Numerical analysis of a retrial system with unreliable servers based on Laplace domain description*

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Abstract. In this paper, we consider a retrial queuing system with unreliable servers and analyze the distribution of the stationary generalized service time which includes also the unavailable periods (setup times) occurring during service of the customer. We consider three service interruption disciplines: preemptive resume (PRS), preemptive repeat different (PRD), and preemptive repeat identical (PRI); and provide the stationary distribution of the generalized service time and the remaining generalized service time for these disciplines in Laplace transform (LT) domain.

The main focus of the paper is on the numerical analysis based on LT domain descriptions, which we evaluate for various numerical examples.
Keywords: Numerical Inverse Laplace transform, preemptive resume, preemptive repeat different, preemptive repeat identical.

1 Introduction

Recent developments in Numerical Inverse Laplace transform (NILT) provide an efficient tool for the analysis of stochastic models [1,4]. In this work, we investigate the applicability of the NILT approach by evaluating the generalized service time distribution and its remaining time distribution (also referred to as equilibrium distribution [5, p. 437] [7, p. 432, 469]) of a retrial queuing system with an unreliable server.

The system behaves as follows. Incoming customers queue up in an infinite buffer. The single server serves the customers in FIFO order. The server is subject

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to breakdown. In case of a server breakdown, the server gets back to operational after an independent, identically distributed (i.i.d.) setup time. If the server was busy at breakdown, it continues the service of the interrupted customer when it gets back to operational according to one of the following three preemption policies: preemptive resume (PRS), preemptive repeat different (PRD), and preemptive repeat identical (PRI). With the PRS policy, the server continues the service of the interrupted customer from the point it was interrupted. With the PRD policy, after an interruption the server restarts the service of the interrupted customer with i.i.d. service time. With the PRI policy the server restarts the service of the interrupted customer, and the service time of the customer in the current operational period of the server is identical to the one of the previous operational period.

We selected this stochastic model, because depending on the applied discipline at server failure the complexity of the Laplace transform (LT) description varies a lot, consequently the applicability of the NILT analysis raises more and more severe numerical and computational complexity issues.

Similarly, the probability distributions used for the (non-generalized) service time and the downtime of the server can significantly affect the computational characteristics of the model. In our numerical experiment we consider a set of distributions with differing degrees of complexity in LT domain description.

Various performance measures of this model have been investigated in preceding papers [2,6]. For example, the LT description of the generalized service time distribution with PRS and PRD policies are available in [2]. But none of the preceding papers considered the NILT based numerical analysis of the generalized service time distribution.

In this paper we extend the LT domain description with the PRI case, but the main focus of the paper is the investigation of the NILT based numerical analysis in case of different preemption policies and service and setup time distributions.

2 Analysis of the generalized service time distribution

The generalized service time, G, is the time from the instant the server starts the service of a customer until it completes the service of that costumer considering the potential breakdown and setup cycles of the server and the applied preemption policy.

The CDF, PDF, and the LT of the (breakdown free) service time, S, are denoted by $F(x) = Pr(S < x)$, $f(x) = dF(x)/dx$, and $f^*(s) = E(e^{-sS})$, respectively. Similarly, the CDF, PDF, and the LT of the setup time, R, and the generalized service time G are denoted by $R(x)$, $r(x)$, $r^*(s)$ and $G(x)$, $g(x)$, $g^*(s)$, respectively.

In this work, we assume that the server breaks down with constant rate ν. That is, when the server is operational, the time to the next breakdown is exponentially distributed with parameter ν (independent of the time of the last breakdown), thus the time of breakdown B has PDF $b(x) = \nu e^{-\nu x}$.

2.1 Preemptive repeat different – PRD

Theorem 1. *[2] In case of PRD preemption policy, the LT of the generalized service time is*

$$g^*(s) = \frac{(s+\nu)f^*(s+\nu)}{(s+\nu) - \nu(1-f^*(s+\nu))r^*(s)}. \tag{1}$$

Proof.

