Moments based characterization of MAPs with reduced rank marginal*

András Mészáros
Department of Networked Systems and Services,
Technical University of Budapest,
Budapest, Hungary
meszarosa@hit.bme.hu

Miklós Telek
Department of Networked Systems and Services,
Technical University of Budapest
ELKH-BME Information Systems Research Group
Budapest, Hungary
telek@hit.bme.hu

ABSTRACT

The moments based characterization of MAPs with full rank marginal (FRM-MAPs) is provided in [11]. MAPs with reduced rank marginal (RRM-MAPs) differ in essential properties from FRM-MAPs [10].

In this work we propose a general procedure for moments based characterization of MAPs which is applicable for both FRM-MAPs and RRM-MAPs, independent of their internal structure. We also show that the procedure terminates in a finite number of steps which is proportional to the order of the MAP.

Keywords: MAP, moments based characterization, MAP with reduced rank marginal.

1. INTRODUCTION

Markovian arrival processes (MAPs) are efficiently used to model point processes with dependent interarrival times. Since their introduction in [9], the properties of MAPs have been studied in many papers and got summarized in recent textbooks [6, 8].

Due to their flexibility and their ability of capturing correlation between consecutive interarrival times, MAPs are also used for approximating real traffic measurements. There are two main approaches to approximate traffic measurements with a MAP, fitting and matching [3, 7]. Fitting intends to optimize a function which represent the difference between the measurement and the MAP. In many cases the likelihood of the measurement is the function to maximize [2, 1]. The other approach, parameter matching, extracts a set of traffic parameters from the measurement and composes a MAP which exhibits exactly the same traffic parameters. When the moments of the interarrival time and the joint moments of consecutive interarrival times are the set of parameters to match, and this set contains enough parameters to uniquely characterize the MAP, the procedure is referred to as moments based characterization.

*This work is partially supported by the OTKA K-138208 project.

MAM11 2022, Seoul, Republic of Korea

For the subset of MAPs with full rank marginal (FRM-MAPs), the moments based characterization is provided in [11]. MAPs with reduced rank marginal (RRM-MAPs) differ in essential properties from FRM-MAPs [10]. For example, an order n FRM-MAP is characterized by n^2 parameters, whereas an order n RRM-MAP is characterized by less than n^2 parameters and the exact number of parameters depends on the internal structure of the MAP. The internal structure of a MAP determines, for example, if the rank degradation of the marginal distribution is due to an observability or controllability reason, or both [4]. For low order MAPs ($n = 2, 3$), various internal structures and related characterizing moments set are provided in [10]. As a consequence, different moments based characterization procedures need to be applied for RRM-MAPs depending on the internal structure of the MAP, and the internal structure needs to be known for selecting the appropriate procedure.

Assuming that all required moments and joint moments of the interarrival times are available (which is a redundant set of information), we propose a general procedure to compute a representation of the MAP, independent of its internal structure.

2. MARKOVIAN ARRIVAL PROCESS

A MAP generates arrivals according to an N-state background continuous time Markov chain (CTMC) with generator $\mathbf{Q}$. While the CTMC is in state i, the MAP generates arrivals according to a Poisson process with rate λ_i. When the CTMC moves from state i to j, an arrival occurs with probability p_{ij}. MAPs are most commonly defined by a pair of matrices $\mathbf{D_0}, \mathbf{D_1}$, which describe the behaviour of the process without and with arrivals and are obtained from $\mathbf{Q}$, λ_i ($i = 1, \ldots, N$) and p_{ij} ($i, j = 1, \ldots, N, i \neq j$) as:

$$\mathbf{D_0}_{ij} = \begin{cases} \mathbf{Q}_{ij}(1 - p_{ij}) & \text{if } i \neq j, \\ \mathbf{Q}_{ii} - \lambda_i & \text{if } i = j, \end{cases} \quad \mathbf{D_1}_{ij} = \begin{cases} \mathbf{Q}_{ij} p_{ij} & \text{if } i \neq j, \\ \lambda_i & \text{if } i = j. \end{cases}$$

