Moments based characterization of MAPs with reduced rank marginal

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ABSTRACT

The moments based characterization of MAPs with full rank marginal (FRM-MAPs) is provided in [11]. MAPs with reduced rank marginal (RRM-MAPs) differ in essential properties from FRM-MAPs [10].

In this work we propose a general procedure for moments based characterization of MAPs which is applicable for both FRM-MAPs and RRM-MAPs, independent of their internal structure. We also show that the procedure terminates in a finite number of steps which is proportional to the order of the MAP.

Keywords: MAP, moments based characterization, MAP with reduced rank marginal.

1. INTRODUCTION

Markovian arrival processes (MAPs) are efficiently used to model point processes with dependent interarrival times. Since their introduction in [9], the properties of MAPs have been studied in many papers and got summarized in recent textbooks [6, 8].

Due to their flexibility and their ability of capturing correlation between consecutive interarrival times, MAPs are also used for approximating real traffic measurements. There are two main approaches to approximate traffic measurements with a MAP, fitting and matching [3, 7]. Fitting intends to optimize a function which represents the difference between the measurement and the MAP. In many cases the likelihood of the measurement is the function to maximize [2, 1]. The other approach, parameter matching, extracts a set of traffic parameters from the measurement and composes a MAP which exhibits exactly the same traffic parameters. When the moments of the interarrival time and the joint moments of consecutive interarrival times are available (which is a redundant set of information), we propose a general procedure to compute a representation of the MAP, independent of its internal structure.

2. MARKOVIAN ARRIVAL PROCESS

A MAP generates arrivals according to an N-state background continuous time Markov chain (CTMC) with generator \(Q\). While the CTMC is in state \(i\), the MAP generates arrivals according to a Poisson process with rate \(\lambda_i\). When the CTMC moves from state \(i\) to \(j\), an arrival occurs with probability \(p_{ij}\). MAPs are most commonly defined by a pair of matrices \(D_0, D_1\), which describe the behaviour of the process without and with arrivals and are obtained from \(Q\), \(\lambda_i (i=1, ..., N)\) and \(p_{ij} (i, j=1, ..., N, i \neq j)\) as:

\[
D_{0ij} = \begin{cases} Q_{ij}(1-p_{ij}) & \text{if } i \neq j, \\ Q_{ii} - \lambda_i & \text{if } i = j, \end{cases} \\
D_{1ij} = \begin{cases} Q_{ij}p_{ij} & \text{if } i \neq j, \\ \lambda_i & \text{if } i = j. \end{cases}
\]

The \((D_0, D_1)\) matrix representation of a MAP is not unique [4]. Infinitely many similar matrix representations describe the same MAP. E.g., if \(B\) is non-singular such that \(B1 = I\), where \(I\) is the column vector of ones of appropriate size, then \((D_0, D_1)\) and \((B^{-1}D_0B, B^{-1}D_1B)\) are different representations of the same MAP.

Let the \(n\)th interarrival time of the MAP with representation \((D_0, D_1)\) be \(X_n\), \(E = (-D_0)^{-1}\) and \(P = ED_1\), where the solution of \(\pi P = \pi, P I = 1\) is unique (i.e., we assume that the modulating CTMC is irreducible). The reduced

\[E \leq 0, B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}
\]

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The important of this decomposition comes from the fact that (3)

\[ \text{rank} \left( H \left( \gamma \right) \right) \leq \text{rank} \left( \begin{bmatrix} u^A_1 & \ldots & u^A_{\nu} \\ \vdots & \ddots & \vdots \\ u^A_{\nu} & \ldots & u^A_{\nu} \end{bmatrix} \right) \cdot \text{rank} \left( \begin{bmatrix} A^0 & A^1 & \ldots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots \end{bmatrix} \right) \].