$$(G|B=h, S=x) = \begin{cases} x & h > x \\ h+R+G & h < x \end{cases}$$

$$E(e^{-sG}|B=h, S=x) = \begin{cases} e^{-sx} & h > x \\ e^{-sh}\underbrace{E(e^{-sR})}_{r^*(s)}\underbrace{E(e^{-sG})}_{g^*(s)} & h < x \end{cases}$$

$$\begin{aligned}
g^*(s) &= E(e^{-sG}) = \int_h b(h)\int_x f(x)E(e^{-sG}|B=h,S=x)dxdh \\
&= \int_{h=0}^{\infty} b(h)\left(\int_{x=0}^{h} f(x)e^{-sx}dx + \int_{x=h}^{\infty} f(x)e^{-sh}r^*(s)g^*(s)dx\right)dh \\
&= \int_{h=0}^{\infty} \nu e^{-\nu h}\left(\int_{x=0}^{h} f(x)e^{-sx}dx + (1-F(h))e^{-sh}r^*(s)g^*(s)\right)dh \\
&= \int_{x=0}^{\infty} f(x)e^{-sx}\underbrace{\int_{h=x}^{\infty}\nu e^{-\nu h}dh}_{e^{-\nu h}}\,dx + \nu\underbrace{\int_{h=0}^{\infty} e^{-(s+\nu)h}(1-F(h))dh}_{\frac{1-f^*(s+\nu)}{s+\nu}}\,r^*(s)g^*(s)
\end{aligned}$$

That is,

$$g^*(s) = f^*(s+\nu) + \nu\frac{1-f^*(s+\nu)}{s+\nu}r^*(s)g^*(s),$$

from which

$$g^*(s) = \frac{f^*(s+\nu)}{1-\nu\frac{1-f^*(s+\nu)}{s+\nu}r^*(s)} = \frac{(s+\nu)f^*(s+\nu)}{(s+\nu)-\nu(1-f^*(s+\nu))r^*(s)}$$

2.2 Preemptive resume – PRS

Theorem 2. *[2] In case of PRS preemption policy, the LT of the generalized service time is*

$$g^*(s) = f^*(s+\nu-\nu r^*(s)). \tag{2}$$

Proof. The number of interruptions during the service time $S = x$ is N_x. N_x is Poisson($x\nu$) distributed, i.e., $Pr(N_x = i) = \frac{(x\nu)^i}{i!} e^{-x\nu}$

$$(G|S = x, N_x = i) = x + \sum_{j=1}^{i} R_j$$

$$E(e^{-sG}|S = x, N_x = i) = e^{-sx}(r^*(s))^i$$

$$\begin{aligned}
g^*(s) &= \int_x \sum_{i=0}^{\infty} Pr(N_x = i) f(x) E(e^{-sG}|S = x, N_x = i) dx \\
&= \int_x \sum_{i=0}^{\infty} \frac{(x\nu)^i}{i!} e^{-x\nu} f(x) e^{-sx} (r^*(s))^i dx \\
&= \int_x f(x) e^{-(s+\nu)x} \underbrace{\sum_{i=0}^{\infty} \frac{(x\nu r^*(s))^i}{i!}}_{e^{x\nu r^*(s)}} dx \\
&= f^*(s + \nu - \nu r^*(s))
\end{aligned}$$

2.3 Preemptive repeat identical – PRI

Theorem 3. *In case of PRI preemption policy, the mean and the LT of the generalized service time are*

$$E(G) = \Big(E(R) + \frac{1}{\nu}\Big)\Big(f^*(-\nu) - 1\Big) \tag{3}$$

and

$$g^*(s) = \frac{(s+\nu)}{(s+\nu) - \nu r^*(s)} \cdot \sum_{j=0}^{\infty} \left(\frac{-\nu r^*(s)}{(s+\nu) - \nu r^*(s)}\right)^j f^*((j+1)(s+\nu)). \tag{4}$$

Before proving the theorem we need the following lemma.

Lemma 1.

$$\sum_{i=j}^{\infty} \binom{i}{j} a^i = a^j (1-a)^{-j-1}$$

Proof (Lemma 1). Using

$$\frac{d}{da}(1-a)^{-j-1} = (j+1)(1-a)^{-j-2},$$
$$\frac{d^n}{da^n}(1-a)^{-j+1} = (j+1)\dots(j+n)(1-a)^{-j-n-1} = \frac{(j+n)!}{j!}(1-a)^{-j-n-1},$$

the Taylor series of $(1-a)^{-j-1}$ is

$$(1-a)^{-j-1} = \sum_{n=0}^{\infty} \frac{(n+j)!}{n!j!} a^n = \sum_{n=0}^{\infty} \binom{j+n}{n} a^n = \sum_{n=0}^{\infty} \binom{j+n}{j} a^n$$

from which

$$\sum_{i=j}^{\infty} \binom{i}{j} a^i = a^j \sum_{i=j}^{\infty} \binom{i}{j} a^{i-j} = a^j \sum_{n=0}^{\infty} \binom{n+j}{j} a^n = a^j (1-a)^{-j-1}.$$