The $(\mathbf{D_0}, \mathbf{D_1})$ matrix representation of a MAP is not unique [4]. Infinitely many similar matrix representations describe the same MAP. E.g., if $\mathbf{B}$ is non-singular such that $\mathbf{B}\mathbb{1} = \mathbb{1}$, where $\mathbb{1}$ is the column vector of ones of appropriate size, then $(\mathbf{D_0}, \mathbf{D_1})$ and $(\mathbf{B}^{-1}\mathbf{D_0}\mathbf{B}, \mathbf{B}^{-1}\mathbf{D_1}\mathbf{B})$ are different representations of the same MAP.

Let the nth interarrival time of the MAP with representation $(\mathbf{D_0}, \mathbf{D_1})$ be X_n, $\mathbf{E} = (-\mathbf{D_0})^{-1}$ and $\mathbf{P} = \mathbf{E}\mathbf{D_1}$, where the solution of $\pi\mathbf{P} = \pi, \pi\mathbb{1} = 1$ is unique (i.e., we assume that the modulating CTMC is irreducible). The reduced

$k+1$ tuple joint moment of the $s_0 = 0 < s_1 < \ldots < s_k$th inter-arrival times is

$$\gamma^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_k} = \frac{E(X_0^{i_0} X_{s_1}^{i_1} \ldots X_{s_k}^{i_k})}{i_0! \ldots i_k!} = \pi \mathbf{E}^{i_0} \mathbf{P}^{a_1} \mathbf{E}^{i_1} \mathbf{P}^{a_2} \ldots \mathbf{P}^{a_k} \mathbf{E}^{i_k} \mathbb{1}, \quad (1)$$

where $a_i = s_i - s_{i-1}$ and equivalently, $s_i = \sum_{j=1}^{i} a_j$. The reduced marginal moments are obtained from $\gamma^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_k}$ at $k = 0$

$$r_i = \frac{E(X_0^i)}{i!} = \pi \mathbf{E}^i \mathbb{1}.$$

Our aim is to obtain a matrix representation of a MAP based on its $\gamma^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_k}$ moments (including the marginal ones).

3. MOMENT BASED CHARACTERIZATION OF MAPS

Hankel matrices help to obtain structural information on a MAP based on its moments.

Definition 1. *The matrix composed by the elements of the series* $\{z_0, z_1, z_2, \ldots\}$ *as*

$$\mathbf{H}(\{z_0, z_1, z_2, \ldots\}) = \begin{bmatrix} z_0 & z_1 & z_2 & \cdots \\ z_1 & z_2 & z_3 & \cdots \\ z_2 & z_3 & z_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is referred to as Hankel matrix.

Definition 2. *The Hankel order of the series* $\{z_0, z_1, \ldots\}$, *denoted as* $\mathrm{HO}(\{z_0, z_1, \ldots\})$, *is the rank of the Hankel matrix* $\mathbf{H}(\{z_0, z_1, \ldots\})$. *I.e.,* $\mathrm{HO}(\{z_0, z_1, \ldots\}) = \mathrm{rank}(\mathbf{H}(\{z_0, z_1, \ldots\}))$.

From the $\gamma^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_k}$ moments, we compose moments series as follows

$$\underline{\gamma}^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_k}(\ell) = \{\gamma^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_{\ell-1},0,i_{\ell+1},\ldots,i_k}, \gamma^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_{\ell-1},1,i_{\ell+1},\ldots,i_k}, \gamma^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_{\ell-1},2,i_{\ell+1},\ldots,i_k}, \ldots\} \quad (2)$$

Theorem 1 ([10] - Corollary 2). *The order of a MAP is the maximum of the Hankel orders of all moments series.*

The Hankel order of the marginal distribution of a MAP is $\mathrm{HO}(\{r_0, r_1, \ldots\})$.