Assuming that \( A \) does not contain a zero eigenvalue (which can be eliminated by size reduction \( \mathbb{R} \)), the rank of the matrices are determined by the zero entries of \( u \) and \( v \). For example, if \( A \) is diagonal, then the rank of the first matrix on the rhs is the number of non-zero elements of \( u \). Since the elements of the Jordan decomposition might be non-positive, we distinguish structural and random zeros in vector \( u \) and \( v \). Structural zeros are the result of the zero – non-zero structure of the Jordan decomposition, while random zeros are the result of matrix multiplications involving non-zero elements. Hereafter we focus on the structural zeros of the representations which are associated with the structural properties of the MAP.
Lemma 1. If the size of the MAP with Jordan representation \((\delta, \Lambda, \tilde{P}, h)\) is \(n\), then for any \(j \in \{1, 2, \ldots, n\}\), there is a \(k \geq 0\) and a set of parameters \(\{i_0, a_1, \ldots, i_k, a_{k+1}\}\) for which the \(j\)th element of \(\tilde{u}\) is not a structural zero.

Proof. We make the indirect assumption that there is a \(j \in \{1, 2, \ldots, n\}\), such that the \(j\)th element of \(\tilde{u}\) is a structural zero for all \(k\) and all \(i, a_1, \ldots, i_k, a_{k+1}\) sets of parameters. Then the \(j\)th column of matrix \(A\) in the decomposition is zero and thus the rank of \(\Lambda\) and, consequently, the rank of \(H(\gamma)\) is less than \(n\), which is in conflict with Theorem 4.

Theorem 2. For any order \(n\) MAP with minimal extended Jordan representation \((\delta, \Lambda, \tilde{P}, h)\), the vector \(\delta(\Lambda + \tilde{P})^{n-1}\) has no structural zeros.

Proof. Let \(z(b)\) denote the set of indices of the structural zero elements of vector \(b\). From the definition of structural zeros, for any pair of \(\hat{b}\) and \(\hat{b}\) vectors we have \(z(\hat{b} + \hat{b}) = z(\hat{b}) \cup z(\hat{b})\). Since the diagonal elements of \(\Lambda\) are non-zero, \(z(b\Lambda) \subseteq z(b)\), consequently

\[
z(b(\Lambda + \tilde{P})) = z(b\Lambda) \cup z(b\tilde{P}) \subseteq z(b\Lambda) \subseteq z(b).
\] (5)

Therefore,

\[
z(\delta(\tilde{P} + \tilde{P})) = z(\delta\tilde{P} + \delta\tilde{P}) = \cdots = z(\delta\tilde{P}^{k+1} + \cdots) \subseteq z(\delta\tilde{P}^{k+1} + \cdots) \subseteq z(\delta) \subseteq z(\delta(\Lambda + \tilde{P})) \subseteq \cdots \subseteq z(\delta(\Lambda + \tilde{P}))^{n-1} = \{\}.
\] (6)

From (6), there exists \(c \in \mathcal{N}\), such that \(z(\delta(\Lambda + \tilde{P})) = \{\}\). Let us denote the smallest such \(c\) by \(c_{\min}\).

For any \(b\) and matrix \(K\), if \(z(b) = z(bK)\), then \(z(b) = z(bK)\), \(\forall i \in \mathcal{N}\) as well. Using (6), this means that \(z(\delta(\Lambda + \tilde{P})) \subseteq \cdots \subseteq z(\delta(\Lambda + \tilde{P}))^{c_{\min}} = \{\} \subseteq \{\}.
= \{\}
\] (7)

That is, the number of structural zeros in \(\delta\) decreases by 1 or more after every multiplication by \((\Lambda + \tilde{P})\) until it has none. Since \(\delta\) must have at least one non-zero element to satisfy Theorem 1, this means that \(c_{\min} \leq n - 1\), i.e., \(\delta(\Lambda + \tilde{P})^{n-1}\) has no structural zeros, which is what we had to prove.

Theorem 3. For any order \(n\) MAP with minimal extended Jordan representation \((\delta, \Lambda, \tilde{P}, h)\), the vector \((\Lambda + \tilde{P})^{n-1}h\) has no structural zeros.

Proof. The proof follows the same arguments as the proof of Theorem 2.

