d = 1 - b$. $\frac{\partial}{\partial d}(\boldsymbol{D_1^{(1)}})_{1,1}\big|_{d=1-b} = \frac{a\alpha c}{c-bc+ab\alpha}$, which is positive if $b < \frac{c}{c-a\alpha}$, but from (A.1) we know that $b < \frac{1}{1-\alpha}$ and $\frac{1}{1-\alpha} < \frac{c}{c-a\alpha}$ for $a\alpha < c < a$, therefore $(\boldsymbol{D_1^{(1)}})_{1,1}$ is positive if

$$d > 1 - b. \tag{38}$$

Let us denote the denominator of $(\boldsymbol{D_1^{(1)}})_{2,1}$ by w_6 and the numerator of $(\boldsymbol{D_0^{(1)}})_{1,2}$ by w_7. By comparing them we get that $w_6 = \frac{w_7}{1-\alpha} + 2c - 2a\alpha > \frac{w_7}{1-\alpha}$ if $c > a\alpha$, therefore w_6 (the denominator of $(\boldsymbol{D_1^{(1)}})_{2,1}$) is positive if w_7 is positive and we know that w_7 is positive, because $(\boldsymbol{D_0^{(1)}})_{1,2}$ is positive. Furthermore the numerator of $(\boldsymbol{D_1^{(1)}})_{2,1}$ is $2c(1-d)(c-a\alpha)$, which is trivially positive if $d < 1$ (because $c > a\alpha$), therefore $(\boldsymbol{D_1^{(1)}})_{2,1}$ is positive if

$$d < 1. \tag{39}$$

It is easy to see that $(\boldsymbol{D_1^{(1)}})_{2,2} > (\boldsymbol{D_1^{(1)}})_{1,1}$ (this also has to hold by the definition of the form) from which $(\boldsymbol{D_1^{(1)}})_{2,2}$ has to be positive when $(\boldsymbol{D_1^{(1)}})_{1,1}$ is positive.

Finally we examine $\boldsymbol{\delta^{(1)}}$. The expression under the square root in $\delta_1^{(1)}$ is the same as in the other elements of the representation, thus if $b < 1$, then this expression is always positive consequently $\delta_1^{(1)}$ is real. If $b > 1$, then

$$d < \frac{a\alpha(2-b) - 2\sqrt{a\alpha(1-b)(c-a\alpha)}}{c} \tag{40}$$

has to hold otherwise $\delta_1^{(1)}$ is complex. Furthermore we get that $\delta_1^{(1)} = 0$ in

$$d = \frac{a(1-b)(1-c)^2 - (1-a)\alpha(c(1-a) - ab(1-c))}{c(1-a)(1-c-\alpha+a\alpha)}, \tag{41}$$

and $\delta_1^{(1)}$ increases in d because

$$\begin{aligned}\frac{\partial \delta_1^{(1)}}{\partial d} =& \frac{c(1-c-\alpha+a\alpha)}{2(c-a\alpha)\sqrt{(2c-cd-ab\alpha)^2 - 4c(1-d)(c-a\alpha)}} \\ &\cdot \left(2a\alpha - ab\alpha - cd + \sqrt{(2c-cd-ab\alpha)^2 - 4c(1-d)(c-a\alpha)}\right),\end{aligned} \tag{42}$$

which is positive when $(\boldsymbol{D_0^{(1)}})_{1,2}$ is positive, because $(-cd + 2a\alpha - ab\alpha + \sqrt{(2c-cd-ab\alpha)^2 - 4c(1-d)(c-a\alpha)})$ (the third term in the numerator of (42)) is the same as the second term in the numerator of $(\boldsymbol{D_0^{(1)}})_{1,2}$ (which is positive) and the other terms are trivially positive when $a\alpha < c < a$. From

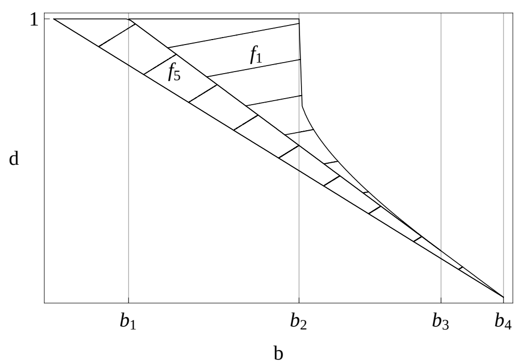


Figure 3: The boundaries of the canonical forms for given a, c, α values ($a\alpha < c < a$)

the above it follows, that $\delta_1^{(1)}$ is positive and real if

$$\frac{a(1-b)(1-c)^2-(1-a)\alpha(c(1-a)-ab(1-c))}{c(1-a)(1-c-\alpha+a\alpha)} < d < \\ < \frac{a\alpha(2-b)-2\sqrt{a\alpha(1-b)(c-a\alpha)}}{c}, \quad (43)$$

and the upper and lower bounds become equal for

$$d = \frac{a(1-c)^2-(1-a)\alpha(a(2-c)+c)}{a(1-c-a+a\alpha)^2}. \quad (44)$$

The obtained constraints of the Markovian region for $a\alpha < c < a$, are illustrated in Figure 3 for fixed a, c and α values. The constraints can be summarized as follows. Form 5 is Markovian if

$$b < b_1 = \frac{a-c}{a(1-c-\alpha+a\alpha)} \text{ and } 1-b < d < 1 \quad (45)$$

or

$$\begin{gathered} b_1 = \frac{a-c}{a(1-c-\alpha+a\alpha)} < b < b_4 = \frac{1-c}{1-c-\alpha+a\alpha} \\ 1-b < d < \frac{a(1-b)(1-c)^2-(1-a)\alpha(c(1-a)-ab(1-c))}{c(1-a)(1-c-\alpha+a\alpha)} \end{gathered} \quad (46)$$