Definition 3 ([5]). *A MAP has a reduced rank marginal if its order is larger than the Hankel order of its marginal. Otherwise the MAP has full rank marginal.*[1]

An order n FRM-MAP is fully characterised by the first $2n-1$ marginal moments, $r_i, i = 1, 2, \ldots, 2n-1$, and the first $(n-1)^2$ joint moments of two consecutive inter-arrivals $\gamma^{(1)}_{ij}, i, j = 1, 2, \ldots, n-1$ [11], and a procedure is provided in the paper for obtaining a matrix representation of the MAP based on those moments. Unfortunately, this property does not hold for RRM-MAPs, thus the corresponding procedure cannot be used for them either [10]. In the following we describe an alternate moment characterization, which can be used for both RRM-MAPS and for FRM-MAPs. In the rest of this section we discuss how one can find a $\underline{\gamma}^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_k}(\ell)$ moments series for any order n MAP, for which $\mathrm{HO}(\underline{\gamma}^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_k}(\ell)) = n$. In Section 4 we will use this result to provide a method that can generate a matrix representation of the MAP using only its moments.

Let us consider a MAP with matrix representation $(\hat{\mathbf{D}}_0, \hat{\mathbf{D}}_1)$ and embedded stationary probability vector $\hat{\pi}$ (which is the normalized solution of $-\hat{\pi}\hat{\mathbf{D}}_0^{-1}\hat{\mathbf{D}}_1 = \hat{\pi}$). Using a non-singular matrix $\mathbf{B}$ we can transform this representation to

$$\delta' = \hat{\pi}\mathbf{B}^{-1}, \ \mathbf{E} = \mathbf{B}(-\hat{\mathbf{D}}_0)^{-1}\mathbf{B}^{-1},$$
$$\mathbf{P} = \mathbf{B}(-\hat{\mathbf{D}}_0)^{-1}\hat{\mathbf{D}}_1\mathbf{B}^{-1}, \ h' = \mathbf{B}\mathbb{1}.$$

We refer to the four-tuple $(\delta', \mathbf{E}, \mathbf{P}, h')$ as extended representation of the MAP.

Definition 4. *Let* $(\delta', \mathbf{E}, \mathbf{P}, h')$ *be an extended representation of a MAP and* $\mathbf{E} = \mathbf{\Gamma}^{-1}\mathbf{\Lambda}\mathbf{\Gamma}$ *the Jordan decomposition of* $\mathbf{E}$. *Applying a similarity transformation with matrix* $\mathbf{\Gamma}$ *on* $(\delta', \mathbf{E}, \mathbf{P}, h')$ *we obtain the extended Jordan representation of the MAP*

$$(\delta'\mathbf{\Gamma}^{-1}, \mathbf{\Gamma}\mathbf{E}\mathbf{\Gamma}^{-1}, \mathbf{\Gamma}\mathbf{P}\mathbf{\Gamma}^{-1}, \mathbf{\Gamma}h') = (\delta, \mathbf{\Lambda}, \hat{\mathbf{P}}, h).$$

Based on the Jordan representation, the moments, defined in (1), can be calculated as

$$\gamma^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_k} = \delta \mathbf{\Lambda}^{i_0} \hat{\mathbf{P}}^{a_1} \mathbf{\Lambda}^{i_1} \ldots \hat{\mathbf{P}}^{a_k} \mathbf{\Lambda}^{\mathbf{i_k}} h.$$

Let $\underline{u} = \delta \mathbf{\Lambda}^{i_0} \hat{\mathbf{P}}^{a_1} \ldots \mathbf{\Lambda}^{i_{\ell-1}} \hat{\mathbf{P}}^{a_\ell}$ and $\underline{v} = \hat{\mathbf{P}}^{a_{\ell+1}} \ldots \mathbf{\Lambda}^{i_{\ell+1}} \ldots \hat{\mathbf{P}}^{a_k} \mathbf{\Lambda}^{i_k} h$. The Hankel matrix of the moments series $\underline{\gamma} = \underline{\gamma}^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_k}(\ell)$ (for notational simplification) can be decomposed as

$$\mathbf{H}\left(\underline{\gamma}\right) = \begin{bmatrix} \underline{u}\mathbf{\Lambda}^0 \\ \underline{u}\mathbf{\Lambda}^1 \\ \cdots \end{bmatrix} \cdot \begin{bmatrix} \mathbf{\Lambda}^0\underline{v} & \mathbf{\Lambda}^1\underline{v} & \ldots \end{bmatrix}. \quad (3)$$

