Moments based matrix representation of Markov and rational arrival processes with reduced rank marginal*

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Abstract

The moments based matrix representation of Markovian and rational arrival processes (MAP/RAPs) with full rank marginal (FRM) is provided in [14]. MAP/RAPs with reduced rank marginal (RRM) differ in essential properties from the ones with FRM [13]. The main difficulty of the moments based matrix representation of MAP/RAPs with RRM comes from the fact that the moments needed to characterize a MAP/RAPs with RRM depends on the internal structure of the MAP/RAP.

In this work, we propose a general procedure for moments based matrix representation that is applicable to MAP/RAPs with both FRM and RRM, independent of their internal structures. We also show that the procedure terminates in a finite number of steps which is proportional to the order of the MAP/RAP.

Keywords: Markovian arrival processes, rational arrival processes, moments based matrix representation, reduced rank marginal.

1 Introduction

Markovian and rational arrival processes (MAP/RAPs) are efficiently used to model point processes with dependent interarrival times. Since their intro-

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duction in [12, 2], the properties of MAP/RAPs have been studied in many papers and got summarized in recent textbooks [8, 11].

Due to their flexibility and their ability to capture correlation between consecutive interarrival times, MAP/RAPs are also used for approximating real traffic measurements. There are two main approaches to approximate traffic measurements with a MAP/RAP: fitting and matching [5, 10]. Fitting intends to optimize a function which represents the difference between the measurement and the MAP/RAP. In many cases, the likelihood of the measurement is the function to maximize [4, 3]. The other approach, parameter matching, extracts a set of traffic parameters from the measurement and composes a MAP/RAP which exhibits exactly the same traffic parameters. When the moments of the interarrival time and the joint moments of consecutive interarrival times (for brevity they are referred to as moments hereafter) are the set of parameters to match, and this set contains enough parameters to uniquely characterize the MAP/RAP, the procedure is referred to as moments based matrix representation of MAP/RAP. It can also be seen as a transformation from the moments set to the matrix representation of the MAP/RAP, as both of them carry all information about the MAP/RAP.

For the subset of MAP/RAPs with full rank marginal (FRM), the moments based matrix representation is provided in [14]. MAP/RAPs with reduced rank marginal (RRM) differ in essential properties from MAP/RAPs with FRM [13]. For example, an order n MAP/RAP with FRM is characterized by n^2 parameters, whereas an order n MAP/RAP with RRM is characterized by less than n^2 parameters and the exact number of parameters depends on the internal structure of the MAP/RAP. The internal structure of a MAP/RAP determines, for example, if the rank degradation of the marginal distribution is due to an observability or controllability reason, or both [6]. For low order MAP/RAPs ($n = 2, 3$), various internal structures and the related characterizing moments sets are provided in [13]. A shortcoming of these results is that different moments based matrix representation procedures need to be applied for MAP/RAPs with RRM depending on the internal structure of the MAP/RAP, and the internal structure needs to be known for selecting the appropriate procedure.

In this work, we propose a general procedure to compute a matrix representation of the MAP/RAP, independent of its internal structure. The procedure does not intend to be optimal with respect to the number of moments used for generating the matrix representation, since this is possible only for a priori known internal structures (as it is in [13]).

2 Markovian and Rational Arrival Process

A MAP of size N generates arrivals according to an N-state background continuous time Markov chain (CTMC). Maps are usually described by a pair of matrices $(\mathbf{D_0}, \mathbf{D_1})$, where $\mathbf{D_0}$ describes the evolution of the modulating Markov chain without an arrival and $\mathbf{D_1}$ describes it when arrival occurs [8, 11].

RAPs of size N are also characterized by a pair of matrices $(\mathbf{D_0}, \mathbf{D_1})$, but they lack the stochastic interpretation of the background process [1, 13]. MAPs of size N is a proper subset of RAPs of size N for $N > 2$.

In this paper, we consider stationary MAP/RAPs, for which all inter-arrival times have the same distribution. That is, the phase distribution (without stochastic interpretation, i.e., with potentially negative elements) at arrival instances can be computed from $\mathbf{D_0}$ and $\mathbf{D_1}$, as the solution of the linear system $\pi(-\mathbf{D_0})^{-1}\mathbf{D_1} = \pi, \pi\mathbb{1} = 1$. Furthermore, we assume that the solution of this linear system is unique.

The $(\mathbf{D_0}, \mathbf{D_1})$ matrix representation of a MAP/RAP is not unique [6]. Infinitely many similar matrix representations describe the same MAP/RAP. E.g., if $\mathbf{B}$ is non-singular such that $\mathbf{B}\mathbb{1} = \mathbb{1}$, where $\mathbb{1}$ is the column vector of ones of appropriate size, then $(\mathbf{D_0}, \mathbf{D_1})$ and $(\mathbf{B}^{-1}\mathbf{D_0}\mathbf{B}, \mathbf{B}^{-1}\mathbf{D_1}\mathbf{B})$ are different representations of the same MAP/RAP with the same closing vector $\mathbb{1}$.

Definition 1. *A* $(\mathbf{D_0}, \mathbf{D_1})$ *matrix representation is Markovian, if* $\mathbf{D_1}$ *is non-negative and the off diagonal elements of* $\mathbf{D_0}$ *are non-negative.*

A RAP of size N with a Markovian $(\mathbf{D_0}, \mathbf{D_1})$ representation is a MAP of size N; and consequently, a RAP of size N which is not a MAP of size N, does not have a Markovian representation of size N. In general, there are infinitely many Markovian and infinitely many non-Makrkovian matrix representations of a MAP.

Let the nth interarrival time of the MAP/RAP with representation $(\mathbf{D_0}, \mathbf{D_1})$ be X_n, $\mathbf{E} = (-\mathbf{D_0})^{-1}$ and $\mathbf{P} = \mathbf{E}\mathbf{D_1}$. The reduced $k+1$-tuple joint moment of the $s_0 = 0 < s_1 < \ldots < s_k$th interarrival times is

$$\gamma_{i_0,\ldots,i_k}^{(a_1,\ldots,a_k)} = \frac{E(X_0^{i_0} X_{s_1}^{i_1} \ldots X_{s_k}^{i_k})}{i_0! \ldots i_k!} = \pi\mathbf{E}^{i_0}\mathbf{P}^{a_1}\mathbf{E}^{i_1}\mathbf{P}^{a_2} \ldots \mathbf{P}^{a_k}\mathbf{E}^{i_k}\mathbb{1}, \qquad (1)$$

where $a_i = s_i - s_{i-1}$ and equivalently, $s_i = \sum_{j=1}^{i} a_j$. The reduced marginal moments are obtained from $\gamma_{i_0,\ldots,i_k}^{(a_1,\ldots,a_k)}$ at $k = 0$, that is,

$$r_i = \frac{E(X_0^i)}{i!} = \pi\mathbf{E}^i\mathbb{1}, \qquad (2)$$

where *reduced* refers to the fact that the ith moment is divided by $i!$. In the following, for the sake of simplicity we refer to reduced moments and reduced joint moments, as moments and joint moments, respectively, or simply moments for brevity. Our aim is to obtain a matrix representation of a MAP/RAP based on its $\gamma_{i_0,\ldots,i_k}^{(a_1,\ldots,a_k)}$ moments (including the marginal ones).