Proof (Theorem 3). Let N_x be the number of interruptions if the service time is $S = x$. N_x is Geometrical(p) distributed, i.e., $Pr(N_x = i) = p(1-p)^i$, with $p = e^{-x\nu}$ and $E(N_x) = \frac{1-p}{p} = \frac{1-e^{-x\nu}}{e^{-x\nu}}$.

$$(G|S = x, N_x = i) = x + \sum_{j=1}^{i} \left(B_j(x) + R_j\right), \tag{5}$$

where the interruption time, $B(x)$, is truncated exponentially distributed, i.e., for $0 < h < x$

$$Pr(B(x) < h) = \frac{1-e^{-h\nu}}{1-e^{-x\nu}} \quad \text{and} \quad \frac{d}{dh} Pr(B(x) < h) = \frac{\nu e^{-h\nu}}{1-e^{-x\nu}}.$$

Consequently,

$$E(B(x)) = \int_{h=0}^{x} h \frac{\nu e^{-h\nu}}{1-e^{-x\nu}} dh = \frac{1}{\nu} - \frac{x e^{-x\nu}}{1-e^{-x\nu}}$$

and

$$\begin{aligned} i^*(s,x) = E(e^{-sB(x)}) &= \int_{h=0}^{x} e^{-sh} \frac{\nu e^{-h\nu}}{1-e^{-x\nu}} dh = \frac{\nu}{1-e^{-x\nu}} \int_{h=0}^{x} e^{-(s+\nu)h} dh \\ &= \frac{\nu}{1-e^{-x\nu}} \cdot \frac{1-e^{-(s+\nu)x}}{s+\nu} = \frac{\nu(1-e^{-(s+\nu)x})}{(s+\nu)(1-e^{-x\nu})} \end{aligned}$$

From (5) we get

$$E(G|S = x, N_x = i) = x + \Big(E(R) + E(B(x))\Big) i,$$

and

$$E(e^{-sG}|S = x, N_x = i) = e^{-sx} (r^*(s))^i (i^*(s,x))^i.$$

From these we get

$$
\begin{aligned}
E(G) &= \int_x f(x) \sum_{i=0}^{\infty} Pr(N_x = i) E(G|S = x, N_x = i) dx \\
&= \int_x f(x) \sum_{i=0}^{\infty} Pr(N_x = i) \Big(x + \big(E(R) + E(B(x))\big) i \Big) dx \\
&= \int_x f(x) \Big(x + \big(E(R) + E(B(x))\big) E(N_x) \Big) dx \\
&= E(S) + \int_x f(x) \Big(E(R) + \frac{1}{\nu} - \frac{x e^{-x\nu}}{1 - e^{-x\nu}} \Big) \frac{1 - e^{-x\nu}}{e^{-x\nu}} dx
\end{aligned}
$$

$$
\begin{aligned}
E(G) &= E(S) + \int_x f(x) \Big(E(R) + \frac{1}{\nu} - \frac{x e^{-x\nu}}{1 - e^{-x\nu}} \Big) \frac{1 - e^{-x\nu}}{e^{-x\nu}} dx \\
&= E(S) + \Big(E(R) + \frac{1}{\nu} \Big) \int_x f(x) \frac{1 - e^{-x\nu}}{e^{-x\nu}} dx - \underbrace{\int_x f(x) x dx}_{E(S)} \\
&= \Big(E(R) + \frac{1}{\nu} \Big) \int_x f(x) (e^{x\nu} - 1) dx \\
&= \Big(E(R) + \frac{1}{\nu} \Big) \Big(f^*(-\nu) - 1 \Big)
\end{aligned}
$$