4. THE PROPOSED MOMENT BASED CHARACTERIZATION METHOD

The pseudocode of the proposed procedure can be found in Algorithm 1. The inputs of the algorithm are the \(\gamma_{n, \ldots, i_k}\) moments, but the algorithm works with complex sums of those moments. To avoid expressing those complex sums, we describe them using the elements of the extended Jordan representation. For example we write \(\delta(\tilde{P} + \Lambda)^{h}\) instead of \(1 + 2r_1 + r_2\) since \(\delta(\tilde{P} + \Lambda)^{h} = \delta(\tilde{P}^2 + \Lambda^2 + 2\tilde{P}A + \Lambda^2 + \tilde{P})(h)\) instead of \(1 + 2r_1 + r_2\).

In the following, we provide an intuitive explanation of Algorithm 1. By substituting \(\bar{u}(k) = \delta(\tilde{P} + \tilde{P})^{h}\) and \(\bar{v}(l) = (\tilde{P} + l)^{h}\) into (2), we can see that the Hankel matrix \(H(\mu)\) is full rank, if \(\bar{u}(k)\) and \(\bar{v}(l)\) have no structural or random zeros. From Theorem 2 and 3 we know that \(\bar{u}(n - 1)\) and \(\bar{v}(n - 1)\) have no structural zeros. In practice, random zeros can be removed by an additional multiplication by \((\tilde{P} + \Lambda)\), therefore, \(\exists k, l : k + l \leq 2n\), such that rank\((\tilde{H}(\mu)) = (\tilde{H}(\mu) = n)\). Algorithm 1 searches for such \(k, l\) while trying to minimize \(c = k + l\). If such \(k, l\) are found, we can generate a matrix exponential distribution with representation \((\alpha, \tilde{D}_0)\) such that \(a_{E}^tI = \mu / i, \forall i = 1, \ldots, 2n - 1, \bar{E} = -\tilde{D}_0^{-1}\) using the same method as in p.10. of [10], indicated by MEFromMoments in line 8 of the algorithm. The MEFromMoments method returns a size \(\tilde{H}(\mu)\) \((\alpha, \tilde{D}_0)\) representation, which is why we need to ensure that \(\tilde{H}(\mu) = n\). Then, from

\[
a_{E}^tI = \frac{1}{\mu_0} \delta(\tilde{P} + \Lambda)^{k} \Lambda^t(\tilde{P} + \Lambda)^{l}h, \forall i.
\] (8)

From (3), we have that \(E\) and \(\Lambda\) are similar and \(\Lambda\) is a Jordan matrix. If \(E = \Gamma^{-1} \Lambda \Gamma\) is the Jordan decomposition of \(E\), then \(c_1 \Gamma^{-1} = \delta(\tilde{P} + \Lambda)^{k}\) and \(c_2 \Gamma^{-1} = \delta(\tilde{P} + \Lambda)^{l}\) such that \(c_1 \cdot c_2 = \mu_0\). Consequently,

\[
\frac{1}{\mu_0} \delta(\tilde{P} + \Lambda)^{k} \Lambda^t(\tilde{P} + \Lambda)^{l}h = a_{E}^t \tilde{P} \bar{E}^t I, \forall i, j,
\]

such that \(\tilde{P}\) is similar to \(\tilde{P}\) with transformation matrix \(\Gamma\), i.e., \(\tilde{P} = \Gamma^{-1} \tilde{P} \Gamma\). Therefore, lines 11-13 can be used to obtain \(\tilde{P}\) with transformation matrix \(\Gamma\), where \(R_{\mu, i}\) is the \(i\)th column of \(L\) and \(R_{\mu, i}\) is \(i\)th row of \(R\). Finally, we can calculate \(\tilde{D}_1\) from \(\tilde{D}_0\) and \(\tilde{P}\) to obtain a \((\tilde{D}_0, \tilde{D}_1)\) representation of the MAP.