2.1 Properties and representations of MAP/RAPs

Hankel matrices help to obtain structural information on a MAP/RAP based on its moments.

Definition 2. *The matrix composed by the elements of the series* $\{z_0, z_1, z_2, \ldots\}$ *as*

$$\mathbf{H}(\{z_0, z_1, z_2, \ldots\}) = \begin{bmatrix} z_0 & z_1 & z_2 & \cdots \\ z_1 & z_2 & z_3 & \cdots \\ z_2 & z_3 & z_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is referred to as Hankel matrix.

Definition 3. *The Hankel order of the series* $\{z_0, z_1, \ldots\}$*, denoted as* $\mathrm{HO}(\{z_0, z_1, \ldots\})$*, is the rank of the Hankel matrix* $\mathbf{H}(\{z_0, z_1, \ldots\})$*. I.e.,* $\mathrm{HO}(\{z_0, z_1, \ldots\}) = \mathrm{rank}(\mathbf{H}(\{z_0, z_1, \ldots\}))$.

From the $\gamma_{i_0,\ldots,i_k}^{(a_1,\ldots,a_k)}$ moments, we compose moments series as follows

$$\begin{aligned}
&\underline{\gamma}_{i_0,\ldots,i_{\ell-1},i_{\ell+1},\ldots,i_k}^{(a_1,\ldots,a_k)}(\ell) \\
&= \left\{ \frac{E(X_0^{i_0} \ldots X_{s_\ell}^{0} \ldots X_{s_k}^{i_k})}{i_0! \ldots 0! \ldots i_k!}, \frac{E(X_0^{i_0} \ldots X_{s_\ell}^{1} \ldots X_{s_k}^{i_k})}{i_0! \ldots 1! \ldots i_k!}, \frac{E(X_0^{i_0} \ldots X_{s_\ell}^{2} \ldots X_{s_k}^{i_k})}{i_0! \ldots 2! \ldots i_k!}, \ldots \right\} \\
&= \left\{ \gamma_{i_0,\ldots,i_{\ell-1},0,i_{\ell+1},\ldots,i_k}^{(a_1,\ldots,a_k)}, \gamma_{i_0,\ldots,i_{\ell-1},1,i_{\ell+1},\ldots,i_k}^{(a_1,\ldots,a_k)}, \gamma_{i_0,\ldots,i_{\ell-1},2,i_{\ell+1},\ldots,i_k}^{(a_1,\ldots,a_k)}, \ldots \right\}, \qquad (3)
\end{aligned}$$

that is, the power of X_{s_ℓ} in the numerator and the related factorial in the denominator runs from 0 to infinity.

Theorem 1 ([13] - Corollary 2)**.** *The order of a MAP/RAP is the maximum of the Hankel orders of all moments series.*

The Hankel order of the marginal distribution of a MAP/RAP is $\mathrm{HO}(\{r_0, r_1, \ldots\})$.

Definition 4 ([7]). *A MAP/RAP has a reduced rank marginal if its order is larger than the Hankel order of its marginal moments series. Otherwise the MAP/RAP has full rank marginal.*[1]

As mentioned above, the $(\mathbf{D_0}, \mathbf{D_1})$ matrix representation is not unique. For a representation which sheds more light on the structure of the MAP/RAPs, we introduce the Jordan representation, which contains also the initial and the closing vectors of the representation.

Definition 5. *The* **Jordan representation** *of a MAP/RAP with matrix representation* $(\mathbf{D_0}, \mathbf{D_1})$ *is* $(\delta, \mathbf{\Lambda}, \hat{\mathbf{P}}, h)$*, where* $\mathbf{E} = \mathbf{G}^{-1}\mathbf{\Lambda}\mathbf{G}$ *is the Jordan decomposition of* $\mathbf{E} = -\mathbf{D_0}^{-1}$*,* $\delta = \pi\mathbf{G}^{-1}$*,* $\hat{\mathbf{P}} = \mathbf{G}\mathbf{P}\mathbf{G}^{-1}$*, and* $h = \mathbf{G}\mathbb{1}$*.*

Based on the Jordan representation, the moments, defined in (1), can be calculated as

$$\gamma_{i_0,\ldots,i_k}^{(a_1,\ldots,a_k)} = \delta\mathbf{\Lambda}^{i_0}\hat{\mathbf{P}}^{a_1}\mathbf{\Lambda}^{i_1}\ldots\hat{\mathbf{P}}^{a_k}\mathbf{\Lambda}^{i_k}h. \tag{4}$$

We use the Jordan representation to make the orthogonal structure of the vectors and the eigenvectors of the matrices in (1) to be explicit. E.g., if π is orthogonal to an eigenvector of matrix $\mathbf{E}$ then in the Jordan representation δ contains a zero associated with that eigenvalue.

To refer to the zero-nonzero structure of a vector, we write $\star$ for the vector elements which can be non-zero and 0 for the vector elements which are restricted to be zero by the structure of the Jordan representation.

For the eigenvalues with multiplicity one, the zero-nonzero structure of δ and h cannot be modified by multiplication with matrix $\mathbf{\Lambda}$. E.g., in case of

$$\delta = \{\star, 0, 0\} \text{ and } \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix},$$

we have $\delta\mathbf{\Lambda} = \{\star, 0, 0\}$, $\delta\mathbf{\Lambda}^2 = \{\star, 0, 0\}$. For the eigenvalues with higher multiplicity, the zero-nonzero structure of δ and h can be modified by multiplication with matrix $\mathbf{\Lambda}$. That is, the zero-nonzero structure of $\delta\mathbf{\Lambda}$ and $\mathbf{\Lambda}h$ could be different from the ones of δ and h, e.g., in case of

$$\delta = \{\star, 0, 0\} \text{ and } \mathbf{\Lambda} = \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix},$$

we have $\delta\mathbf{\Lambda} = \{\star, \star, 0\}$, $\delta\mathbf{\Lambda}^2 = \{\star, \star, \star\}$. Therefore, [13] and Table 1 refer to the zero-nonzero structure of $\delta\mathbf{\Lambda}^{n-1}$ and $\mathbf{\Lambda}^{n-1}h$.

[1] A different terminology was applied in [13], MAP/RAP with RRM was referred to as redundant MAP/RAP and MAP/RAP with FRM as non-redundant MAP/RAP.