The importance of this decomposition comes from the fact that

$$\mathrm{rank}\left(\mathbf{H}\left(\underline{\gamma}^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_k}(\ell)\right)\right) \le \mathrm{rank}\left(\begin{bmatrix} \underline{u}\mathbf{\Lambda}^0 \\ \underline{u}\mathbf{\Lambda}^1 \\ \cdots \end{bmatrix}\right) \cdot \mathrm{rank}\left(\begin{bmatrix} \mathbf{\Lambda}^0\underline{v} & \mathbf{\Lambda}^1\underline{v} & \ldots \end{bmatrix}\right). \quad (4)$$

Assuming that $\mathbf{\Lambda}$ does not contain a zero eigenvalue (which can be eliminated by size reduction [4]), the rank of the matrices are determined by the zero entries of $\underline{u}$ and $\underline{v}$. For example, if $\mathbf{\Lambda}$ is diagonal, then the rank of the first matrix on the rhs is the number of non-zero elements of $\underline{u}$. Since the elements of the Jordan decomposition might be non-positive, we distinguish structural and random zeros in vector $\underline{u}$ and $\underline{v}$ [10]. Structural zeros are the result of the zero – non-zero structure of the Jordan decomposition, while random zeros are the result of matrix multiplications involving non-zero elements. Hereafter we focus on the structural zeros of the representations which are associated with the structural properties of the MAP.

[1] A different terminology was applied in [10], RRM-MAP was referred to as redundant MAP and FRM-MAP as non-redundant MAP.

LEMMA 1. *If the size of the MAP with Jordan representation $(\delta, \mathbf{\Lambda}, \hat{\mathbf{P}}, h)$ is n, then for any $j \in \{1, 2, \ldots, n\}$, there is a $k \geq 0$ and a set of parameters $\{i_0, a_1, \ldots, i_k, a_{k+1}\}$ for which the jth element of $\underline{u}$ is not a structural zero.*

PROOF. We make the indirect assumption that there is a $j \in \{1, 2, \ldots, n\}$, such that the jth element of $\underline{u}$ is a structural zero for all k and all $i_1, a_1, \ldots, i_k, a_{k+1}$ sets of parameters. Then the jth column of matrix $\mathbf{A}$ in the

$$\mathbf{H}\left(\underline{\gamma}\right) = \underbrace{\begin{bmatrix} \underline{u}\mathbf{\Lambda}^0 \\ \underline{u}\mathbf{\Lambda}^1 \\ \ldots \end{bmatrix}}_{\mathbf{A}} \cdot \left[\mathbf{\Lambda}^0 \underline{v} \quad \mathbf{\Lambda}^1 \underline{v} \quad \ldots\right]$$

decomposition is zero and thus the rank of $\mathbf{A}$ and, consequently, the rank of $\mathbf{H}(\underline{\gamma})$ is less than n, which is in conflict with Theorem 1. $\square$

THEOREM 2. *For any order n MAP with minimal extended Jordan representation $(\delta, \mathbf{\Lambda}, \hat{\mathbf{P}}, h)$, the vector $\delta(\mathbf{\Lambda} + \hat{\mathbf{P}})^{n-1}$ has no structural zeros.*

PROOF. Let $z(b)$ denote the set of indices of the structural zero elements of vector b. From the definition of structural zeros, for any pair of $\hat{b}$ and $\check{b}$ vectors we have $z(\hat{b} + \check{b}) = z(\hat{b}) \cap z(\check{b})$. Since the diagonal elements of $\mathbf{\Lambda}$ are non-zero, $z(b\mathbf{\Lambda}) \subseteq z(b)$, consequently

$$z(b(\mathbf{\Lambda} + \hat{\mathbf{P}})) = z(b\mathbf{\Lambda}) \cap z(b\hat{\mathbf{P}}) \subseteq z(b\mathbf{\Lambda}) \subseteq z(b). \tag{5}$$