Form 1 is Markovian if

$$\begin{gathered} b_1 = \frac{a-c}{a(1-c-\alpha+a\alpha)} < b < b_2 = 1, \\ \frac{a(1-b)(1-c)^2-(1-a)\alpha(c(1-a)-ab(1-c))}{c(1-a)(1-c-\alpha+a\alpha)} < d < 1, \end{gathered} \quad (47)$$

or if

$$b_2 = 1 < b < b_3 = \frac{a(1-c)^2 - (1-a)\alpha(a(2-c)+c)}{a(1-c-a+a\alpha)^2},$$
$$\frac{a(1-b)(1-c)^2 - (1-a)\alpha(c(1-a) - ab(1-c))}{c(1-a)(1-c-\alpha+a\alpha)} < d < \tag{48}$$
$$< \frac{2a\alpha - ab\alpha}{c} - 2\sqrt{\frac{a\alpha(a\alpha - c)(1-b)}{c^2}}.$$

Which together give the TRAP(2) constraints (A.2) – (A.5) presented in Appendix A. □

5. The uniqueness of the Markovian forms in Theorem 1

In the previous section we proved the equivalence between TMAP(2) and TRAP(2) by showing that the TRAP(2) set can be covered using five different Markovian forms. In this section we show that these forms cover non-overlapping areas.

Theorem 2. *If an order 2 transient rational arrival process has a Markovian representation in one of the five forms presented in Theorem 1 then it does not have a Markovian representation in any of the other four forms.*

The theorem is proved through the following lemmas.

Lemma 7. *Form 1 and 2 cover a disjoint TMAP(2) area compared to Form 3 and 4.*

Proof. The eigenvalues of $\boldsymbol{H_0}, \boldsymbol{H_1}$ and $\boldsymbol{P}$ are maintained by similarity transformation. For Form 1 the product of the two eigenvalues of $\boldsymbol{P^{(1)}} = (-\boldsymbol{D_0^{(1)}})^{-1}\boldsymbol{D_1^{(1)}}$ is $\hat{a}\hat{b}$ and for Form 2 it is $\frac{\bar{a}\bar{b}}{1-\bar{c}(1-\bar{b}-\bar{d})}$, which are both positive for any Markovian representation, where the related $a, b, c, d, 1-a-c, 1-b-d$ parameters are between 0 and 1. For Form 3 the product of the two eigenvalues of $\boldsymbol{P^{(3)}}$ is $-\check{a}\check{d}$ and for From 4 it is $-\frac{\tilde{a}\tilde{d}}{1-\tilde{c}(1-\tilde{b}-\tilde{d})}$ which are both negative. □

As a consequence a TRAP(2) can only be represented either in one or more of Form 1, 2, 5 (if $\gamma > 0$) or in one or more of Form 3, 4, 5 (if $\gamma < 0$). In the following lemmas we prove that the elements of these triplets also correspond to disjoint areas. Like before, we will use the $\lambda_2 = 1$ assumption, which does not change the proof in any significant way.

Lemma 8. *If an order 2 transient rational arrival process has a Markovian representation in Form 1 of Theorem 1, then it cannot have a Markovian representation in Form 2.*

Proof. From the fact that Form 1 is Markovian we have $a > 0, b > 0, c > 0, d > 0, 1-a-c > 0, 1-b-d > 0$ and from the eigenvalue and element constraints of Form 1 we additionally have $0 < \alpha = \lambda_1/\lambda_2 < 1$ and $a\alpha < b$.

Just like before, we use the similarity transformation from (5) to transform a Markovian TMAP of Form 1 to Form 2. The $\boldsymbol{T}$ matrix can be obtained using the constraints of Form 2, i.e., $(\boldsymbol{T}\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{1})}\boldsymbol{T}^{-1})_{1,2} = 0$ and $(\boldsymbol{T}(\boldsymbol{D}_{\mathbf{0}}^{(\mathbf{1})}+\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{1})})\mathbb{1})_2 = 0$. The solution of this pair of equations which maintains the eigenvalue constraint of Form 2 is

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 \\ \frac{1-b-d}{(1-b-d)-\alpha(1-a-c)} & -\frac{\alpha(1-a-c)}{(1-b-d)-\alpha(1-a-c)} \end{bmatrix}.$$

Based on this transformation the Form 2 representation is

$$\pi^{(1)}\boldsymbol{T}^{-1} = \left[\bullet, (1-\pi_1^{(1)})\frac{\alpha(1-a-c)-(1-b-d)}{1-b-d}\right],$$

$$\boldsymbol{T}\boldsymbol{D}_{\mathbf{0}}^{(\mathbf{1})}\boldsymbol{T}^{-1} = \begin{bmatrix} -\frac{\alpha(1-a-c)-c(1-b-d)}{1-a-c} & \frac{\alpha c(1-a-c)-c(1-b-d)}{1-a-c} \\ \frac{(1-b-d)((1-\alpha)(1-a-c)+c(1-b-d))}{(1-a-c)((1-b-d)-\alpha(1-a-c))} & -\frac{(1-a-c)-c(1-b-d)}{1-a-c} \end{bmatrix},$$