Algorithm 1 Moments based matrix representation of MAP/RAPs with FRM

Input: n, $r_i, i = 1, \ldots, 2n-1$, $\gamma_{ij}^{(1)}, i, j = 1, \ldots, n-1$

1: **if** $HO(\underline{r}) == n$ **then**
2: $\quad \alpha, \bar{\mathbf{D}}_{\mathbf{0}} \leftarrow \text{MEFromMoments}(\underline{r})$
3: $\quad \bar{\mathbf{E}} \leftarrow -\bar{\mathbf{D}}_{\mathbf{0}}^{-1}$
4: $\quad \mathbf{M} : \mathbf{M}_{i,j} \leftarrow \gamma_{i,j}^{(1)},\ i, j = 0, \ldots, n-1$
5: $\quad \mathbf{L} : \mathbf{L}_{i,\_} \leftarrow \alpha \bar{\mathbf{E}}^i, i = 0, \ldots, n-1$
6: $\quad \mathbf{R} : \mathbf{R}_{\_,i} \leftarrow \bar{\mathbf{E}}^i \mathbb{1}, i = 0, \ldots, n-1$
7: $\quad \bar{\mathbf{P}} \leftarrow \mathbf{L}^{-1} \mathbf{M} \mathbf{R}^{-1}$
8: $\quad \bar{\mathbf{D}}_{\mathbf{1}} \leftarrow -\bar{\mathbf{D}}_{\mathbf{0}} \bar{\mathbf{P}}$
9: $\quad$ **return** $\bar{\mathbf{D}}_{\mathbf{0}}, \bar{\mathbf{D}}_{\mathbf{1}}$
10: **end if**
11: **return** Moments cannot be represented by size n MAP/RAP

2.2 Moments based matrix representation of MAP/RAPs with FRM

An order n MAP/RAP with FRM is fully characterised by the first $2n-1$ marginal moments, $r_i, i = 1, 2, \ldots, 2n-1$, and the first $(n-1)^2$ joint moments of two consecutive interarrivals $\gamma_{ij}^{(1)}, i, j = 1, 2, \ldots, n-1$ [14], where $r_0 = 1$ and $\gamma_{i0}^{(1)} = \gamma_{0i}^{(1)} = r_i$, are also known by definition. Based on these moments, Algorithm 1 provides a $(\mathbf{D}_{\mathbf{0}}, \mathbf{D}_{\mathbf{1}})$ matrix representation of the MAP/RAP with FRM [14]. In line 2 of the algorithm, the `MEFromMoments` function is based on the procedure discussed in [14], which is a modified version of the one in [15], and is implemented in the BuTools package [9]. It generates a vector-matrix pair such that $\alpha \bar{\mathbf{E}}^i \mathbb{1} = \underline{r}_i, \forall i = 0, \ldots, 2n-1$, where $\bar{\mathbf{E}} = -\bar{\mathbf{D}}_{\mathbf{0}}^{-1}$. We note that the generated $\bar{\mathbf{D}}_{\mathbf{0}}$ matrix of size n is non-singular, when the Hankel order of $\underline{r}$ is n.

The structural property, which is utilized in line 7 of Algorithm 1, is the non-singularity of matrices $\mathbf{L}$ and $\mathbf{R}$ of size n. This non-singularity is also a consequence of the fact that the Hankel order of $\underline{r}$ is n.

Unfortunately, when the Hankel order of $\underline{r}$ is less than n and the order of the MAP/RAP is n, then at least one of matrices $\mathbf{L}$ and $\mathbf{R}$ of size n is singular and Algorithm 1 is not applicable. The aim of this paper is to provide a variant of Algorithm 1, which is applicable irrespective of the structure of the MAP/RAP, that is, it can be applied for both, FRM and RRM.

2.3 Examples

The above general statements on MAP/RAPs with FRM and RRM can be exemplified in order 3, because there is no order 2 MAP/RAP with RRM according to [13]. The structurally different order 3 MAP/RAPs are summarized in Table 1, where we neglect the cases which can be obtained by swapping the role of δ and h (i.e., swapping the zero-nonzero structures in row 2 and row 3 in Table 1) or by reordering rows and columns of the representation (i.e., permuting the elements of the vectors in row 2 and row 3 in Table 1).

Case a) is the MAP/RAP with FRM. This type of order 3 MAP/RAPs are characterized by $3^2 = 9$ parameters and the last row (referred to as `basic moments set`) presents the 9 moments, which allow the matrix representation of the MAP. I.e., based on these 9 moments Algorithm 1 generates a $(\mathbf{D_0}, \mathbf{D_1})$ matrix representation of the MAP. There are other sets of 9 independent moments which characterize these MAP/RAPs, but the set of moments in the basic moments set are the ones with lowest order.

Case b) is a MAP/RAP with RRM. The zero-nonzero structures of $\delta\mathbf{\Lambda}^{n-1}$ and $\mathbf{\Lambda}^{n-1}h$ are provided in the second and third row of the table. This type of order 3 MAP/RAPs are characterized by 8 ($< 3^2 = 9$) parameters and [13] discusses how to obtain a matrix representation based on the set of moments in its `basic moments set`. This procedure and the set of moments in the `basic moments set` is specific to Case b). Example 1 presents a $(\mathbf{D_0}, \mathbf{D_1})$ matrix representation with this structure. A directly visible property of the MAP/RAP in Example 1 is that the third eigenvalue in $\mathbf{\Lambda}$ does not contribute to the reduced moment series $r_i = \frac{E(X_0^i)}{i!} = \delta\mathbf{\Lambda}^i h$. Since $\delta = \{\star, \star, 0\}$ and $\mathbf{\Lambda}$ is diagonal. Consequently, the marginal distribution is order 2.

Case c) is a MAP/RAP with RRM as well and Example 2 presents a $(\mathbf{D_0}, \mathbf{D_1})$ matrix representation with this structure. The characterizing feature of this case is that its marginal distribution is order 1, which means that the stationary interarrival time is exponentially distributed, while the consecutive interarrivals are correlated. This type of order 3 MAP/RAPs are characterized by 8 parameters. Since the marginal distribution is order 1, only the mean of the stationary interarrival time (r_1) appears in its `basic moments set`.

Case d) is a MAP/RAP with RRM, which differs from Case b) in the zero-nonzero structure of $\mathbf{\Lambda}^{n-1}h$. It is worth noting that this change in the zero-nonzero structure further reduces the number of parameters that characterizes the model. This type of order 3 MAP/RAPs are characterized by 7 ($< 3^2 = 9$) parameters, e.g., by the 7 moments of its `basic moments set`. Example 3 presents a MAP with this structure.

We note that Examples 1-3 are from [13] and the $(\mathbf{D_0}, \mathbf{D_1})$ matrix representations are not Markovian in case of Example 1 and 2.