and

$$
\begin{aligned}
g^*(s) &= \int_x f(x) \sum_{i=0}^{\infty} Pr(N_x = i) E(e^{-sG}|S = x, N_x = i) dx \\
&= \int_x \sum_{i=0}^{\infty} e^{-x\nu} (1 - e^{-x\nu})^i f(x) e^{-sx} (i^*(s, x))^i (r^*(s))^i dx \\
&= \int_x \sum_{i=0}^{\infty} e^{-x\nu} (1 - e^{-x\nu})^i f(x) e^{-sx} \left(\frac{\nu (1 - e^{-(s+\nu)x})}{(s + \nu)(1 - e^{-x\nu})} \right)^i (r^*(s))^i dx \\
&= \int_x f(x) e^{-(s+\nu)x} \sum_{i=0}^{\infty} \left(\frac{\nu r^*(s)}{s + \nu} \right)^i (1 - e^{-(s+\nu)x})^i dx \\
&= \int_x f(x) e^{-(s+\nu)x} \sum_{i=0}^{\infty} \left(\frac{\nu r^*(s)}{s + \nu} \right)^i \sum_{j=0}^{i} \binom{i}{j} (-e^{-(s+\nu)x})^j dx \\
&= \int_x f(x) e^{-(s+\nu)x} \sum_{i=0}^{\infty} \left(\frac{\nu r^*(s)}{s + \nu} \right)^i \sum_{j=0}^{i} \binom{i}{j} (-1)^j e^{-j(s+\nu)x} dx
\end{aligned}
$$

Further on

$$
\begin{aligned}
g^*(s) &= \int_x f(x) e^{-(s+\nu)x} \sum_{j=0}^{\infty} (-1)^j e^{-j(s+\nu)x} \underbrace{\sum_{i=j}^{\infty} \binom{i}{j} \left(\frac{\nu r^*(s)}{s+\nu}\right)^i} dx \\
&= \int_x f(x) \sum_{j=0}^{\infty} (-1)^j e^{-(j+1)(s+\nu)x} \frac{\left(\frac{\nu r^*(s)}{s+\nu}\right)^j}{\left(1-\frac{\nu r^*(s)}{s+\nu}\right)^{j+1}} dx \\
&= \sum_{j=0}^{\infty} (-1)^j \frac{(s+\nu)\,(\nu r^*(s))^j}{((s+\nu)-\nu r^*(s))^{j+1}} \int_x f(x) e^{-(j+1)(s+\nu)x} dx \\
&= \sum_{j=0}^{\infty} \frac{(s+\nu)\,(-\nu r^*(s))^j}{((s+\nu)-\nu r^*(s))^{j+1}} f^*((j+1)(s+\nu)) \\
&= \frac{(s+\nu)}{(s+\nu)-\nu r^*(s)} \cdot \sum_{j=0}^{\infty} \left(\frac{-\nu r^*(s)}{(s+\nu)-\nu r^*(s)}\right)^j f^*((j+1)(s+\nu)),
\end{aligned}
$$

where we used Lemma 1 to rewrite the expression highlighted by the brace under it.

The infinite summation in (4), provides the complete description of the generalized service time distribution, but in order to compute $g^*(s)$ based on (4), we need to truncate the summation at a given threshold.

The region of convergence for the $f^*(s) = \int_{x=0}^{\infty} e^{-sx} f(x) dx$ integral is always of the form $\{s : Re(s) > a\}$ (possibly including some points of the boundary line $\{Re(s) = a\}$), or empty ($a = \infty$), or the entire complex plane ($a = -\infty$). The real constant a is known as the abscissa of absolute convergence.

Corollary 1. *With PRI policy, the mean generalized service time in* (3) *is finite when the abscissa of absolute convergence of the service time distribution is less than* $-\nu$*, that is,*

$$
f^*(-\nu) = \int_0^{\infty} f(x) e^{x\nu} dx < \infty.
$$

2.4 Remaining time distribution

The LT domain description of the remaining time distribution of the generalized service time when the server is busy, $h^*(s)$, and for an arriving customer, $\hat{h}^*(s)$, are

$$
h^*(s) = \frac{1-g^*(s)}{sE(G)}, \qquad \hat{h}^*(s) = (1-p_{busy}) + p_{busy}\frac{1-g^*(s)}{sE(G)},
$$

where $E(G)$ is the mean of the generalized service time, that is $E(G) = -\frac{d}{ds} g^*(s)|_{s=0}$ and $p_{busy} = \lambda_{arr} E(G)$ is the probability that an arriving customer finds the server busy, with λ_{arr} being the arrival rate of customers.

