M/M/1/N queues with energy required service and phase type vacation time

January 19, 2024

Nikhil.A.P [a], Miklos Telek [b,c], and Deepak.T.G [a]

[a] Indian Institute of Space Science and Technology, Department of Mathematics,Thiruvananthapuram, India,
[b]MTA-BME Information Systems Research Group, Budapest, Hungary
[c]Budapest University of Technology and Economics, Department of Networked Systems and Services, Budapest, Hungary

Fluid queues

A usual fluid flow model comprises a buffer, finite or infinite, of fluid where the rates at which the fluid flows to the buffer (inflow rates) and the rates at which the fluid flows out of the buffer (outflow rates) are governed by a continuous time Markov chain.

Fluid queues

A usual fluid flow model comprises a buffer, finite or infinite, of fluid where the rates at which the fluid flows to the buffer (inflow rates) and the rates at which the fluid flows out of the buffer (outflow rates) are governed by a continuous time Markov chain.

They arise as models of real-world systems that deal with the processing of continuous entities such as the ones used in the petroleum and chemical industries.

Fluid queues

A usual fluid flow model comprises a buffer, finite or infinite, of fluid where the rates at which the fluid flows to the buffer (inflow rates) and the rates at which the fluid flows out of the buffer (outflow rates) are governed by a continuous time Markov chain.

They arise as models of real-world systems that deal with the processing of continuous entities such as the ones used in the petroleum and chemical industries.

They are also used as models of the asymptotic behavior of queues in heavy traffic.

Fluid queues

A usual fluid flow model comprises a buffer, finite or infinite, of fluid where the rates at which the fluid flows to the buffer (inflow rates) and the rates at which the fluid flows out of the buffer (outflow rates) are governed by a continuous time Markov chain.

They arise as models of real-world systems that deal with the processing of continuous entities such as the ones used in the petroleum and chemical industries.

They are also used as models of the asymptotic behavior of queues in heavy traffic.

Fluid-flow models also provide an important tool for the performance analysis of high-speed data networks, or large-scale production systems where a large number of relatively small jobs are processed.

Fluid queues

A usual fluid flow model comprises a buffer, finite or infinite, of fluid where the rates at which the fluid flows to the buffer (inflow rates) and the rates at which the fluid flows out of the buffer (outflow rates) are governed by a continuous time Markov chain.

They arise as models of real-world systems that deal with the processing of continuous entities such as the ones used in the petroleum and chemical industries.

They are also used as models of the asymptotic behavior of queues in heavy traffic.

Fluid-flow models also provide an important tool for the performance analysis of high-speed data networks, or large-scale production systems where a large number of relatively small jobs are processed.

In such a fluid flow model, the only state variables present are the fluid level and the state of the Markov chain.

Fluid queues

A usual fluid flow model comprises a buffer, finite or infinite, of fluid where the rates at which the fluid flows to the buffer (inflow rates) and the rates at which the fluid flows out of the buffer (outflow rates) are governed by a continuous time Markov chain.

They arise as models of real-world systems that deal with the processing of continuous entities such as the ones used in the petroleum and chemical industries.

They are also used as models of the asymptotic behavior of queues in heavy traffic.

Fluid-flow models also provide an important tool for the performance analysis of high-speed data networks, or large-scale production systems where a large number of relatively small jobs are processed.

In such a fluid flow model, the only state variables present are the fluid level and the state of the Markov chain.

Proposed model

Here we are considering an M/M/1/N queueing model where each customer needs a random amount of fluid for its service.

Proposed model

Here we are considering an M/M/1/N queueing model where each customer needs a random amount of fluid for its service.
Here fluid may be termed as energy or power required for performing service.

Proposed model

Here we are considering an M/M/1/N queueing model where each customer needs a random amount of fluid for its service.
Here fluid may be termed as energy or power required for performing service.
Since the time required for a service is random, the amount of fluid (energy) required for a service is also a random variable.

Proposed model

Here we are considering an M/M/1/N queueing model where each customer needs a random amount of fluid for its service.

Here fluid may be termed as energy or power required for performing service.

Since the time required for a service is random, the amount of fluid (energy) required for a service is also a random variable.

This assumption is in line with a queueing -inventory model where each customer requires an inventoried item (like raw material) for its service.

Proposed model

Here we are considering an M/M/1/N queueing model where each customer needs a random amount of fluid for its service.

Here fluid may be termed as energy or power required for performing service.

Since the time required for a service is random, the amount of fluid (energy) required for a service is also a random variable.

This assumption is in line with a queueing -inventory model where each customer requires an inventoried item (like raw material) for its service.

Here, instead of inventoried items, the server requires a random amount of fluid for completing a service.

Proposed model

Here we are considering an M/M/1/N queueing model where each customer needs a random amount of fluid for its service.

Here fluid may be termed as energy or power required for performing service.

Since the time required for a service is random, the amount of fluid (energy) required for a service is also a random variable.