Therefore,

$$\begin{aligned} & z(\delta(\mathbf{\Lambda} + \hat{\mathbf{P}})^{i_0 + a_1 + \ldots + i_k + a_{k+1}}) \\ & = z(\delta \mathbf{\Lambda}^{i_1} \hat{\mathbf{P}}^{a_1} \ldots \mathbf{\Lambda}^{i_k} \hat{\mathbf{P}}^{a_{k+1}} + \ldots) \\ & \subseteq z(\delta \mathbf{\Lambda}^{i_1} \hat{\mathbf{P}}^{a_1} \ldots \mathbf{\Lambda}^{i_k} \hat{\mathbf{P}}^{a_{k+1}}) \end{aligned}$$

For $j \in \{1, 2, \ldots, n\}$ let $k(j)$ and $\{i_0(j), a_1(j), \ldots, i_k(j), a_{k+1}(j)\}$ be the set of parameters which makes the jth element of $\underline{u}$ to be structurally different from zero. Let $k = \max\{k(1), \ldots, k(n)\}$, $i_0 = \max\{i_0(1), \ldots, i_0(n)\}$, $a_1 = \max\{a_1(1), \ldots, a_1(n)\}$, $\ldots$, $a_{k+1} = \max\{a_{k+1}(1), \ldots, a_{k+1}(n)\}$, then according to (5)

$$z(\delta(\mathbf{\Lambda} + \hat{\mathbf{P}})^{i_0 + a_1 + \ldots + i_k + a_{k+1}}) = \{\}. \tag{6}$$

From (6), there exists $c \in \mathcal{N}$, such that $z(\delta(\mathbf{\Lambda} + \hat{\mathbf{P}})^c) = \{\}$. Let us denote the smallest such c by c_{min}.

From the definition of structural zeros it also follows, that for any vector b and matrix $\mathbf{K}$, if $z(b) = z(b\mathbf{K})$, then $z(b) = z(b\mathbf{K}^i), \forall i \in \mathcal{N}$ as well. Using (5), this means that

$$z(\delta) \subset z(\delta(\mathbf{\Lambda} + \hat{\mathbf{P}})) \subset \ldots \subset z(\delta(\mathbf{\Lambda} + \hat{\mathbf{P}})^{c_{min}}) = \{\}. \tag{7}$$

That is, the number of structural zeros in δ decreases by 1 or more after every multiplication by $(\mathbf{\Lambda} + \hat{\mathbf{P}})$ until it has none. Since δ must have at least one non-zero element to satisfy Theorem 1, this means that $c_{min} \leq n - 1$, i.e., $\delta(\mathbf{\Lambda} + \hat{\mathbf{P}})^{n-1}$ has no structural zeros, which is what we had to prove. $\square$

THEOREM 3. *For any order n MAP with minimal extended Jordan representation $(\delta, \mathbf{\Lambda}, \hat{\mathbf{P}}, h)$, the vector $(\mathbf{\Lambda} + \hat{\mathbf{P}})^{n-1}h$ has no structural zeros.*

PROOF. The proof follows the same arguments as the proof of Theorem 2. $\square$

4. THE PROPOSED MOMENT BASED CHARACTERIZATION METHOD

The pseudocode of the proposed procedure can be found in Algorithm 1. The inputs of the algorithm are the $\gamma_{i_0,\ldots,i_k}^{(a_1,\ldots,a_k)}$ moments, but the algorithm works with complex sums of those moments. To avoid expressing those complex sums, we describe them using the elements of the extended Jordan representation. For example we write $\delta(\hat{\mathbf{P}} + \mathbf{\Lambda})^2 h$ instead of $1 + 2r_1 + r_2$ since $\delta(\hat{\mathbf{P}} + \mathbf{\Lambda})^2 h = \delta(\hat{\mathbf{P}}^2 + \mathbf{\Lambda}^2 + \hat{\mathbf{P}}\mathbf{\Lambda} + \mathbf{\Lambda}\hat{\mathbf{P}})h = 1 + 2r_1 + r_2$.