$$\boldsymbol{T}\boldsymbol{D}_{\mathbf{1}}^{(\mathbf{1})}\boldsymbol{T}^{-1} = \begin{bmatrix} \alpha a & 0 \\ \bullet & b \end{bmatrix}, \quad \boldsymbol{T}\boldsymbol{\delta}^{(\mathbf{1})}\boldsymbol{T}^{-1} = \begin{bmatrix} \alpha(1-a-c) \\ 0 \end{bmatrix},$$

where $\bullet$ denotes an irrelevant term. From the non-negativity of the second element of the initial vector of the Form 2 representation we have $1-b-d < \alpha(1-a-c)$, from which it follows that $(\boldsymbol{T}\boldsymbol{D}_{\mathbf{0}}^{(\mathbf{1})}\boldsymbol{T}^{-1})_{2,1}$ is negative, because the numerator is positive and the denominator is negative when $1-b-d < \alpha(1-a-c)$, thus the resulting representation is not Markovian. $\square$

Lemma 9. *If an order 2 transient rational arrival process has a Markovian representation in Form 1 of Theorem 1, then it cannot have a Markovian representation in Form 5.*

Proof. Similarly to the previous lemma from the fact that Form 1 is Markovian we have $a > 0, b > 0, c > 0, d > 0, 1-a-c > 0, 1-b-d > 0, 0 < \alpha < 1$, and from the additional constraint on Form 1 we have $a\alpha < b$. We use the similarity transformation from (5) again to transform a Markovian TMAP of Form 1 to Form 5.

The $\boldsymbol{T}$ matrix can be obtained using the constraints of Form 2, i.e., $(\boldsymbol{T}\boldsymbol{D}_{\mathbf{0}}^{(\mathbf{1})}\boldsymbol{T}^{-1})_{2,1} = 0$ and $(\boldsymbol{T}(\boldsymbol{D}_{\mathbf{0}}^{(\mathbf{1})} + \boldsymbol{D}_{\mathbf{1}}^{(\mathbf{1})})\mathbb{1})_1 = 0$. The solution of this pair of equations which maintains the eigenvalue constraint of Form 5 is

$$\boldsymbol{T} = \begin{bmatrix} \frac{1-\hat{b}-\hat{d}}{(1-\hat{b}-\hat{d})-\alpha(1-\hat{a}-\hat{c})} & \frac{\alpha(1-\hat{a}-\hat{c})}{(1-\hat{b}-\hat{d})-\alpha(1-\hat{a}-\hat{c})} \\ 0 & 1 \end{bmatrix}.$$

Based on this transformation the Form 5 representation is

$$
\begin{aligned}
\pi^{(1)}\boldsymbol{T}^{-1} &= \left[\pi_1^{(1)} \frac{(1-\hat{b}-\hat{d}) - \alpha(1-\hat{a}-\hat{c})}{1-\hat{b}-\hat{d}}, \bullet\right], \\
\boldsymbol{T}\boldsymbol{D}_{\boldsymbol{0}}^{(\boldsymbol{1})}\boldsymbol{T}^{-1} &= \begin{bmatrix} -\alpha & \frac{\alpha(1-\hat{a}-\hat{c}(\hat{b}+\hat{d})-\alpha(1-\hat{a}-\hat{c}))}{(1-\hat{b}-\hat{d})-\alpha(1-\hat{a}-\hat{c})} \\ 0 & -1 \end{bmatrix}, \quad \boldsymbol{T}\boldsymbol{\delta}^{(\boldsymbol{1})}\boldsymbol{T}^{-1} = \begin{bmatrix} 0 \\ 1-\hat{b}-\hat{d} \end{bmatrix}, \\
\boldsymbol{T}\boldsymbol{D}_{\boldsymbol{1}}^{(\boldsymbol{1})}\boldsymbol{T}^{-1} &= \begin{bmatrix} \frac{\alpha(\hat{a}(1-\hat{b})-\hat{d}(1-\hat{c}))}{1-\hat{b}-\hat{d}} & \frac{\alpha(1-\hat{a}-\hat{c})(\hat{b}^2-(1-\hat{d}+\alpha\hat{a})\hat{b}+\alpha(\hat{a}-\hat{d}(1-\hat{c})))}{(1-\hat{b}-\hat{d})(1-\hat{b}-\hat{d}-\alpha(1-\hat{a}-\hat{c}))} \\ \frac{\hat{d}(1-\hat{b}-\hat{d}-\alpha(1-\hat{a}-\hat{c}))}{1-\hat{b}-\hat{d}} & \frac{-\hat{b}^2+(1-\hat{d})\hat{b}+\hat{d}\alpha(1-\hat{a}-\hat{c})}{1-\hat{b}-\hat{d}} \end{bmatrix},
\end{aligned}
$$

from which $(1-\hat{b}-\hat{d}) > \alpha(1-\hat{a}-\hat{c})$ has to hold, otherwise the first element of the initial vector of the Form 5 representation is negative. From the $(1-\hat{b}-\hat{d}) > \alpha(1-\hat{a}-\hat{c})$ inequality it is easy to see that the $(2,1)$ element of $\boldsymbol{T}\boldsymbol{D}_{\boldsymbol{1}}^{(\boldsymbol{1})}\boldsymbol{T}^{-1}$ ($(\boldsymbol{T}\boldsymbol{D}_{\boldsymbol{1}}^{(\boldsymbol{1})}\boldsymbol{T}^{-1})_{2,1}$) is positive. To prove the theorem we show that $(\boldsymbol{T}\boldsymbol{D}_{\boldsymbol{1}}^{(\boldsymbol{1})}\boldsymbol{T}^{-1})_{1,1}$ and $(\boldsymbol{T}\boldsymbol{D}_{\boldsymbol{1}}^{(\boldsymbol{1})}\boldsymbol{T}^{-1})_{1,2}$ cannot be positive at the same time.

