Numerical analysis of a retrial system with unreliable servers based on Laplace transform description

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Abstract

In this paper, we consider a retrial queuing system with unreliable servers and analyse the distribution of the stationary generalized service time which includes also the unavailable periods (setup times) occurring during service of the customer. We consider three service interruptions disciplines: preemptive resume (PRS), preemptive repeat different (PRD) and preemptive repeat identical (PRI); and provide the stationary distribution of the generalized service time and the remaining generalized service time with these disciplines in Laplace transform (LT) or Laplace-Stieltjes transform (LST) domain.

The main focus of the paper is on the numerical analysis based on LST domain descriptions, which we evaluate for various numerical examples.

Keywords: Numerical Inverse Laplace transform, preemptive resume, preemptive repeat different, preemptive repeat identical.

1. Introduction

Recent developments in Numerical Inverse Laplace transform (NILT) provide an efficient tool for the analysis of stochastic models [1, 4]. In this work, we investigate the applicability of the NILT approach by evaluating the generalized service time distribution and its remaining time distribution (also referred to as equilibrium distribution [5, p. 437] [7, p. 432, 469]) of a retrial queuing system with unreliable server.

The system behaves as follows. Incoming customers queues up in an infinite buffer. The single server serves the customers in FIFO order. The server is subject to break down. In case of a server break down, the server gets back to operational after an independent, identically distributed (i.i.d.) setup time. When the server was busy at break down, it continuous the service of the interrupted customer when it gets back to operational according to one of the following three preemption policies: preemptive resume (PRS), preemptive repeat different (PRD) and preemptive repeat identical (PRI). With the PRS policy the server continuous the service of the interrupted customer from the point it was interrupted. With the PRD policy the server drops the interrupted customer and starts the service of the next customer (with i.i.d. service time). With the PRI policy the server restarts the service of the interrupted customer, which means that the service time of the customer in the current operational period of the server is identical to the one of the previous operational period.

We selected this stochastic model, because depending on the applied discipline at server failure the complexity of the Laplace transform (LT) or Laplace-Stieltjes transform (LST) description varies a lot and consequently, the applicability of the NILT analysis raises more and more severe numerical and computational complexity issues.

Other important properties of the considered model are the service time distribution when the server is always on and the down time distribution of the server, which we refer to as setup time distribution. In our numerical experiment we consider a set of distributions with various difficulties in LST domain description.

Various performance measures of this model has been investigated in preceding papers [2, 6]. For example, the LT description of the generalized service time distribution with PRS and PRD policies are available at [2]. But non of the preceding papers considered the NILT based numerical analysis of the generalized service time distribution.

In this paper we extend the LT domain description with the PRI case, but the main focus of the paper is the investigation of the NILT based numerical analysis in case of different preemption policies and service and setup time distributions.

The rest of the paper is organized as follow.....

2. Analysis of the generalized service time distribution

The generalized service time, G, is the time from the instant the server starts the service of a customer until it completes the service of that costumer considering the potential break down and setup cycles of the server and the applied preemption policy.

The CDF, PFD and the LT of the (break down free) service time, S, are denoted by $F(x) = Pr(S < x)$, $f(x) = dF(x)/dx$ and $f^*(s) = E(e^{-sS})$ respectively. Similarly, the CDF, PFD and the LT of the setup time, R, and the generalized service time G are denoted by $R(x)$, $r(x)$, $r^*(s)$ and $G(x)$, $g(x)$, $g^*(s)$, respectively.

In this work, we assume that the server breaks down with constant rate ν. That is, when the server is operational the time to the next break down is exponentially distributed with parameter ν (independent of the time of the last breakdown).

2.1. Preemptive repeat different – PRD.

Theorem 1. [2] In case of PRD preemption policy the LT of the generalized service time is

$$g^*(s) = \frac{(s+\nu)f^*(s+\nu)}{(s+\nu) - \nu(1 - f^*(s+\nu))r^*(s)}.$$

2.2. Preemptive resume – PRS.

Theorem 2. [2] In case of PRS preemption policy the LT of the generalized service time is

$$g^*(s) = f^*(s + \nu - \nu r^*(s)).$$

2.3. Preemptive repeat identical – PRI.

Theorem 3. In case of PRI preemption policy the mean and LT of the generalized service time are

$$E(G) = E(S) - \nu + \Big(E(R) + \frac{1}{\nu}\Big)\Big(f^*(-\nu) - 1\Big) \tag{1}$$

and

$$g^*(s) = \frac{(s+\nu)}{(s+\nu) - \nu r^*(s)} \cdot \sum_{j=0}^{\infty} \left(\frac{-\nu r^*(s)}{(s+\nu) - \nu r^*(s)}\right)^j f^*((j+1)(s+\nu)) \tag{2}$$

The region of convergence for the $f^*(s) = \int_{x=0}^{\infty} e^{-sx} f(x)dx$ integral is always of the form $\{s : Re(s) > a\}$ (possibly including some points of the boundary line $\{Re(s) = a\}$), or empty ($a = \infty$), or the entire complex plane ($a = -\infty$). The real constant a is known as the abscissa of absolute convergence.

Corollary 1. With PRI policy, the mean generalized service time is finite when the abscissa of absolute convergence of the service time distribution is less than $-\nu$. I.e.,

$$f^*(-\nu) = \int_0^{\infty} f(x)e^{x\nu}dx < \infty.$$

2.4. Remaining time distribution. The LT domain description of the remaining time distribution of the generalized service time can be obtained as

$$h^*(s) = \frac{1 - g^*(s)}{s\mu},$$

where μ is the mean of the generalized service time, that is $\mu = -\frac{d}{ds}g^*(s)|_{s=0}$.