3 NILT using Abate-Whitt framework methods

In this work we restrict our attention to NILT methods in the Abate-Whitt framework [1]. In this framework, the order N approximate of $f(t)$ at point $t = T$ is obtained based on $f^*(s) = \int_t f(t)e^{-st}dt$ as

$$f(T) \approx f_N(T) := \sum_{n=1}^{N} \frac{\eta_n}{T} f^*\left(\frac{\beta_n}{T}\right), \tag{6}$$

where the coefficients η_n and β_n are determined by the order (N) and the NILT method (e.g., Euler [1], Gaver [3], Talbot [8], CME [4]) and they are independent of the function $f^*(s)$. We assume that the η_n and β_n coefficients are available with negligible computational cost, since, at worst, they can be calculated and stored in advance. This way, the computational complexity of computing $f_N(T)$ based on (6) is approximately N times the computational cost of evaluating $f^*(s)$ at potentially complex points.

4 Laplace transform of positive distributions

We consider the following list of service time and setup time distributions.

- Weibull distribution with density $f(t) = \alpha\lambda(\lambda t)^{\alpha-1}e^{-(\lambda t)^\alpha}$:
  The complexity of $f^*(s) = \int_t f(t)e^{-st}dt$ depends on α.
  - When α is irrational, $f^*(s)$ does not have a closed form expression.
  - When α is rational, $f^*(s)$ can be described with generalized hypergeometric functions. The complexity of the hypergeometric function depends on α. The following two cases result in the simplest LT expressions
    - heavy tailed case ($\alpha = 1/2$): $f^*(s) = \frac{\sqrt{\pi\lambda}}{2\sqrt{s}} e^{\frac{\lambda}{4s}} Erfc\left(\frac{\sqrt{\lambda/s}}{2}\right)$,
    - light tailed case ($\alpha = 2$): $f^*(s) = 1 - \frac{s\sqrt{\pi}}{2\lambda} e^{\frac{s^2}{4\lambda^2}} Erfc\left(\frac{s}{2\lambda}\right)$,

    where $Erfc$ is the complementary error function defined as $Erfc(z) = \frac{2}{\sqrt{\pi}} \int_{t=z}^{\infty} e^{-t^2} dt$. The hypergeometric function and its special case, the $Erfc$ function, are integral functions and the computational complexity of the evaluation of these functions depends on their implementation.
    For $\alpha < 1$ the abscissa of absolute convergence of the Weibull distributed density is $a = 0$ and for $\alpha > 1$ it is $a = -\infty$.
- Gamma distribution with density $f(t) = \frac{\lambda^\alpha t^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)}$:
  $f^*(s)$ has the analytic form

  $$f^*(s) = (1 + s/\lambda)^{-\alpha},$$

  which can be calculated with low computational cost. The abscissa of absolute convergence is $a = -\lambda$. The special case when α is a positive integer gives the Erlang distribution and when $\alpha = 1$ gives the exponential distribution.

- Pareto distribution with density $f(t) = \alpha(t+1)^{-(\alpha+1)}$ and support on $(0, \infty)$: $f^*(s)$ can be expressed with the use of the exponential integral function $E_x(s) = \int_1^\infty t^{-x} e^{-st} dt$ as

$$f^*(s) = \alpha e^s E_{\alpha+1}(s).$$

  The abscissa of absolute convergence is $a = 0$.
- Lognormal distribution with density $f(t) = \frac{1}{\sigma t \sqrt{2\pi}} e^{\frac{-(\log(t)-\mu)^2}{2\sigma^2}}$: $f^*(s)$ has no closed form. $f^*(s) = \int_t f(t) e^{-st} dt$ needs to be evaluated. The abscissa of absolute convergence is $a = 0$.

5 Numerical experiments

The previous sections introduced LT domain distributions with various complexity. We investigate their numerical behaviour in this section. We used the order $N = 60$ CME method [4] in all cases.

5.1 Weibull distributed service time

In the following we consider the case of PRS, PRD, and PRI preemption with light and heavy tailed Weibull distributed service times.

Figure 1 plots the PDF of the generalized service time distribution and the remaining generalized service time distribution for the three preemption policies - calculated from $h^*(s)$ - with light and heavy tailed Weibull service time distribution (defined in Section 4 with $\alpha = 2$ and $1/2$, respectively). The setup time is exponentially distributed with parameter 4 ($r^*(s) = \frac{4}{s+4}$) and the server break down rate is $\nu = 2$. The shape of the service time distribution has a profound effect on the shape of the generalized service time distribution, but this effect vanishes in the remaining generalized service time distribution. In the heavy tailed case, $a = 0$ and the mean generalized service time with PRI policy is infinite, that is why we cannot plot the PDF of the remaining generalized service time, while the PDF of the generalized service time is still computable. In the light tailed case, $a = -\infty$ and the mean generalized service time with PRI policy is finite. In this case both densities of the PRI policy are available.