Example 1. *An example of Case b) order 3 matrix representation (with $h = \mathbb{1}$) is*

$$\mathbf{D_0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \mathbf{D_1} = \begin{pmatrix} 1/10 & 4/5 & 1/10 \\ 1/220 & 491/990 & -1/1980 \\ 2/5 & 3/5 & 1 \end{pmatrix}, \tag{5}$$

whose Jordan representation is $\delta = (1/100, 99/100, 0)$,

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \ \hat{\mathbf{P}} = \begin{pmatrix} 1/10 & 4/5 & 1/10 \\ 1/110 & 491/495 & -1/990 \\ 1/5 & 3/10 & 1/2 \end{pmatrix}, \ h = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Example 2. *An example of Case c) order 3 matrix representation is*

$$\mathbf{D_0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 1 & 0 & -2 \end{pmatrix}, \quad \mathbf{D_1} = \begin{pmatrix} 1/10 & 4/5 & 1/10 \\ -1/10 & 9/20 & 3/20 \\ 71/10 & -36/5 & 11/10 \end{pmatrix}, \tag{6}$$

whose Jordan representation is $\delta = (1, 0, 1/5)$,

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \ \hat{\mathbf{P}} = \begin{pmatrix} 1/5 & 4/5 & 1/10 \\ 1/10 & 9/10 & 3/10 \\ 4 & -4 & 1/2 \end{pmatrix}, \ h = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Example 3. *An example of Case d) order 3 matrix representation is*

$$\mathbf{D_0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3/5 & 1/10 \\ 0 & 1/15 & -17/30 \end{pmatrix}, \quad \mathbf{D_1} = \begin{pmatrix} 1/5 & 18/25 & 2/25 \\ 1/9 & 1/3 & 1/18 \\ 0 & 2/45 & 41/90 \end{pmatrix}. \tag{7}$$

whose Jordan representation is $\delta = (1/10, 9/10, 0)$,

$$\mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{pmatrix}, \ \hat{\mathbf{P}} = \begin{pmatrix} 1/5 & 4/5 & 1/3 \\ 4/45 & 41/45 & -1/27 \\ 1/5 & -1/5 & 1/2 \end{pmatrix}, \ h = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Unlike the previous examples, this $(\mathbf{D_0}, \mathbf{D_1})$ *representation is Markovian, and consequently, the associated process is a MAP of size* 3.

Table 1: Various cases of order 3 MAP/RAPs

Case	a)	b)	c)	d)
$\delta\mathbf{\Lambda}^{n-1}$ $(\mathbf{\Lambda}^{n-1}h)^T$	$\{\star,\star,\star\}$ $\{\star,\star,\star\}$	$\{\star,\star,0\}$ $\{\star,\star,\star\}$	$\{\star,0,\star\}$ $\{\star,\star,0\}$	$\{\star,\star,0\}$ $\{\star,\star,0\}$
number of parameters	9	8	8	7
$\mathrm{HO}\left(\underline{r_i}\right)$ $\left(\mathrm{HO}\left(\underline{\gamma_{i1}^{(1)}}\right), \mathrm{HO}\left(\underline{\gamma_{1i}^{(1)}}\right)\right)$ $\mathrm{HO}\left(\underline{\gamma_{1i1}^{(11)}}\right)$	3 (3,3) 3	2 (2,3) 3	1 (2,2) 3	2 (2,2) 3
required information	lag-1 mom.	lag-2 mom.	triple mom.	triple mom.
basic moments set	$r_1,\ldots,r_5,$ $\gamma_{11},\gamma_{12},$ γ_{21},γ_{22}	$r_1,r_2,r_3,$ $\gamma_{11},\gamma_{12},\gamma_{13},$ $\gamma_{11}^{(2)},\gamma_{12}^{(2)}$	$r_1,\gamma_{11},\gamma_{12},\gamma_{21},$ $\gamma_{11}^{(2)},\gamma_{11}^{(3)}$ $\gamma_{11}^{(4)},\gamma_{111}^{(11)}$	$r_1,r_2,r_3,$ $\gamma_{11},\gamma_{11}^{(2)}$ $\gamma_{11}^{(3)},\gamma_{111}^{(11)}$

3 Moments based matrix representation of MAP/RAPs

In the following we describe a general moments matrix representation, which can be used for both MAP/RAPs with RRM and with FRM. In the rest of this section we discuss how one can find a $\underline{\gamma}^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_{\ell-1},i_{\ell+1},\ldots,i_k}(\ell)$ moments series for any order n MAP/RAP, for which $\mathrm{HO}(\underline{\gamma}^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_{\ell-1},i_{\ell+1},\ldots,i_k}(\ell)) = n$. In Section 4, we will use this result to provide a method that can generate a matrix representation of the MAP/RAP using only its moments.

Let $\underline{u} = \delta\mathbf{\Lambda}^{i_0}\hat{\mathbf{P}}^{a_1}\ldots\mathbf{\Lambda}^{i_{\ell-1}}\hat{\mathbf{P}}^{a_\ell}$ and $\underline{v} = \hat{\mathbf{P}}^{a_{\ell+1}}\ldots\mathbf{\Lambda}^{i_{\ell+1}}\ldots\hat{\mathbf{P}}^{a_k}\mathbf{\Lambda}^{i_k}h$. The Hankel matrix of the moments series $\underline{\gamma} = \underline{\gamma}^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_{\ell-1},i_{\ell+1},\ldots,i_k}(\ell)$ (for notational simplification) can be decomposed as

$$\mathbf{H}\left(\underline{\gamma}\right) = \begin{bmatrix} \underline{u}\mathbf{\Lambda}^0 \\ \underline{u}\mathbf{\Lambda}^1 \\ \cdots \end{bmatrix} \cdot \begin{bmatrix} \mathbf{\Lambda}^0\underline{v} & \mathbf{\Lambda}^1\underline{v} & \ldots \end{bmatrix}. \tag{8}$$

The importance of this decomposition comes from the fact that

$$\mathrm{rank}\left(\mathbf{H}\left(\underline{\gamma}\right)\right) \leq \min\left(\mathrm{rank}\left(\begin{bmatrix} \underline{u}\mathbf{\Lambda}^0 \\ \underline{u}\mathbf{\Lambda}^1 \\ \cdots \end{bmatrix}\right), \mathrm{rank}\left(\begin{bmatrix} \mathbf{\Lambda}^0\underline{v} & \mathbf{\Lambda}^1\underline{v} & \ldots \end{bmatrix}\right)\right). \tag{9}$$

Assuming that $\mathbf{\Lambda}$ does not contain a zero eigenvalue (which can be eliminated by size reduction [6]), the rank of the matrices are determined by the zero entries of $\underline{u}$ and $\underline{v}$. For example, if $\mathbf{\Lambda}$ is diagonal, then the rank of the first matrix on the rhs is the number of non-zero elements of $\underline{u}$. Since the elements of the Jordan decomposition might be non-positive, we distinguish structural and random zeros in vector $\underline{u}$ and $\underline{v}$ [13]. Structural zeros are the result of the zero – non-zero structure of the Jordan decomposition, while random zeros are the result of matrix multiplications involving non-zero elements. That is, random zeros are such that modifying a non-zero element of the Jordan representation, the number can become non-zero, while modifying any non-zero element of the Jordan representation leaves the value of the structural zeros untouched. Hereafter we focus on the structural zeros of the representations which are associated with the structural properties of the MAP.