For $(\boldsymbol{T}\boldsymbol{D}_{\boldsymbol{1}}^{(\boldsymbol{1})}\boldsymbol{T}^{-1})_{1,1} > 0$ we have $\hat{a}(1-\hat{b}) > \hat{d}(1-\hat{c})$. For $(\boldsymbol{T}\boldsymbol{D}_{\boldsymbol{1}}^{(\boldsymbol{1})}\boldsymbol{T}^{-1})_{1,2} > 0$ we need that the quadratic term, $\hat{b}^2 - (1-\hat{d}+\alpha\hat{a})\hat{b} + \alpha(\hat{a}-\hat{d}(1-\hat{c}))$, is positive in the admissible region of variable $\hat{b}$, which is $\alpha\hat{a} < \hat{b} < 1-\hat{d}$. To investigate the sign of the quadratic term we first bound the discriminant $D = (1-\hat{d}+\alpha\hat{a})^2 - 4\alpha(\hat{a}-\hat{d}(1-\hat{c}))$. From $\hat{a}(1-\hat{b}) > \hat{d}(1-\hat{c})$ and $\alpha\hat{a} < \hat{b}$ we have

$$
\begin{aligned}
D &= (1-\hat{d}+\alpha\hat{a})^2 - 4\alpha(\hat{a}-\hat{d}(1-\hat{c})) < (1-\hat{d}+\alpha\hat{a})^2 - 4\alpha(\hat{a}-\hat{a}(1-\hat{b})) \\
&= (1-\hat{d}+\alpha\hat{a})^2 - 4\alpha\hat{a}\hat{b} < (1-\hat{d}+\alpha\hat{a})^2.
\end{aligned}
$$

On the other hand, from $\hat{c} < 1-\hat{a}$ we have

$$
\begin{aligned}
D &= (1-\hat{d}+\alpha\hat{a})^2 - 4\alpha(\hat{a}-\hat{d}(1-\hat{c})) > (1-\hat{d}+\alpha\hat{a})^2 - 4\alpha(\hat{a}-\hat{d}\hat{a}) \\
&= (1-\hat{d}-\alpha\hat{a})^2 > 0.
\end{aligned}
$$

Since the discriminant is positive the quadratic term is positive when $\hat{b} < \frac{1}{2}(1-\hat{d}+\alpha\hat{a}-\sqrt{D})$ or when $\hat{b} > \frac{1}{2}(1-\hat{d}+\alpha\hat{a}+\sqrt{D})$. For these limits we have

$$
\hat{b} < \frac{1}{2}\left(1-\hat{d}+\alpha\hat{a}-\sqrt{D}\right) < \frac{1}{2}\left(1-\hat{d}+\alpha\hat{a}-\sqrt{(1-\hat{d}+\alpha\hat{a})^2}\right) = \alpha\hat{a}
$$

and

$$
\hat{b} > \frac{1}{2}\left(1-\hat{d}+\alpha\hat{a}+\sqrt{D}\right) > \frac{1}{2}\left(1-\hat{d}+\alpha\hat{a}+\sqrt{(1-\hat{d}-\alpha\hat{a})^2}\right) = 1-\hat{d},
$$

but both of these are outside of the $\alpha a < b < 1-d$ admissible region, which means that $(\boldsymbol{T}\boldsymbol{D}_{\boldsymbol{1}}\boldsymbol{T}^{-1})_{1,2}$ is never positive in the admissible region of $\hat{b}$. $\square$

Lemma 10. *If an order 2 transient rational arrival process has a Markovian representation in Form 2 of Theorem 1, then it cannot have a Markovian representation in Form 5.*

Proof. From the fact that Form 2 is Markovian we have $\bar{a} > 0, \bar{b} > 0, \bar{c} > 0, \bar{d} > 0, 1 - \bar{a} - \bar{c} > 0, 1 - \bar{b} - \bar{d} > 0$, and from the additional constraint on Form 2 we have $\bar{a}\bar{\alpha} = \bar{a}\bar{\lambda_1}/\bar{\lambda_2} < \bar{b}\bar{d}$. We use the similarity transformation from (5) again to transform a Markovian TMAP of Form 2 to Form 5.

The $\boldsymbol{T}$ matrix can be obtained using the constraints of Form 5, i.e., $(\boldsymbol{T}\boldsymbol{D_0^{(2)}}\boldsymbol{T}^{-1})_{2,1} = 0$ and $(\boldsymbol{T}(\boldsymbol{D_0^{(2)}} + \boldsymbol{D_1^{(2)}})\mathbb{1})_1 = 0$. The solution of this pair of equations which maintains the eigenvalue constraint of Form 5 is

$$\boldsymbol{T} = \begin{bmatrix} \frac{1-\bar{\alpha}+\sqrt{(1-\bar{\alpha})^2+4\bar{c}\bar{\alpha}(1-\bar{b})}}{2(1-\bar{b})} & \frac{1-2\bar{b}+\bar{\alpha}-\sqrt{(1-\bar{\alpha})^2+4\bar{c}\bar{\alpha}(1-\bar{b})}}{2(1-\bar{b})} \\ 0 & 1 \end{bmatrix}.$$