5.2 Computational complexity

There are two main factors that affect the computational cost of the calculation of the PDF of the generalized service time ($g(t)$) via NILT: the order of the NILT (N) and the complexity of the Laplace domain function of the generalized service time ($g^*(s)$). To obtain the time domain function ($g(t)$) in a single point, $g^*(s)$ has to be evaluated in N points. Assuming that the evaluation of $g^*(s)$ dominates the complexity of NILT and that the complexity of the evaluation of $g^*(s)$ is independent of the point it is evaluated in, the computational complexity of the NILT is N times the computational complexity of evaluating $g^*(s)$. To

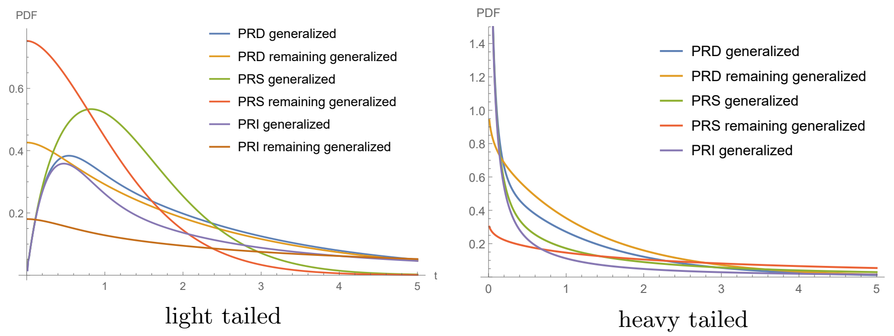


Fig. 1: Density function of the generalized service time distribution and the remaining generalized service time distribution with PRD, PRS and PRI policies, when the service time is light ($\lambda = 1, \alpha = 2$) and heavy ($\lambda = 1, \alpha = 1/2$) tailed Weibull distributed, the setup time is exponentially distributed with parameter 4 and the failure rate of the server is $\nu = 2$.

evaluate $g^*(s)$ in one point, in the PRD and the PRS case we have to evaluate $r^*(s)$ in one point and $f^*(s)$ in one point (Theorem 1 and 2), in the PRI case, if we truncate the infinite sum at $j = k - 1$, then we have to evaluate $r^*(s)$ in one point, and $f^*(s)$ in k points (Theorem 3). In the following we only discuss the computational cost of the evaluation of a Laplace transform PDF in a single point (e.g., the evaluation of $f^*(s)$ or $r^*(s)$), from which the cost of NILT of $g^*(s)$ can be easily calculated. E.g., according to (1), 60 $f^*(s)$ evaluations and 60 $r^*(s)$ evaluations are needed to obtain $g(t)$ in a single point in the PRD case with $N = 60$.

The computational complexity of the evaluation of a Laplace transform PDF (say $f^*(s)$) is a nuanced question, a detailed discussion of which is out of the scope of this paper. To give a practical perspective, we investigated the computational time of the evaluation of such PDFs using Wolfram Mathematica for the distributions listed in Section 4. For these we measured the average evaluation time of $f^*(s)$ using 100 random complex s values. The evaluation times in seconds can be seen in Table 1. Unlike other distributions, the lognormal distribution does not have a closed form Laplace transform, therefore in this case numerical integration is needed, which is considered as part of the evaluation of $f^*(s)$.

In accordance with the expectations, the Laplace transform of the gamma distribution can be evaluated extremely fast. The Laplace transforms of the other distributions do not have closed forms, but both the hypergeometric function (whose special case is the $Erfc$ function) in the Laplace transform of the Weibull distribution and the exponential integral function in the Laplace transform of the Pareto distribution can be calculated efficiently, in general. More precisely, the order of the hyper geometric function in case of the Weibull distribution depends on $\max\{a, b\}$ for the rational $\alpha = a/b$, where a and b are relative primes.

For large a or b (integer) parameters the hypergeometric function is of high order and the evaluation of $f^*(s)$ can become quite complex, which explains the high computational cost for $\alpha = 11/100$ in Table 1. Finally, the Laplace transform of the lognormal distribution requires numerical integration, thus the related computational time is higher than most other cases.