Lemma 1. *If the size of the MAP/RAP with Jordan representation $(\delta, \mathbf{\Lambda}, \hat{\mathbf{P}}, h)$ is n, then for any $j \in \{1, 2, \ldots, n\}$, there is a $k \geq 0$ and a set of parameters $\{i_0, a_1, \ldots, i_k, a_{k+1}\}$ for which the jth element of $\underline{u}$ is not a structural zero.*

Proof. We make the indirect assumption that there is a $j \in \{1, 2, \ldots, n\}$, such that the jth element of $\underline{u}$ is a structural zero for all k and all $i_1, a_1, \ldots, i_k, a_{k+1}$ sets of parameters. Then the jth column of matrix $\mathbf{A}$ in the

$$\mathbf{H}\left(\underline{\gamma}\right) = \underbrace{\begin{bmatrix} \underline{u}\mathbf{\Lambda}^0 \\ \underline{u}\mathbf{\Lambda}^1 \\ \cdots \end{bmatrix}}_{\mathbf{A}} \cdot \begin{bmatrix} \mathbf{\Lambda}^0 \underline{v} & \mathbf{\Lambda}^1 \underline{v} & \ldots \end{bmatrix}$$

decomposition is zero and thus the rank of $\mathbf{A}$ and, consequently, the rank of $\mathbf{H}(\underline{\gamma})$ is less than n, which is in conflict with Theorem 1. $\square$

Theorem 2. *For any order n MAP/RAP with minimal Jordan representation $(\delta, \mathbf{\Lambda}, \hat{\mathbf{P}}, h)$, the vector $\delta(\mathbf{\Lambda} + \hat{\mathbf{P}})^{n-1}$ has no structural zeros.*

Proof. Let $z(b)$ denote the set of indices of the structural zero elements of vector b. From the definition of structural zeros, for any pair of $\hat{b}$ and $\check{b}$ vectors we have $z(\hat{b} + \check{b}) = z(\hat{b}) \cap z(\check{b})$. Since the diagonal elements of $\mathbf{\Lambda}$ are non-zero, $z(b\mathbf{\Lambda}) \subseteq z(b)$, consequently

$$z(b(\mathbf{\Lambda} + \hat{\mathbf{P}})) = z(b\mathbf{\Lambda}) \cap z(b\hat{\mathbf{P}}) \subseteq z(b\mathbf{\Lambda}) \subseteq z(b). \tag{10}$$

Therefore,

$$\begin{aligned}
&z(\delta(\mathbf{\Lambda}+\hat{\mathbf{P}})^{i_0+a_1+\ldots+i_k+a_{k+1}}) \\
&= z(\delta\mathbf{\Lambda}^{i_0}\hat{\mathbf{P}}^{a_1}\ldots\mathbf{\Lambda}^{i_k}\hat{\mathbf{P}}^{a_{k+1}}+\ldots) \\
&\subseteq z(\delta\mathbf{\Lambda}^{i_0}\hat{\mathbf{P}}^{a_1}\ldots\mathbf{\Lambda}^{i_k}\hat{\mathbf{P}}^{a_{k+1}})
\end{aligned}$$

For $j \in \{1, 2, \ldots, n\}$ let $k(j)$ and $\{i_0(j), a_1(j), \ldots, i_k(j), a_{k+1}(j)\}$ be the set of parameters which makes the jth element of $\underline{u}$ to be structurally different from zero. Let $k = \max\{k(1), \ldots, k(n)\}$, $i_0 = \max\{i_0(1), \ldots, i_0(n)\}$, $a_1 = \max\{a_1(1), \ldots, a_1(n)\}$, $\ldots$, $a_{k+1} = \max\{a_{k+1}(1), \ldots, a_{k+1}(n)\}$, then according to (10)

$$z(\delta(\mathbf{\Lambda}+\hat{\mathbf{P}})^{i_0+a_1+\ldots+i_k+a_{k+1}}) = \emptyset. \tag{11}$$

From (11), there exists $c \in \mathcal{N}$, such that $z(\delta(\mathbf{\Lambda}+\hat{\mathbf{P}})^c) = \emptyset$. Let us denote the smallest such c by c_{min}.

From the definition of structural zeros it also follows that for any vector b and matrix $\mathbf{K}$, if $z(b) = z(b\mathbf{K})$, then $z(b) = z(b\mathbf{K}^i), \forall i \in \mathcal{N}$ as well. Therefore, if $z(\delta) \neq \emptyset$ and $z(\delta(\mathbf{\Lambda}+\hat{\mathbf{P}})^{c_{min}}) = \emptyset$, then $z(\delta(\mathbf{\Lambda}+\hat{\mathbf{P}})^{c-1}) \supset z(\delta(\mathbf{\Lambda}+\hat{\mathbf{P}})^c), \forall 1 \le c < c_{min}$, otherwise there would exist $1 < c < c_{min}$, such that

$$z(\delta(\mathbf{\Lambda}+\hat{\mathbf{P}})^{c-1}) = z(\delta(\mathbf{\Lambda}+\hat{\mathbf{P}})^c) = z(\delta(\mathbf{\Lambda}+\hat{\mathbf{P}})^{c+1}) = \ldots = z(\delta(\mathbf{\Lambda}+\hat{\mathbf{P}})^{c_min}) \neq \emptyset,$$

which is impossible. Thus, if $z(\delta) \neq \emptyset$, then the number of structural zeros in δ decreases by 1 or more after every multiplication by $(\mathbf{\Lambda}+\hat{\mathbf{P}})$ until it has none. Since δ must have at least one non-zero element to satisfy Theorem 1, this means that $c_{min} \le n-1$, i.e., $\delta(\mathbf{\Lambda}+\hat{\mathbf{P}})^{n-1}$ has no structural zeros, which is what we had to prove. □

Theorem 3. *For any order n MAP/RAP with minimal Jordan representation $(\delta, \mathbf{\Lambda}, \hat{\mathbf{P}}, h)$, the vector $(\mathbf{\Lambda}+\hat{\mathbf{P}})^{n-1}h$ has no structural zeros.*

Proof. The proof follows the same arguments as the proof of Theorem 2. □

4 The proposed moments based matrix representation method

The pseudocode of the proposed procedure can be found in Algorithm 2. The inputs of the algorithm are the $\gamma_{i_0,\ldots,i_k}^{(a_1,\ldots,a_k)}$ moments, but the algorithm works with complex sums of those moments. To avoid expressing those complex sums, we describe them using the elements of the Jordan representation. For example we write $\delta(\hat{\mathbf{P}}+\boldsymbol{\Lambda})^2 h$ instead of $1+2r_1+r_2$, since $\delta(\hat{\mathbf{P}}+\boldsymbol{\Lambda})^2 h = \delta(\hat{\mathbf{P}}^2+\boldsymbol{\Lambda}^2+\hat{\mathbf{P}}\boldsymbol{\Lambda}+\boldsymbol{\Lambda}\hat{\mathbf{P}})h = 1+2r_1+r_2$.