Using the notations $w_8 = 1 - \bar{\alpha} - \sqrt{(1-\bar{\alpha})^2 + 4\bar{c}\bar{\alpha}(1-\bar{b})}$ and $w_9 = 1 - \bar{\alpha} + \sqrt{(1-\bar{\alpha})^2 + 4\bar{c}\bar{\alpha}(1-\bar{b})}$ we have

$$\boldsymbol{T}\boldsymbol{D_0^{(2)}}\boldsymbol{T}^{-1} = \begin{bmatrix} -\frac{w_8+2\bar{\alpha}}{2} & \frac{w_8+2\bar{\alpha}-2\bar{b}}{2} \\ 0 & -\frac{w_9+2\bar{\alpha}}{2} \end{bmatrix}, \quad \boldsymbol{T}\boldsymbol{\delta^{(2)}} = \begin{bmatrix} 0 \\ \frac{2\bar{\alpha}(1-\bar{a}-\bar{c})(1-\bar{b}-\bar{d})}{2(1-\bar{b}-\bar{d})-w_9} \end{bmatrix},$$

$$\boldsymbol{T}\boldsymbol{D_1^{(2)}}\boldsymbol{T}^{-1} = \begin{bmatrix} \bar{b} + \frac{\bar{d}w_9}{2(1-\bar{b}-\bar{d})} & \bar{d} - \frac{\bar{d}w_8}{2(1-\bar{b}-\bar{d})} \\ \frac{w_9\left(2(\bar{a}\bar{\alpha}-\bar{b})(1-\bar{b}-\bar{d})-\bar{d}w_8\right)}{2(1-\bar{b}-\bar{d})\left(2(1-\bar{b}-\bar{d})-w_9\right)} & \bar{a}\bar{\alpha} - \frac{\bar{d}w_9}{2(1-\bar{b}-\bar{d})} \end{bmatrix}.$$

The numerator of the second element of $\boldsymbol{T}\boldsymbol{\delta^{(2)}}$ is positive, because $1 - \bar{a} - \bar{c} > 0, 1 - \bar{b} - \bar{d} > 0$, therefore the denominator has to be positive as well. Consequently the denominator of $(\boldsymbol{T}\boldsymbol{D_1^{(2)}}\boldsymbol{T}^{-1})_{2,1}$ has to be positive as well. However the numerator of $(\boldsymbol{T}\boldsymbol{D_1^{(2)}}\boldsymbol{T}^{-1})_{2,1}$ is negative because w_9 is positive, and in the second factor both terms are negative ($\bar{a}\bar{\alpha} < \bar{b}$ and $1 - \bar{b} - \bar{d} > 0$ are constraints from Form 2 and it is easy to see that w_8 is positive). Consequently $(\boldsymbol{T}\boldsymbol{\delta^{(2)}})_2$ and $(\boldsymbol{T}\boldsymbol{D_1^{(2)}}\boldsymbol{T}^{-1})_{2,1}$ cannot be positive at the same time. □

The proof for the triplet, Form 3, 4 and 5, where γ can be negative follow similar pattern and is omitted here due to space limitation. The proof for that triplet is available at [13].

In this section we proved that the Markovian forms proposed in Theorem 1 cover disjoint subsets of TRAP(2)s and in the previous section we proved that these forms cover the whole TRAP(2) class. Theorem 1 and 2 imply that every TRAP(2) can be represented in exactly one of the Markovian forms of Theorem 1, which we propose to be the canonical representation for the TRAP(2) class.

6. Conclusion

The paper presents a canonical representation of the TMAP(2) class and shows that the TRAP(2) class is identical with the TMAP(2) class. The identity of the TMAP(2) and TRAP(2) classes makes the use of TRAP(2) irrelevant in practice. The availability of a canonical representation makes the TMAP(2) class easy to apply in practical computations as it was the case for similar model classes with canonical representation. For example, one can restrict the attention to the presented Markovian canonical forms when fitting a transient point process with TRAP(2) and the number of considered cases can be further restricted when the sign of the correlation parameter is known.

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Appendix A. Feasible TRAP(2) regions resulted by the direct and the iterative parameter constraints

In this section we summarize the constraints of TRAP(2) for different values of γ and c. We present the derivation of these constraints for $\gamma > 0,\ c < 0$ in more detail. The constraints for the other cases can be derived in a very similar fashion.

Iterative parameter constraints for $\gamma > 0,\ c < 0$

From Section 3.1 we have the following direct parameter constraints

$$0 < a < 1, \quad 0 < b < \frac{1}{1-\alpha}, \quad 0 < \alpha < 1. \tag{A.1}$$

Using the constraints of b in Lemma 3 the behaviour of $u(x)$ can be analysed based on the value of c. This analysis will give us five different cases: $c < 0$, $0 < c < a\alpha$, $a\alpha < c < a$, $a < c < 1$, and $c > 1$. (The b parameter has the same constraints for $0 < c < a\alpha$ and $a\alpha < c < a$. The additional split of the $0 < c < a$ interval is due to the fact that the fixed points are on different sides of the $u(x)$ hyperbola for the two different cases.)

The respective parts of the above constraints have to hold in every case, we will focus on the additional bounds. In what follows we show how the iterative constraints can be derived through the specific case of $\gamma > 0,\ c < 0$. The same method can be used for the other cases for which we only state the results in the next subsection.