Comparing the result for Weibull distribution with $\alpha = 11/100$ and lognormal distribution (based on numerical integration) suggests, that the computational complexity of Weibull PDF with $\alpha = 11/100$ using high order hyper geometric function could be larger than the evaluation of the numerical integral according to $f^*(s) = \int_t f(t)e^{-st}dt$.

Table 1: Evaluation time of $f^*(s)$ in a single point for different distributions

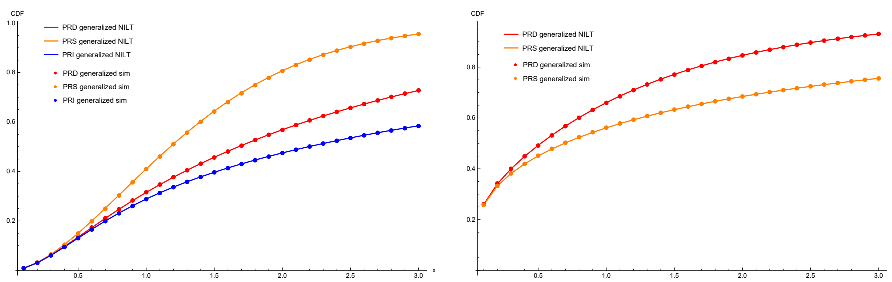
	Weibull			Gamma	Pareto	Lognormal
	$\alpha = 1/2$	$\alpha = 2$	$\alpha = 11/100$	$\alpha = 5/2, \lambda = 1$	$\alpha = 2$	$\alpha = 2, \lambda = 1$
time	$7.34 \cdot 10^{-4}$	$1.56 \cdot 10^{-4}$	$6.40 \cdot 10^{-1}$	$< 10^{-6}$	$1.56 \cdot 10^{-5}$	$4.68 \cdot 10^{-3}$

5.3 Accuracy of the NILT results

The distribution of the service time and generalized service time cannot be calculated analytically for the more complex functions of this paper. Therefore, to verify the accuracy of NILT, we implemented the models with PRD, PRS, and PRI preemption using a discrete event simulator. The model specific features of the applied simulation tool are as follows:

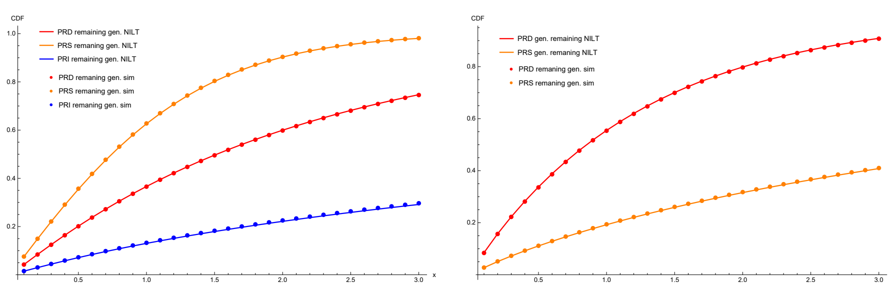
- Based on the the PASTA property, the *remaining generalized service time* is measured at independent Poisson instants that arrive at a constant rate.
- Utilizing that the stationary distributions of the *elapsed time* and the *remaining time* are identical, the simulation collects statistics on the *elapsed time* for implementation convenience.
- In case of the heavy-tailed Weibull distributed service time with PRI policy, the mean generalized service time has an infinite mean, which requires a special simulation approach of the generalized service time, which utilizes the fact that we are interested in the CDF until a known upper bound. Consequently, the simulation follows the life of customers only until their system time reaches the upper bound.

We ran simulations for PRD, PRS, and PRI preemption using Weibull distributed service time distribution and exponential setup and server break down distributions using the same parameters as in Section 5.1. We compared the CDF of the generalized service times as well as the remaining generalized service times obtained using simulation and the NILT of the corresponding formulas. We obtained the empirical CDF (ECDF) curves as the average of 200 simulation runs for each interruption mode with 1000 served customer in each run and also calculated their 95% confidence intervals. The results are presented in Figure 2 and Figure 3. To approximate the infinite sum for the PRI preemption, we used the first 21 terms ($j_{max} = 20$). The figures show that the simulation and the NILT give almost identical results. Because the confidence intervals are highly tight, we did not plot them as they would not be informative. We state, however, that the mean length of the intervals vary from 0.003 to 0.004, with the CDF obtained using NILT always lying within the interval bounds. These results verify that the NILT based approach is safely applicable for the complex functions discussed in this paper.