Here, we provide an intuitive explanation of Algorithm 2. Let $\hat{\mu} = \delta(\hat{\mathbf{P}}+\boldsymbol{\Lambda})^{k+l}h$ and $\underline{\mu}$ be the series with elements $\mu_i = \frac{1}{\hat{\mu}}\delta(\hat{\mathbf{P}}+\boldsymbol{\Lambda})^k\boldsymbol{\Lambda}^i(\hat{\mathbf{P}}+\boldsymbol{\Lambda})^l h$. By substituting $\underline{u} = \delta(\boldsymbol{\Lambda}+\hat{\mathbf{P}})^k$ and $\underline{v} = (\boldsymbol{\Lambda}+\hat{\mathbf{P}})^l h$ into (9), we can see that the Hankel matrix $\mathbf{H}(\underline{\mu})$ is rank n, if $\underline{u}$ and $\underline{v}$ of size n have no structural or random zeros. From Theorem 2 and 3, we know that $\underline{u}$ and $\underline{v}$ have no structural zeros when $k = l = n-1$. In practice, random zeros can be removed by an additional multiplication by $(\hat{\mathbf{P}}+\boldsymbol{\Lambda})$, therefore $\exists k,l : k+l \leq 2n$, such that $\text{rank}(\mathbf{H}(\underline{\mu})) = HO(\underline{\mu}) = n$. Algorithm 2 searches for such k,l while trying to minimize $e = k+l$. If such k,l are found, we can generate a matrix exponential distribution with representation $(\alpha, \bar{\mathbf{D}}_\mathbf{0})$ such that $\alpha\bar{\mathbf{E}}^i\mathbb{1} = \mu_i, \forall i = 0,1,\ldots,2n-1$, where $\bar{\mathbf{E}} = -\bar{\mathbf{D}}_\mathbf{0}^{-1}$, using the same `MEFromMoments` method as in line 3 of Algorithm 1. The `MEFromMoments` method returns a size $HO(\underline{\mu})$ $(\alpha, \bar{\mathbf{D}}_\mathbf{0})$ representation, which is why we need to ensure that $HO(\underline{\mu}) = n$. Then, from

$$\mu_i = \alpha\bar{\mathbf{E}}^i\mathbb{1} = \frac{1}{\hat{\mu}}\delta(\hat{\mathbf{P}}+\boldsymbol{\Lambda})^k\boldsymbol{\Lambda}^i(\hat{\mathbf{P}}+\boldsymbol{\Lambda})^l h, \quad \forall i \geq 0, \tag{12}$$

we have that $\bar{\mathbf{E}}$ and $\boldsymbol{\Lambda}$ are similar and $\boldsymbol{\Lambda}$ is a Jordan matrix. If $\bar{\mathbf{E}} = \boldsymbol{\Gamma}^{-1}\boldsymbol{\Lambda}\boldsymbol{\Gamma}$ is the Jordan decomposition of $\bar{\mathbf{E}}$, then $c_1\alpha\boldsymbol{\Gamma}^{-1} = \delta(\hat{\mathbf{P}}+\boldsymbol{\Lambda})^k$ and $c_2\boldsymbol{\Gamma}^{-1}\mathbb{1} = (\hat{\mathbf{P}}+\boldsymbol{\Lambda})^l h$ such that $c_1 \cdot c_2 = \hat{\mu}$. Consequently,

$$\frac{1}{\hat{\mu}}\delta(\hat{\mathbf{P}}+\boldsymbol{\Lambda})^k\boldsymbol{\Lambda}^i\hat{\mathbf{P}}\boldsymbol{\Lambda}^j(\hat{\mathbf{P}}+\boldsymbol{\Lambda})^l h = \alpha\bar{\mathbf{E}}^i\bar{\mathbf{P}}\bar{\mathbf{E}}^j\mathbb{1}, \quad \forall i,j \geq 0, \tag{13}$$

such that $\bar{\mathbf{P}}$ is similar to $\hat{\mathbf{P}}$ with transformation matrix $\boldsymbol{\Gamma}$, i.e., $\bar{\mathbf{P}} = \boldsymbol{\Gamma}^{-1}\hat{\mathbf{P}}\boldsymbol{\Gamma}$. Therefore, lines 9-11 can be used to obtain $\bar{\mathbf{P}}$, where $\mathbf{R}_{\_,i}$ is the ith column of $\mathbf{R}$ and $\mathbf{L}_{i,\_}$ is ith row of $\mathbf{L}$. Since the Hankel order of the $\underline{\mu}$ series is n, matrix $\mathbf{R}$ and $\mathbf{L}$ of size $n \times n$ are non-singular and matrix $\bar{\mathbf{P}}$ can be computed in line 11.

In line 12, we calculate $\bar{\mathbf{D}}_\mathbf{1}$ from $\bar{\mathbf{D}}_\mathbf{0}$ and $\bar{\mathbf{P}}$ to obtain a $(\bar{\mathbf{D}}_\mathbf{0}, \bar{\mathbf{D}}_\mathbf{1})$ representation of the MAP. Unfortunately, the closing vector of this representation is not vector $\mathbb{1}$, when $l > 0$. In general, the closing vector is the solution of $h = \bar{\mathbf{P}}h$. To transform the obtained representation to a matrix representation whose closing vector is $\mathbb{1}$, we apply a similarity transform with matrix $\mathbf{B}$, where matrix $\mathbf{B}$ is non-singular and satisfies $\mathbf{B}\mathbb{1} = h$. There are many

Algorithm 2 Moment based matrix representation of MAP/RAPs

Input: n, $\gamma_{i_0,\ldots,i_k}^{(a_1,\ldots,a_k)}$ $\forall k < n, i_0, \ldots, i_k, a_1, \ldots, a_k$