In the following, we provide an intuitive explanation of Algorithm 1. By substituting $\underline{u}(k) = \delta(\mathbf{\Lambda} + \hat{\mathbf{P}})^k$ and $\underline{v}(l) = (\mathbf{\Lambda} + \hat{\mathbf{P}})^l h$ into (4) we can see that the Hankel matrix $\mathbf{H}(\underline{\mu})$ is full rank, if $\underline{u}(k)$ and $\underline{v}(l)$ have no structural or random zeros. From Theorem 2 and 3 we know that $\underline{u}(n-1)$ and $\underline{v}(n-1)$ have no structural zeros. In practice, random zeros can be removed by an additional multiplication by $(\hat{\mathbf{P}} + \mathbf{\Lambda})$, therefore $\exists k, l : k + l \leq 2n$, such that $\text{rank}(\mathbf{H}(\underline{\mu})) = HO(\underline{\mu}) = n$. Algorithm 1 searches for such k, l while trying to minimize $e = k + l$. If such k, l are found, we can generate a matrix exponential distribution with representation $(\alpha, \bar{\mathbf{D}}_{\mathbf{0}})$ such that $\alpha \bar{\mathbf{E}}^i \mathbb{1} = \underline{\mu}_i / i!, \forall i = 1, \ldots, 2n-1$, $\bar{\mathbf{E}} = -\bar{\mathbf{D}}_{\mathbf{0}}^{-1}$ using the same method as in p.10. of [10], indicated by MEFromMoments in line 8 of the algorithm. The MEFromMoments method returns a size $HO(\underline{\mu})$ $(\alpha, \bar{\mathbf{D}}_{\mathbf{0}})$ representation, which is why we need to ensure that $HO(\underline{\mu}) = n$. Then, from

$$\alpha \bar{\mathbf{E}}^i \mathbb{1} = \frac{1}{\mu_0} \delta(\hat{\mathbf{P}} + \mathbf{\Lambda})^k \mathbf{\Lambda}^i (\hat{\mathbf{P}} + \mathbf{\Lambda})^l h, \forall i. \tag{8}$$

From (8), we have that $\bar{\mathbf{E}}$ and $\mathbf{\Lambda}$ are similar and $\mathbf{\Lambda}$ is a Jordan matrix. If $\bar{\mathbf{E}} = \mathbf{\Gamma}^{-1} \mathbf{\Lambda} \mathbf{\Gamma}$ is the Jordan decomposition of $\bar{\mathbf{E}}$, then $c_1 \alpha \mathbf{\Gamma}^{-1} = \delta(\mathbf{P} + \mathbf{\Lambda})^k$ and $c_2 \mathbf{\Gamma}^{-1} \mathbb{1} = \delta(\mathbf{P} + \mathbf{\Lambda})^l h$ such that $c_1 \cdot c_2 = \mu_0$. Consequently,

$$\frac{1}{\mu_0} \delta(\hat{\mathbf{P}} + \mathbf{\Lambda})^k \mathbf{\Lambda}^i \hat{\mathbf{P}} \mathbf{\Lambda}^j (\hat{\mathbf{P}} + \mathbf{\Lambda})^l h = \alpha \bar{\mathbf{E}}^i \bar{\mathbf{P}} \bar{\mathbf{E}}^j \mathbb{1}, \text{for} \forall i, j,$$