(a) light tailed ($\alpha = 2$) (b) heavy tailed

Fig. 2: NILT and simulation of generalized service time, for light tailed Weibull distributed service time



(a) light tailed ($\alpha = 2$) (b) heavy tailed

Fig. 3: NILT and simulation of remaining generalized service time, for light tailed Weibull distributed service time

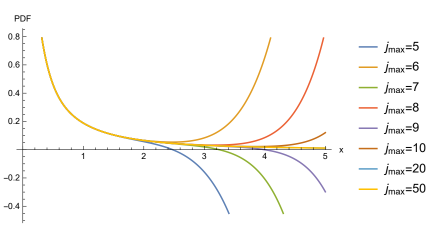
5.4 Truncation of the infinite summation in the PRI case

According to Theorem 3, the LT of the generalized service time with PRI policy is obtained as a result of the infinite summation in (4). In practice, we approximate the LT as

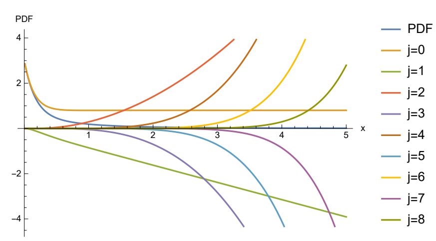
$$g^*_{j_{max}}(s) = \frac{(s+\nu)}{(s+\nu)-\nu r^*(s)} \cdot \sum_{j=0}^{j_{max}} \left(\frac{-\nu r^*(s)}{(s+\nu)-\nu r^*(s)} \right)^j f^*((j+1)(s+\nu)), \quad (7)$$

i.e, we truncate the infinite sum at $j = j_{max}$.

Figure 4a and 5a demonstrate the behaviour of the obtained finite approximation of the PDF and the CDF of the generalized service time for various values of j_{max}. These show that the initial part of the PDF and CDF can be approximated well using lower j_{max}, but higher j_{max} is needed for a good approximation of their tail. It is hard to determine an exact threshold when the error of the PDF and the CDF become significant (e.g., higher than a predefined ϵ value) for a given j_{max}. For the PDF the only certain threshold is when the approximation becomes negative. For the CDF, we have two such thresholds: one when the approximation of the CDF starts decreasing (which is identical with negative PDF) and the other when it becomes larger than 1. Figure 5a indicates that the CDF satisfies these two error criteria for larger and larger intervals with increasing j_{max}. It is also visible that for odd j_{max} the CDF violates the first error criterion, while for even j_{max} it violates the second one. This odd-even behaviour already suggests that the odd and even terms of the summations in (7) has different sign and Figure 4b and 5b verify this behaviour.



(a) Result of truncation at different j_{max} values

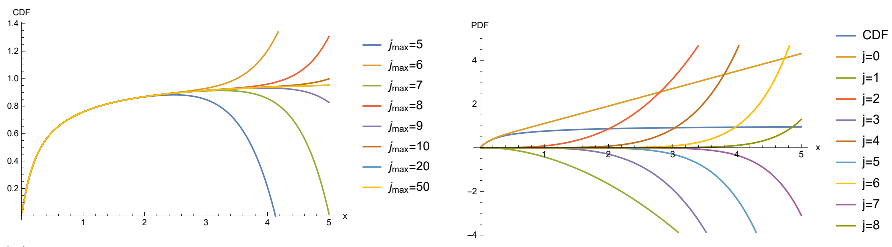


(b) NILT of different terms

Fig. 4: Density function of the generalized service time distribution with PRI policy, when the service time and the setup time are exponentially distributed with parameter 3 and 4 and the failure rate of the server is $\nu = 2$.

6 Conclusion

In this work, we investigated the applicability of numerical inverse Laplace transformation based analysis of complex queueing systems, where various distribu-



(a) Result of truncation at different j_{max} values

(b) NILT of different terms

Fig. 5: Cumulative density function of the generalized service time distribution with the same settings as in Figure 4.

tions and preemption policies characterize the complexity of the Laplace transform expression of the measure of interest.

Based on a wide set numerical experimentation, we conclude, that the NILT is generally stable, and the computational complexity of the analysis comes from the evaluation of the LT transform function. There are positive distributions, often considered in qeueuing models (e.g. the lognormal distribution) for which no closed form transform domain description is available, but the Laplace domain expression can be evaluate via numerical integration.

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