1: **for** $e = 0, \ldots, 2n$ **do**
2:     **for** $k = 0, \ldots, e$ **do**
3:         $l \leftarrow e - k$
4:         $\hat{\mu} \leftarrow \delta(\hat{\mathbf{P}} + \mathbf{\Lambda})^{k+l} h$
5:         $\underline{\mu} : \mu_i \leftarrow \frac{1}{\hat{\mu}} \delta(\hat{\mathbf{P}} + \mathbf{\Lambda})^k \mathbf{\Lambda}^i (\hat{\mathbf{P}} + \mathbf{\Lambda})^l h,\ i = 1, \ldots, 2n-1$
6:         **if** $HO(\underline{\mu}) == n$ **then**
7:             $\alpha, \bar{\mathbf{D}}_{\mathbf{0}} \leftarrow \texttt{MEFromMoments}(\underline{\mu})$
8:             $\mathbf{M} : \mathbf{M}_{i,j} \leftarrow \frac{1}{\hat{\mu}} \delta(\hat{\mathbf{P}} + \mathbf{\Lambda})^k \mathbf{\Lambda}^i \hat{\mathbf{P}} \mathbf{\Lambda}^j (\hat{\mathbf{P}} + \mathbf{\Lambda})^l h,\ i, j = 0, \ldots, n-1$
9:             $\mathbf{L} : \mathbf{L}_{i,\_} \leftarrow \alpha(-\bar{\mathbf{D}}_{\mathbf{0}})^{-i}, i = 0, \ldots, n-1$
10:             $\mathbf{R} : \mathbf{R}_{\_,i} \leftarrow (-\bar{\mathbf{D}}_{\mathbf{0}})^{-i} \mathbb{1}, i = 0, \ldots, n-1$
11:             $\bar{\mathbf{P}} \leftarrow \mathbf{L}^{-1} \mathbf{M} \mathbf{R}^{-1}$
12:             $\bar{\mathbf{D}}_{\mathbf{1}} \leftarrow -\bar{\mathbf{D}}_{\mathbf{0}} \bar{\mathbf{P}}$
13:             **if** $l > 0$ **then**
14:                 $h \leftarrow \texttt{Solve}(h = \bar{\mathbf{P}} h, h\mathbb{1} = 1)$
15:                 $\mathbf{B} \leftarrow \mathbf{I} + \texttt{diag}(h) - \texttt{PermutationMatrix}(i \rightarrow \mathrm{Mod}(i+1, n))$
16:                 $\bar{\mathbf{D}}_{\mathbf{0}}, \bar{\mathbf{D}}_{\mathbf{1}} \leftarrow \mathbf{B}^{-1} \bar{\mathbf{D}}_{\mathbf{0}} \mathbf{B}, \mathbf{B}^{-1} \bar{\mathbf{D}}_{\mathbf{1}} \mathbf{B}$
17:             **end if**
18:             **return** $\bar{\mathbf{D}}_{\mathbf{0}}, \bar{\mathbf{D}}_{\mathbf{1}}$
19:         **end if**
20:     **end for**
21: **end for**
22: **return** Moments cannot be represented by size n MAP

such matrices. In the procedure we use

$$\mathbf{B} = \begin{pmatrix} 1+h_1 & -1 & & & \\ & \ddots & \ddots & & \\ & & 1+h_{n-1} & -1 \\ -1 & & & 1+h_n \end{pmatrix}, \tag{14}$$

which readably satisfies $\mathbf{B}\mathbb{1} = h$. In this case, the matrix representation with the proper closing vector is obtained as $(\mathbf{B}^{-1}\bar{\mathbf{D}}_{\mathbf{0}}\mathbf{B}, \mathbf{B}^{-1}\bar{\mathbf{D}}_{\mathbf{1}}\mathbf{B})$, which is set in line 16.

5 Numerical examples

In this section we demonstrate the operation of Algorithm 2 through numerical examples. First, we consider the order 3 MAP/RAPs with RRM from Example 1 and 3 to illustrate the workings of finding the optimal k and l values, then we present a MAP/RAP that results from the analysis of a MAP/MAP/1 queue.

5.1 Order 3 MAP/RAPs with RRM

5.1.1 Analysis of Example 1

Let us consider Example 1 with the $(\mathbf{D_0}, \mathbf{D_1})$ matrix representation of (5). Here, and in the following examples, the input of Algorithm 2 is computed from the $(\mathbf{D_0}, \mathbf{D_1})$ representation according to (1).

The Hankel order of the reduced marginal moments series of this MAP/RAP, $\mathrm{HO}(\{r_1, \ldots, r_n\})$ is 2, and the order of the MAP/RAP is 3, as it is in Table 1 case b). The Hankel order $\mathrm{HO}(\underline{\mu})$ computed by Algorithm 2 is 2, when $k < 2$, but it increases to 3, when $k = 2,\ l = 0$. This is in line with the $\left(\mathrm{HO}\left(\underline{\gamma^{(1)}_{\mathtt{i1}}}\right), \mathrm{HO}\left(\underline{\gamma^{(1)}_{\mathtt{1i}}}\right)\right)$ row of Table 1, since based on line 5 of Algorithm 2, this means that

$$\underline{\mu} = \left\{ \frac{1}{\hat{\mu}} \delta(\hat{\mathbf{P}} + \mathbf{\Lambda})^2 \mathbf{\Lambda}^i h,\ i = 1, \ldots, 5 \right\} \tag{15}$$

has a Hankel order of 3, where $\hat{\mu} = \delta(\hat{\mathbf{P}} + \mathbf{\Lambda})^2 h$, and the expressions in (15) contain, e.g., the

$$\gamma^1_{1i} = \delta \mathbf{\Lambda} \hat{\mathbf{P}} \mathbf{\Lambda}^i h,\ i = 1, \ldots 5$$

moments. From the moments computed from the $(\mathbf{D_0}, \mathbf{D_1})$ representation, Algorithm 2 obtains the following order 3 non-Markovian MAP/RAP representation at $k = 2$ and $l = 0$:

$$\bar{\mathbf{D}}_\mathbf{0} = \begin{pmatrix} 5.0662 & 11.8777 & -17.4469 \\ 2.58316 & 6.35342 & -9.43882 \\ 4.53175 & 9.88534 & -14.9196 \end{pmatrix},$$

$$\bar{\mathbf{D}}_\mathbf{1} = \begin{pmatrix} 10.3763 & 9.97501 & -19.8484 \\ 2.22544 & -0.440123 & -1.28309 \\ 5.32673 & 3.51606 & -8.34025 \end{pmatrix}.$$

Calculating the respective moments according to (1) verifies that $(\bar{\mathbf{D}}_\mathbf{0}, \bar{\mathbf{D}}_\mathbf{1})$ is indeed a (non-Markovian) representation of the original MAP/RAP with representation $(\mathbf{D_0}, \mathbf{D_1})$. That is, $(\bar{\mathbf{D}}_\mathbf{0}, \bar{\mathbf{D}}_\mathbf{1})$ and $(\mathbf{D_0}, \mathbf{D_1})$ are two similar representations of the same MAP.

The algorithm uses the $r_i,\ i = 1, \ldots, 2n+1$, $\gamma^1_{1i},\ i = 1, \ldots, 2n-1$ moments for obtaining a Hankel matrix of order 3, and the $\gamma^1_{ij},\ i = 1, \ldots, n+2, j = 1, \ldots n$ and $\gamma^{11}_{1ij},\ i, j = 1, \ldots, n$ moments for calculating $\hat{P}$, where $n = 3$. This set of moments (with $12 + 19 = 31$ different elements) is much larger than the `basic moments set` (of 8 elements) in Table 1. This is the price to pay for the generality of the procedure.