such that $\bar{\mathbf{P}}$ is similar to $\hat{\mathbf{P}}$ with transformation matrix Γ, i.e., $\bar{\mathbf{P}} = \mathbf{\Gamma}^{-1} \hat{\mathbf{P}} \mathbf{\Gamma}$. Therefore, lines 11-13 can be used to obtain $\bar{\mathbf{P}}$, where $\mathbf{R}_{-,i}$ is the ith column of $\mathbf{L}$ and $\mathbf{R}_{-,i}$ is ith row of $\mathbf{R}$. Finally, we can calculate $\bar{\mathbf{D}}_{\mathbf{1}}$ from $\bar{\mathbf{D}}_{\mathbf{0}}$ and $\bar{\mathbf{P}}$ to obtain a $(\bar{\mathbf{D}}_{\mathbf{0}}, \bar{\mathbf{D}}_{\mathbf{1}})$ representation of the MAP.

5. NUMERICAL EXAMPLE

In this section we apply the proposed method to the numerical example from Section 8 of [10], which is based on an ETAQA output process approximation of a MAP/MAP/1 queue [12].

$$\mathbf{D_0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -17 & 3 & 1 & 0 \\ 0 & 0 & 0 & -6 & 0 & 1 \\ 0 & 0 & 0 & 0 & -16 & 3 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{pmatrix},$$

$$\mathbf{D_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 6.5 & 6.5 & 0 & 0 & 0 & 0 \\ 2.5 & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 6 & 0.5 & 0.5 \\ 0 & 0 & 2 & 2 & 0.5 & 0.5 \end{pmatrix}.$$

Algorithm 1 Moment based characterization

Input: $\gamma^{(a_1,\dots,a_k)}_{i_0,\dots,i_k}\ \forall k, i_0,\dots,i_k, a_1,\dots,a_k$

for $e = 0,\dots,2n$ **do**
  for $k = 0,\dots,e$ **do**
    $l \leftarrow e - k$
    $\mu_0 \leftarrow \delta(\hat{\mathbf{P}}+\mathbf{\Lambda})^k(\hat{\mathbf{P}}+\mathbf{\Lambda})^l h$
    $\underline{\mu}' : \mu'_i \leftarrow i!\delta(\hat{\mathbf{P}}+\mathbf{\Lambda})^k\mathbf{\Lambda}^i(\hat{\mathbf{P}}+\mathbf{\Lambda})^l h,\ i = 1,\dots,2n-1$
    $\underline{\mu} \leftarrow \frac{1}{\mu_0}\underline{\mu}'$
    if $HO(\underline{\mu}) == n$ **then**
      $\alpha, \bar{\mathbf{D}}_\mathbf{0} \leftarrow \text{MEFromMoments}(\underline{\mu})$
      $\bar{\mathbf{E}} \leftarrow -\bar{\mathbf{D}}_\mathbf{0}^{-1}$
      $\mathbf{M} : \mathbf{M}_{i,j} \leftarrow \frac{1}{\mu_0}\delta(\hat{\mathbf{P}}+\mathbf{\Lambda})^k\mathbf{\Lambda}^i\hat{\mathbf{P}}\mathbf{\Lambda}^j(\hat{\mathbf{P}}+\mathbf{\Lambda})^l h,\ i,j = 1,\dots,n$
      $\mathbf{L} : \mathbf{L}_{i,\_} \leftarrow \alpha\bar{\mathbf{E}}^i, i = 1,\dots,n$
      $\mathbf{R} : \mathbf{R}_{\_,i} \leftarrow \bar{\mathbf{E}}^i\mathbb{1}, i = 1,\dots,n$
      $\bar{\mathbf{P}} \leftarrow \mathbf{L}^{-1}\mathbf{M}\mathbf{R}^{-1}$
      $\bar{\mathbf{D}}_\mathbf{1} \leftarrow -\bar{\mathbf{D}}_\mathbf{0}\bar{\mathbf{P}}$
      return $\bar{\mathbf{D}}_\mathbf{0}, \bar{\mathbf{D}}_\mathbf{1}$
    end if
  end for
end for