3. NILT using Abate-Whitt framework methods

In this work we restrict our attention to the NILT methods belong to the Abate-Whitt framework [1]. According to this framework, the order N approximate of $f(t)$ at point T is obtained based on $f^*(s) = \int_t f(t)e^{-st}dt$ as

$$f(T) \approx f_N(T) := \sum_{n=1}^{N} \frac{\eta_n}{T} f^* \left(\frac{\beta_n}{T} \right), \tag{3}$$

where the coefficients η_n and β_n are determined by the order (N) and the NILT method (e.g., Euler [1], Gaver [3], Talbot [8], CME [4]) and they are independent of the function $f^*(s)$. We assume that the η_n and β_n coefficients are available with negligible computational cost. This way, the computational complexity of computing $f_N(T)$ based on (3) is N times the computational cost of evaluating $f^*(s)$ at potentially complex points.

4. Laplace transform of positive distributions

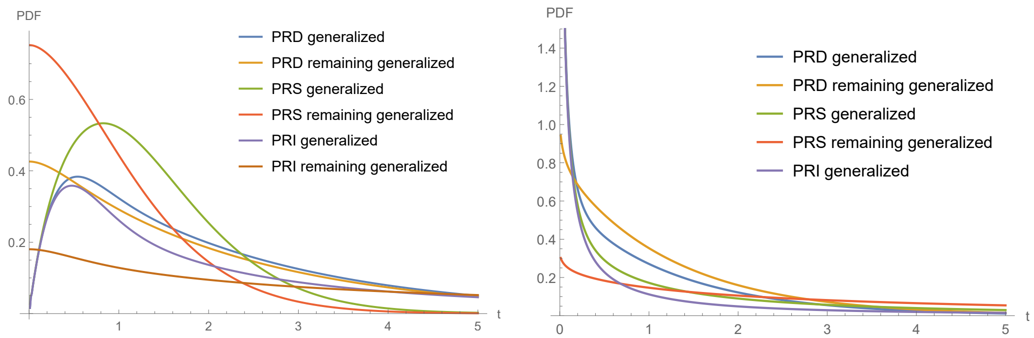
We consider the following list of service time and setup time distributions.

- Weibull distribution with density $f(t) = \alpha\lambda(\lambda t)^{\alpha-1}e^{-(\lambda t)^\alpha}$:
  The complexity of $f^*(s) = \int_t f(t)e^{-st}dt$ depends on α.
  - When α is irrational, $f^*(s)$ does not exhibit a closed form expression.
  - When α is rational, $f^*(s)$ can be described with generalized hypergeometric functions. The complexity of the hypergeometric function depends on α. The following two cases results in the simplest LT expressions
    * heavy tailed case ($\alpha = 1/2$): $f^*(s) = \frac{\sqrt{\pi\lambda}}{2\sqrt{s}} e^{\frac{\lambda}{4s}} Erfc\left(\frac{\sqrt{\lambda/s}}{2}\right)$,
    * light tailed case ($\alpha = 2$): $f^*(s) = 1 - \frac{s\sqrt{\pi}}{2\lambda} e^{\frac{s^2}{4\lambda^2}} Erfc\left(\frac{s}{2\lambda}\right)$.

  In any case, the hypergeometric function and its special case the $Erfc$ function, are integral functions and the computational complexity of the evaluation of these function depends on their implementation.
  With $\alpha < 1$ the abscissa of absolute convergence of the Weibull distributed density is $a = 0$ and with $\alpha > 1$ it is $a = -\infty$.
- Gamma distribution with density $f(t) = \frac{\lambda^\alpha t^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)}$:
  $f^*(s)$ has a simple to compute analytic form

  $$f^*(s) = (1 + s/\lambda)^{-\alpha}.$$

  The abscissa of absolute convergence is $a = -\lambda$. The special case when α is a positive integer gives the Erlang distribution and when $\alpha = 1$ gives the exponential distribution.



light tailed service time distribution — heavy tailed service time distribution

Fig. 1. Density function of the generalized service time distribution and the remaining generalized service time distribution with PRD, PRS and PRI policies, when the service time is light ($\lambda = 1, \alpha = 2$) and heavy ($\lambda = 1, \alpha = 1/2$) tailed Weibull distributed, the setup time is exponentially distributed with parameter 4 and the failure rate of the server is $\nu = 2$.

- Pareto distribution with density $f(t) = \alpha(t+1)^{-(\alpha+1)}$ and support on $(0, \infty)$: $f^*(s)$ can be expressed with the use of the exponential integral function $E_x(s) = \int_1^\infty t^{-x} e^{-st} dt$ as
$$f^*(s) = \alpha e^s E_{\alpha+1}(s).$$
The abscissa of absolute convergence is $a = 0$.
- Lognormal distribution with density $f(t) = \frac{e^{-(\log(t)-2)^2/2}}{\sqrt{2\pi}t}$: $f^*(s)$ has no closed form. $f^*(s) = \int_t f(t)e^{-st}dt$ needs to be evaluated. The abscissa of absolute convergence is $a = 0$.

5. Numerical experiments

Figure 1 plots the density function of the generalized service time distribution with light and heavy tailed Weibull service time distribution. The shape of the service time has a prevalent effect on the shape of the generalized service time distribution, but this effect vanishes in the remaining generalized service time distribution. In the heavy tailed case, $a = 0$ and the mean generalized service time with PRI policy is infinite, that is why we cannot plot the density of the remaining generalized service time, while the density of the generalized service time is still computable. In the light tailed case, $a = -\infty$ and the mean generalized service time with PRI policy is finite. In this case both densities of the PRI policy are available.

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