Let us examine the $u(x)$ hyperbola for the $\gamma > 0,\ c < 0$ case. From (16a) we have that $\pi_0 > v$, therefore π_0 is on the right side of $u(x)$ and x has to stay on that side as well. Because $\gamma \geq 0$, the right side of $u(x)$ opens downwards. This means that $\gamma > 0,\ c < 0$ corresponds to Fig. 1d. As mentioned before, the x_u unstable fixed point cannot be between π_0 and the x_s stable fixed point, consequently the following constraints can be established:

1. $\gamma > 0$, from which $-c(1-b-d) < 0$, consequently $1-b-d < 0$,
2. x_u and x_s have to be real, from which $(2c-cd-ab\alpha)^2-4(c-cd)(c-a\alpha) \geq 0$ (see (20) and (21)),

3. the x_s stable fixed point has to be in the permissible range, that is $0 < \frac{c}{c-a\alpha} \leq x_s \leq \frac{1-c}{1-c-\alpha+a\alpha}$ (from (16a)) (recall that the constraints for π_0 have to hold for $u(\pi_0), u(u(\pi_0))$ etc., thus for the stable fixed point as well),

4. from the last constraint $x_s > \frac{c}{c-a\alpha} = v$, therefore x_s has to be on the right side of the hyperbola (Fig. 1d), from which $v < h$,

5. π_0 has to be greater than the unstable fixed point, that is, $\pi_0 > x_u$,

6. b also has to be greater than the unstable fixed point, that is, $b > x_u$,

7. π_0 has to be in the permissible range defined by (16a), that is $\frac{c}{c-a\alpha} < \pi_0 \leq \frac{1-c}{1-c-\alpha+a\alpha}$,

8. b also has to be in the permissible range defined by Lemma 3, that is $\frac{c}{c-a\alpha} < b \leq \frac{1-c}{1-c-\alpha+a\alpha)}$.

Out of these constraints some are irrelevant. Constraints 1, 6, and the right side of 3 bound d, constraints 5 and the right side of 7 bound π_0 and constraint 8 and 6 bound b. Combining these boundaries we get the following TRAP(2) regions.

- Region 1a ($\gamma > 0$, $c < 0$ and $b < 1$)

$$\begin{aligned}
&\frac{c}{c-a\alpha} < b < 1,\\
&\frac{2a\alpha - ab\alpha}{c} + 2\sqrt{\frac{a\alpha(a\alpha-c)(1-b)}{c^2}} < d < 1-b,\\
&\frac{cd + ab\alpha - 2c - \sqrt{(2c-cd-ab\alpha)^2 + 4c(1-d)(a\alpha-c)}}{a\alpha - c} < \pi_0 <\\
&\qquad < \frac{1-c}{1-c-\alpha+a\alpha}.
\end{aligned}$$

- Region 1b ($\gamma > 0$, $c < 0$ and $b > 1$)

$$\begin{aligned}
&1 < b < \frac{1-c}{1-c-\alpha+a\alpha},\\
&\frac{a(1-b)(1-c)^2 - (1-a)\alpha(c(1-a) - ab(1-c))}{c(1-a)(1-c-\alpha+a\alpha)} < d < 1-b,\\
&\frac{cd + ab\alpha - 2c - \sqrt{(2c-cd-ab\alpha)^2 + 4c(1-d)(a\alpha-c)}}{a\alpha - c} < \pi_0 <\\
&\qquad < \frac{1-c}{1-c-\alpha+a\alpha}.
\end{aligned}$$

Boundaries for $\gamma > 0,\ 0 < c < a\alpha$

For $\gamma > 0$, $0 < c < a\alpha$ (Fig. 1d) the fixed points are on the right side of the hyperbola again and the relevant constraints are

$$\begin{aligned} &\gamma > 0, \quad \{x_s, x_u\} \in \mathbb{R}, \quad 0 < x_s < \frac{1-c}{1-c-\alpha+a\alpha}, \\ &\max(0, x_u) < \{\pi_0,\ b\} < \frac{1-c}{1-c-\alpha+a\alpha}, \end{aligned}$$

from which we obtain the following feasible TMAP(2) regions

- Region 2a ($\gamma > 0$, $0 < c < a\alpha$ and $b < 1$, $d < 1$)

$$\begin{aligned} 0 < b &< 1, \\ 1 - b < d &< 1, \\ 0 < \pi_0 &< \frac{1-c}{1-c-\alpha+a\alpha}. \end{aligned}$$

- Region 2b ($\gamma > 0$, $0 < c < a\alpha$ and $b < 1$, $d > 1$)

$$\begin{aligned} &0 < b < 1, \\ &0 < c < ab\alpha, \\ &1 < d < \frac{2a\alpha - ab\alpha}{c} - 2\sqrt{\frac{a\alpha(a\alpha - c)(1-b)}{c^2}} \\ &\frac{cd + ab\alpha - 2c - \sqrt{(2c - cd - ab\alpha)^2 + 4c(1-d)(a\alpha - c)}}{a\alpha - c} < \pi_0 < \\ &\qquad < \frac{1-c}{1-c-\alpha+a\alpha}. \end{aligned}$$

- Region 2c ($\gamma > 0$, $0 < c < a\alpha$ and $1 < b < \frac{a-c}{a(1-c-\alpha+a\alpha)}$, $d < 1$)

$$\begin{aligned} 1 < b &< \frac{a-c}{a(1-c-\alpha+a\alpha)}, \\ 1 - b < d &< 1, \\ 0 < \pi_0 &< \frac{1-c}{1-c-\alpha+a\alpha}. \end{aligned}$$