5.1.2 Analysis of Example 2

Next, we consider Case c) of Table 1 in Example 2, with the $(\mathbf{D_0}, \mathbf{D_1})$ matrix representation of (6).

An important difference compared to Example 1 is that this MAP/RAP can only be described using more complex joint moments, as $\max\left\{\mathrm{HO}\left(\underline{\gamma^{(1)}_{i1}}\right), \mathrm{HO}\left(\underline{\gamma^{(1)}_{1i}}\right)\right\} = 2$. This is reflected in the results of Algorithm 2 as well, since the following order 3 representation, which represents the MAP/RAP as $(\bar{\mathbf{D}}_\mathbf{0}, \bar{\mathbf{D}}_\mathbf{1})$, is found at $k = 2$ and $l = 2$.

$$\bar{\mathbf{D}}_\mathbf{0} = \begin{pmatrix} 2.23986 & 2.91203 & -6.10961 \\ 0.787256 & 1.39588 & -3.11987 \\ 3.11521 & 3.06964 & -7.13573 \end{pmatrix},$$
$$\bar{\mathbf{D}}_\mathbf{1} = \begin{pmatrix} 18.5484 & -4.72676 & -12.8639 \\ 3.21957 & -2.12352 & -0.159319 \\ 20.8932 & -5.16746 & -14.7748 \end{pmatrix}.$$

Since the order 3 Hankel matrix is obtained at $k = 2$ and $l = 2$, the algorithm uses the $r_i,\ i = 1, \ldots, 2n+3$, $\gamma^1_{1i},\ i = 1, \ldots, 2n+1$, $\gamma^1_{i1},\ i = 1, \ldots, 2n+1$, $\gamma^1_{1i1},\ i = 1, \ldots, 2n-1$ moments for obtaining a Hankel matrix of order 3, and the $\gamma^1_{ij},\ i, j = 1, \ldots, n+2$, $\gamma^{11}_{1ij},\ i = 1, \ldots, n, j = 1, \ldots, n+2$, $\gamma^{11}_{ij1},\ i = 1, \ldots, n+2, j = 1, \ldots, n$, $\gamma^{111}_{1ij1},\ i, j = 1, \ldots, n$ moments for calculating $\hat{P}$. That is, $27 + 57 = 84$ different moments are used instead of the 8 moments of the `basic moments set` in Table 1.

5.1.3 Analysis of Example 3

In Example 3, the Case d) MAP/RAP of Table 1 is given by the Markovian $(\mathbf{D_0}, \mathbf{D_1})$ matrix representation of (7). Also in this case $\max\left\{\mathrm{HO}\left(\underline{\gamma^{(1)}_{i1}}\right), \mathrm{HO}\left(\underline{\gamma^{(1)}_{1i}}\right)\right\} = 2$ and Algorithm 2 finds the following order 3 representation at $k = 2$ and $l = 2$.

$$\bar{\mathbf{D}}_{\mathbf{0}} = \begin{pmatrix} -0.094932 & 5.15566 & -5.59293 \\ -0.192175 & 2.76268 & -3.09503 \\ 0.135884 & 4.1707 & -4.83441 \end{pmatrix},$$
$$\bar{\mathbf{D}}_{\mathbf{1}} = \begin{pmatrix} -31.441 & -45.46 & 77.4332 \\ -0.0034144 & -3.40527 & 3.93322 \\ -13.6486 & -21.6588 & 35.8352 \end{pmatrix}.$$

Since Algorithm 2 obtains the order 3 Hankel matrix at $k = 2$ and $l = 2$, the same $27 + 57 = 84$ different moments are used for computing the representation as in Example 2. It is interesting to note that exactly the same moments are used by Algorithm 2 for Example 3 as in case of Example 2, while in Example 3 the number of parameters is 7 and in Example 2 it is 8.

5.2 ETAQA model

Finally, we apply the proposed method to the numerical example from Section 8 of [13], which is based on an ETAQA output process approximation of a MAP/MAP/1 queue [16].

$$\mathbf{D}_{\mathbf{0}} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -17 & 3 & 1 & 0 \\ 0 & 0 & 0 & -6 & 0 & 1 \\ 0 & 0 & 0 & 0 & -16 & 3 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{pmatrix},$$
$$\mathbf{D}_{\mathbf{1}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 6.5 & 6.5 & 0 & 0 & 0 & 0 \\ 2.5 & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 6 & 0.5 & 0.5 \\ 0 & 0 & 2 & 2 & 0.5 & 0.5 \end{pmatrix}.$$

The size of the above MAP/RAP representation is 6, the Hankel order of its reduced marginal moments series $\mathrm{HO}(\{r_1, \ldots, r_n\})$ is 3, while the order of the MAP/RAP is 5. The Hankel order $\mathrm{HO}(\underline{\mu})$ is 3, for $k = 0, l = 0$ and $k = 1, l = 0$, but it is 5, for $k = 0, l = 2$. We note here that the determinant of the corresponding Hankel matrix is very small ($\sim 10^{-24}$), therefore high-precision arithmetic is needed for its calculation. Algorithm 2 provides the following non-Markovian representation

$$\mathbf{\bar{D}_0} = \begin{pmatrix} 32511.1 & 466.427 & -1786.37 & 42129.6 & -73322.4 \\ 19107.7 & 274.041 & -1049.93 & 24760.9 & -43094. \\ 9200.42 & 131.046 & -505.557 & 11921.6 & -20748.7 \\ 15519.5 & 223.181 & -853.431 & 20111.3 & -35001.8 \\ 23250.1 & 333.74 & -1277.46 & 30128.1 & -52435.9 \end{pmatrix},$$

$$\mathbf{\bar{D}_1} = \begin{pmatrix} -8056.92 & -52.705 & 684.039 & -10898.9 & 18326.1 \\ -4838.85 & -30.5861 & 401.79 & -6529.47 & 10998.5 \\ -1984.01 & -15.3832 & 192.164 & -2725.92 & 4534.3 \\ -3636.87 & -25.8652 & 329.025 & -4955.69 & 8290.67 \\ -5645.91 & -38.0601 & 490.348 & -7657. & 12852. \end{pmatrix}.$$

Again, the moments computed according to (1) verifies that $(\mathbf{\bar{D}_0}, \mathbf{\bar{D}_1})$ is indeed a similar (non-Markovian) representation of the original $(\mathbf{D_0}, \mathbf{D_1})$ MAP from which the input of Algorithm 2 is computed.

6 Conclusion

The moments based matrix representation of MAP/RAPs with RRM is a difficult task, because the moments which represent the MAP/RAP with RRM and the procedure to obtain a matrix representation based on the moments, depends on the internal structure of the MAP/RAP with RRM. This paper presents a general moments based matrix representation method, which can be applied for all MAP/RAPs independent of their internal structure.

The core step of the moments based matrix representation of MAP/RAPs with RRM is to find a moment series with maximal Hankel order. The paper proves that such moments series can be found in at most n^2 steps if the order of the MAP/RAP is n.

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