The size if the above MAP representation is 6, the Hankel order of its reduced marginal moments series $\text{HO}(\{r_1,\dots,r_n\})$ is 3, while the order of the MAP is 5. The Hankel order $\text{HO}(\mu)$ is 3, for $k = 0, l = 0$ and $k = 1, l = 0$, but it is 5, for $k = 0, l = 2$. We note here that the determinant the corresponding Hankel matrix is very small ($\sim 10^{-24}$), therefore high-precision arithmetic is needed for its calculation. After the MEFromMoments procedure we get

$$\alpha = [0.2, 0.2, 0.2, 0.2, 0.2],$$

$$\bar{\mathbf{D}}_\mathbf{0} = \begin{pmatrix} -120.91 & 137.64 & -797.51 & 20849 & -20069 \\ -83.864 & 94.8 & -548.70 & 14347 & -13810 \\ 27.694 & -31.873 & 179.0 & -4686.8 & 4511.2 \\ 27.686 & -31.863 & 184.44 & -4838.6 & 4657.6 \\ 27.686 & -31.863 & 184.46 & -4821.8 & 4640.8 \end{pmatrix}.$$

Executing the rest of the algorithm we get

$$\bar{\mathbf{D}}_\mathbf{1} = \begin{pmatrix} 13.698 & -15.84 & 336.31 & -5182.1 & 4850.6 \\ 9.0133 & -10.239 & 232.76 & -4057.9 & 3828.6 \\ -2.8398 & 3.8063 & -78.824 & 1337.7 & -1259.2 \\ -2.6852 & 3.6136 & -76.537 & 1360.6 & -1284.4 \\ -2.6889 & 3.6182 & -76.601 & 1360.5 & -1284.2 \end{pmatrix}.$$

Calculating the respective moments verifies that $(\mathbf{D}_\mathbf{0}, \mathbf{D}_\mathbf{1})$ is indeed a valid (non-Markovian) representation of the original $(\mathbf{D}_\mathbf{0}, \mathbf{D}_\mathbf{1})$ MAP.

6. REFERENCES

[1] M. Bražėnas, G. Horváth, and M. Telek. Parallel algorithms for fitting Markov arrival processes. *Performance Evaluation*, 123-124:50 – 67, 2018.

[2] P. Buchholz. An EM-algorithm for MAP fitting from real traffic data. In *Computer Performance Evaluation. Modelling Techniques and Tools*, pages 218–236. Springer, 2003.

[3] P. Buchholz, J. Kriege, and I. Felko. *Input Modeling with Phase-Type Distributions and Markov Models - Theory and Applications*. Springer, 2014.

[4] P. Buchholz and M. Telek. On minimal representation of rational arrival processes. *Annals of Operations Research*, 202(1):35–58, 2013.

[5] S. Hautphenne and M. Telek. Extension of some MAP results to transient MAPs and Markovian binary trees. *Performance Evaluation*, 70(9):607 – 622, 2013. Advances in Matrix-Analytic Methodology - Modeling, Estimation and Numerical Analysis.

[6] Q.-M. He. *Fundamentals of Matrix-Analytic Methods*. Springer, 2014.

[7] J. Kriege and P. Buchholz. An empirical comparison of MAP fitting algorithms. In *15th International GI/ITG Conference, MMB&DFT 2010, Essen, Germany*, pages 259–273. Springer, 2010.

[8] L. Lakatos, L. Szeidl, and M. Telek. *Introduction to Queueing Systems with Telecommunication Applications, 2nd edition*. Springer, 2019.

[9] M. F. Neuts. A versatile Markovian point process. *Journal of Applied Probability*, 16(4):764–779, 1979.

[10] M. Telek. Some structural properties of Markov and rational arrival processes. *Stochastic Models*, 27:723–746, 2011.

[11] M. Telek and G. Horváth. A minimal representation of Markov arrival processes and a moments matching method. *Performance Evaluation*, 64(9-12):1153–1168, 2007.

[12] Q. Zhang, A. Heindl, and E. Smirni. Characterizing the BMAP/MAP/1 departure process via the ETAQA truncation. *Stochastic Models*, 21(2-3):821–846, 2005.