- Region 2d ($\gamma > 0$, $0 < c < a\alpha$ and $1 < b < \frac{a-c}{a(1-c-\alpha+a\alpha)}$, $d > 1$)

$$\begin{aligned} &1 < b < \frac{a-c}{a(1-c-\alpha+a\alpha)}, \\ &1 < d < \frac{a(1-b)(1-c)^2 + (1-a)\alpha(ab(1-c) - c(1-a))}{(1-a)c(1-c-\alpha+a\alpha)}, \\ &\frac{cd + ab\alpha - 2c - \sqrt{(2c - cd - ab\alpha)^2 + 4c(1-d)(a\alpha - c)}}{a\alpha - c} < \pi_0 < \\ &\qquad < \frac{1-c}{1-c-\alpha+a\alpha}. \end{aligned}$$

- Region 2e ($\gamma > 0$, $0 < c < a\alpha$ and $b > \frac{a-c}{a(1-c-\alpha+a\alpha)}$)

$$\begin{aligned}
&\frac{a-c}{a(1-c-\alpha+a\alpha)} < b < \frac{1-c}{1-c-a+a\alpha},\\
&1-b < d < \frac{a(1-b)(1-c)^2 + (1-a)\alpha(ab(1-c) - c(1-a))}{(1-a)c(1-c-\alpha+a\alpha)},\\
&0 < \pi_0 < \frac{1-c}{1-c-\alpha+a\alpha}.
\end{aligned}$$

Boundaries for $\gamma > 0,\ a\alpha < c < a$

For $\gamma > 0,\ a\alpha < c < a$ we have $v > h$ (Fig. 1c). The fixed points are on the left side of the hyperbola, which changes some of the constraints. The relevant constraints are

$$\begin{aligned}
&\gamma > 0, \quad \{x_s, x_u\} \in \mathbb{R}, \quad x_s < \frac{1-c}{1-c-\alpha+a\alpha},\\
&0 < \{\pi_0,\ b\} < \min\left(\frac{1-c}{1-c-a+a\alpha}, x_u\right),
\end{aligned}$$

where the notation in the last inequality means that both π_0 and b satisfy the inequality. Combining these we get that one of the following sets of constraints has to hold

- Region 3a ($\gamma > 0,\ a\alpha < c < a$ and $b < \frac{a-c}{1-c-a+a\alpha}$)

$$\begin{aligned}
&0 < b < \frac{a-c}{1-c-a+a\alpha},\\
&0 < \pi_0 < \frac{1-c}{1-c-a+a\alpha},\\
&1-b < d < 1.
\end{aligned} \tag{A.2}$$

- Region 3b ($\gamma > 0,\ a\alpha < c < a$ and $\frac{a-c}{1-c-a+a\alpha} < b < \frac{1-c}{1-c-a+a\alpha}$)

$$\begin{aligned}
&\frac{a-c}{1-c-a+a\alpha} < b < \frac{1-c}{1-c-a+a\alpha},\\
&0 < \pi_0 < \frac{1-c}{1-c-a+a\alpha},\\
&1-b < d < \frac{a(1-b)(1-c)^2 + (1-a)\alpha(ab(1-c) - c(1-a))}{(1-a)c(1-c-\alpha+a\alpha)}.
\end{aligned} \tag{A.3}$$

- Region 3c ($\gamma > 0,\ a\alpha < c < a$ and $\frac{a-c}{1-c-a+a\alpha} < b < 1$)

$$\begin{aligned}
&\frac{a-c}{1-c-a+a\alpha} < b < 1,\\
&0 < \pi_0 < \frac{2c - cd - ab\alpha + \sqrt{(2c-cd-ab\alpha)^2 + 4c(1-d)(a\alpha - c)}}{c - a\alpha},\\
&\frac{a(1-b)(1-c)^2 + (1-a)\alpha(ab(1-c) - c(1-a))}{(1-a)c(1-c-\alpha+a\alpha)} < d < 1.
\end{aligned} \tag{A.4}$$

- Region 3d ($\gamma > 0,\ a\alpha < c < a$ and $1 < b$)

$$
\begin{aligned}
&1 < b < \frac{a(1-c)^2 - (1-a)\alpha(a(2-c)+c)}{a(1-c-a+a\alpha)^2},\\
&0 < \pi_0 < \frac{2c - cd - ab\alpha + \sqrt{(2c-cd-ab\alpha)^2 + 4c(1-d)(a\alpha - c)}}{c - a\alpha},\\
&\frac{a(1-b)(1-c)^2 + (1-a)\alpha(ab(1-c) - c(1-a))}{(1-a)c(1-c-\alpha+a\alpha)} < d <\\
&\qquad < \frac{2a\alpha - ab\alpha}{c} - 2\frac{\sqrt{a\alpha(a\alpha-c)(1-b)}}{c}.
\end{aligned} \tag{A.5}
$$

Boundaries for $\gamma > 0,\ a < c < 1$

For $\gamma > 0,\ a < c < 1$ we have $v > h$ (Fig. 1c) once again. The relevant constraints are

$$
\begin{aligned}
&\gamma > 0, \quad \{x_s, x_u\} \in \mathbb{R}, \quad x_s < \frac{c}{c - a\alpha},\\
&0 < \{\pi_0,\ b\} < \min\left(\frac{c}{c-a\alpha}, x_u\right).
\end{aligned}
$$