5. NUMERICAL EXAMPLE

In this section we apply the proposed method to the numerical example from Section 8 of [10], which is based on an ETAQA output process approximation of a MAP/MAP/1 queue [12].
Algorithm 1 Moment based characterization

\begin{algorithm}
\textbf{Input:} $\gamma_{i_0,\ldots,i_k}$ \forall$k, i_0, \ldots, i_k, a_1, \ldots, a_k$
\begin{algorithmic}
\State 1: for $e = 0, \ldots, 2n$
\State 2: \hspace{1em} for $k = 0, \ldots, e$
\State 3: \hspace{2em} $l \leftarrow e - k$
\State 4: \hspace{2em} $\mu_0 \leftarrow \delta(\hat{P} + \Lambda)^k(\hat{P} + \Lambda)^d$
\State 5: \hspace{2em} $\mu' \leftarrow l(\hat{P} + \Lambda)^{k}\Lambda(\hat{P} + \Lambda)^d$, $i = 1, \ldots, 2n - 1$
\State 6: \hspace{2em} $\mu \leftarrow \frac{1}{\rho_0} \delta(\hat{P} + \Lambda)^k\Lambda(\hat{P} + \Lambda)^d$, $i, j = 1, \ldots, n$
\State 7: \hspace{2em} if $\text{HO}(\mu) == n$ then
\State 8: \hspace{3em} $\alpha, \bar{D}_0 \leftarrow \text{MEFromMoments}(\mu)$
\State 9: \hspace{3em} $\bar{E} \leftarrow -\bar{D}_0^{-1}$
\State 10: \hspace{3em} $M : M_{i,j} \leftarrow \frac{1}{\rho_0} \delta(\hat{P} + \Lambda)^k\Lambda(\hat{P} + \Lambda)^d$, $i, j = 1, \ldots, n$
\State 11: \hspace{3em} $L : L_{i,j} \leftarrow \alpha \bar{E}^i$, $i = 1, \ldots, n$
\State 12: \hspace{3em} $R : R_{i,j} \leftarrow \bar{E}^{i} L$, $i = 1, \ldots, n$
\State 13: \hspace{3em} $\bar{P} \leftarrow L^{-1} MR^{-1}$
\State 14: \hspace{3em} $\bar{D}_1 \leftarrow -\bar{D}_0 \bar{P}$
\State 15: \hspace{3em} return $\bar{D}_0, \bar{D}_1$
\State 16: \hspace{3em} end if
\State 17: \hspace{1em} end for
\State 18: \hspace{1em} end for
\end{algorithmic}
\end{algorithm}

The size if the above MAP representation is 6, the Hankel order of its reduced marginal moments series $\text{HO}(\{r_1, \ldots, r_n\})$ is 3, while the order of the MAP is 5. The Hankel order $\text{HO}(\mu)$ is 3, for $k = 0, l = 0$ and $k = 1, l = 0$, but it is 5, for $k = 0, l = 2$. We note here that the determinant the corresponding Hankel matrix is very small ($\sim 10^{-24}$), therefore high-precision arithmetic is needed for its calculation. After the MEFromMoments procedure we get

$$\alpha = [0.2, 0.2, 0.2, 0.2, 0.2].$$

$$\bar{D}_0 = \begin{bmatrix}
-120.91 & 137.64 & -797.51 & 20849 & -20069 \\
-83.864 & 94.8 & -548.70 & 14347 & -13810 \\
27.694 & -31.873 & 179.0 & -4686.8 & 4511.2 \\
27.668 & -31.863 & 184.44 & -4838.6 & 4657.6 \\
27.668 & -31.863 & 184.46 & -4821.8 & 4640.8 
\end{bmatrix}.$$ 

Executing the rest of the algorithm we get

$$\bar{D}_1 = \begin{bmatrix}
13.698 & -15.84 & 336.31 & -5182.1 & 4850.6 \\
9.0133 & -10.239 & 232.76 & -4057.9 & 3828.6 \\
-2.8398 & 3.8063 & -78.824 & 1337.7 & -1259.2 \\
-2.6852 & 3.6136 & -76.537 & 1360.6 & -1284.4 \\
-2.6889 & 3.6182 & -76.601 & 1360.5 & -1284.2 
\end{bmatrix}.$$ 

Calculating the respective moments verifies that $(\bar{D}_0, \bar{D}_1)$ is indeed a valid (non-Markovian) representation of the original $(\hat{D}_0, \hat{D}_1)$ MAP.

6. REFERENCES