Combining these constraints we get the following feasible regions

- Region 4a ($\gamma > 0,\ a < c < 1$ and $b < 1$)

$$
\begin{aligned}
&0 < b < 1,\\
&1 - b < d < 1,\\
&0 < \pi_0 < \frac{2c - cd - ab\alpha + \sqrt{(2c-cd-ab\alpha)^2 + 4c(1-d)(a\alpha - c)}}{c - a\alpha}.
\end{aligned}
$$

- Region 4b ($\gamma > 0,\ a < c < 1$ and $b > 1$)

$$
\begin{aligned}
&1 < b < \frac{c}{c - a\alpha},\\
&1 - b < d < \frac{2a\alpha - ab\alpha}{c} - 2\frac{\sqrt{a\alpha(a\alpha-c)(1-b)}}{c},\\
&0 < \pi_0 < \frac{2c - cd - ab\alpha + \sqrt{(2c-cd-ab\alpha)^2 + 4c(1-d)(a\alpha - c)}}{c - a\alpha}.
\end{aligned}
$$

Boundaries for $\gamma > 0,\ c > 1$

For $\gamma > 0,\ c > 1$ we also have $v > h$ (Fig. 1c). The relevant constraints are

$$
\begin{aligned}
&\gamma > 0, \quad \frac{1-c}{1-c-\alpha+a\alpha} < x_s < \frac{c}{c-a\alpha},\\
&b < x_u, \quad \frac{1-c}{1-\alpha-c+a\alpha} < \pi_0 < x_u.
\end{aligned}
$$

Combining these constraints we get the following sets of boundaries

- Region 5a ($\gamma > 0,\ c > 1$ and $b < 1$)

$$\begin{aligned}
&\frac{1-c}{1-c-\alpha+a\alpha} < b < 1,\\
&1-b < d < \frac{a(1-b)(1-c)^2 + (1-a)\alpha(ab(1-c) - c(1-a))}{(1-a)c(1-c-\alpha+a\alpha)},\\
&\frac{1-c}{1-c-\alpha+a\alpha} < \pi_0 <\\
&\qquad < \frac{2c - cd - ab\alpha + \sqrt{(2c-cd-ab\alpha)^2 + 4c(1-d)(a\alpha - c)}}{c - a\alpha}.
\end{aligned}$$

- Region 5b ($\gamma > 0,\ c > 1$ and $b > 1$)

$$\begin{aligned}
&1 < b < \frac{c}{c-a\alpha},\\
&1-b < d < \frac{2a\alpha - ab\alpha}{c} - 2\frac{\sqrt{a\alpha(a\alpha-c)(1-b)}}{c},\\
&\frac{1-c}{1-c-\alpha+a\alpha} < \pi_0 <\\
&\qquad < \frac{2c - cd - ab\alpha + \sqrt{(2c-cd-ab\alpha)^2 + 4c(1-d)(a\alpha - c)}}{c - a\alpha}.
\end{aligned}$$

Boundaries for $\gamma < 0$

The $u(x)$ hyperbolas corresponding to $\gamma < 0$ look like the ones in Fig.1a and 1b, thus the stable and the unstable fixed points are on different sides of the vertical axes of the hyperbolas. Because of this the $\{\pi_0, b\} < x_u$ or ($\{\pi_0, b\} > x_u$) constraints are never relevant, instead we only have to consider (16a)-(16d) and Lemma 3. Furthermore, the lower (upper) constraint of d for $c < 0$ ($c > 0$) comes from $c(1-b-d) < 0$, and the upper (lower) constraint of d for $c < 0$ ($c > 0$) comes from $\left|\frac{\partial u(x)}{\partial x}\right|_{x=x_s} < 1$. (Recall from Section 3.2 that the x_i fixed point is stable only if $\left|\frac{\partial u(x)}{\partial x}\right|_{x=x_i} < 1$.)

From these considerations we get the following constraints.

- Region 6 ($\gamma < 0,\ c < 0$)

$$\begin{aligned}
\frac{c}{c-a\alpha} &< b < \frac{1-c}{1-c-\alpha+a\alpha},\\
1-b &< d < -\frac{ab\alpha}{c},\\
\frac{c}{c-a\alpha} &< \pi_0 < \frac{1-c}{1-c-\alpha+a\alpha}.
\end{aligned}$$

- Region 7 ($\gamma < 0,\ 0 < c < a$)

$$\begin{aligned} 0 < b &< \frac{1-c}{1-\alpha-c+a\alpha}, \\ -\frac{ab\alpha}{c} < d &< 1-b, \\ 0 < \pi_0 &< \frac{1-c}{1-\alpha-c+a\alpha}. \end{aligned}$$

- Region 8 ($\gamma < 0,\ a < c < 1$)

$$\begin{aligned} -\frac{ab\alpha}{c} < d &< 1-b, \\ 0 < \pi_0 &< \frac{c}{c-a\alpha}, \\ 0 < b &< \frac{c}{c-a\alpha}. \end{aligned}$$

- Region 9 ($\gamma < 0,\ c > 1$)

$$\begin{aligned} -\frac{ab\alpha}{c} < d &< 1-b, \\ \frac{1-c}{1-c-\alpha+a\alpha} < \pi_0 &< \frac{c}{c-a\alpha}, \\ \frac{1-c}{1-c-\alpha+a\alpha} < b &< \frac{c}{c-a\alpha}. \end{aligned}$$