This assumption is in line with a queueing -inventory model where each customer requires an inventoried item (like raw material) for its service.

Here, instead of inventoried items, the server requires a random amount of fluid for completing a service.

Like an ordinary fluid flow model, fluid inflow is governed by a Markov process and the server consumes fluid at a fixed rate during the service time.

Proposed model

Here we are considering an M/M/1/N queueing model where each customer needs a random amount of fluid for its service.

Here fluid may be termed as energy or power required for performing service.

Since the time required for a service is random, the amount of fluid (energy) required for a service is also a random variable.

This assumption is in line with a queueing -inventory model where each customer requires an inventoried item (like raw material) for its service.

Here, instead of inventoried items, the server requires a random amount of fluid for completing a service.

Like an ordinary fluid flow model, fluid inflow is governed by a Markov process and the server consumes fluid at a fixed rate during the service time.

So in this model, apart from fluid level and state of the underlying Markov process as state variables , we need to take the number of customers in the system also as state variable. This is the basic difference between a usual fluid flow model and the proposed model.

Literature Review

- Ramaswami (1999) used renewal argument for analysing fluid flow models, which led to solution in matrix-exponential form.
- Da Silva Soares and Latouche (2006) used matrix analytic methods and obtained a representation of fluid model in terms of a QBD.
- Computational approach to find various performance measures of fluid models have been improved by N. Bean et al (2005).
- Horvath and Telek [2015], and Saffer and Telek [2016] studied fluid vacation models in detail under exhaustive disciplines. They followed a new methodology based on the matrix analytic analysis of the Markov fluid queues.
- The detailed analysis of our proposed model is very much inspired by the same in Horvath and Telek [2015] and Saffer and Telek [2016] for fluid vacation models.

Model description

Assume that the fluid flow is governed by a Markov process $\{\varphi(t)\}$, having finite state space $S_{\varphi(t)}$ and generator matrix $\mathbf{Q}_{\varphi(t)}$.

Model description

Assume that the fluid flow is governed by a Markov process $\{\varphi(t)\}$, having finite state space $S_{\varphi(t)}$ and generator matrix $\mathbf{Q}_{\varphi(t)}$.
It is assumed that while $\varphi(t) = k$, fluid level increases at rate $c_k > 0$ and during customer service, the server consumes fluid (energy) at rate $d > 0$.

Model description

Assume that the fluid flow is governed by a Markov process $\{\varphi(t)\}$, having finite state space $S_{\varphi(t)}$ and generator matrix $\mathbf{Q}_{\varphi(t)}$.
It is assumed that while $\varphi(t) = k$, fluid level increases at rate $c_k > 0$ and during customer service, the server consumes fluid (energy) at rate $d > 0$. Service will go on continuously as long as customers are there in the finite capacity waiting space and the fluid buffer is nonempty.

Model description

Assume that the fluid flow is governed by a Markov process $\{\varphi(t)\}$, having finite state space $S_{\varphi(t)}$ and generator matrix $\mathbf{Q}_{\varphi(t)}$.
It is assumed that while $\varphi(t) = k$, fluid level increases at rate $c_k > 0$ and during customer service, the server consumes fluid (energy) at rate $d > 0$.
Service will go on continuously as long as customers are there in the finite capacity waiting space and the fluid buffer is nonempty.
At each time the buffer becomes empty, the server waits for a random amount of time to accumulate the fluid in order to restart the service of the customer for which the service got interrupted due to lack of fluid (energy).

Model description

Assume that the fluid flow is governed by a Markov process $\{\varphi(t)\}$, having finite state space $S_{\varphi(t)}$ and generator matrix $\mathbf{Q}_{\varphi(t)}$.
It is assumed that while $\varphi(t) = k$, fluid level increases at rate $c_k > 0$ and during customer service, the server consumes fluid (energy) at rate $d > 0$.
Service will go on continuously as long as customers are there in the finite capacity waiting space and the fluid buffer is nonempty.
At each time the buffer becomes empty, the server waits for a random amount of time to accumulate the fluid in order to restart the service of the customer for which the service got interrupted due to lack of fluid (energy).
Let us call this time duration as the *vacation period* of the server and assume that it is PH distributed with representation $(\alpha, \mathbf{A})$ of order n_{PH} .

Model description

Assume that the fluid flow is governed by a Markov process $\{\varphi(t)\}$, having finite state space $S_{\varphi(t)}$ and generator matrix $\mathbf{Q}_{\varphi(t)}$.
It is assumed that while $\varphi(t) = k$, fluid level increases at rate $c_k > 0$ and during customer service, the server consumes fluid (energy) at rate $d > 0$.
Service will go on continuously as long as customers are there in the finite capacity waiting space and the fluid buffer is nonempty.
At each time the buffer becomes empty, the server waits for a random amount of time to accumulate the fluid in order to restart the service of the customer for which the service got interrupted due to lack of fluid (energy).
Let us call this time duration as the *vacation period* of the server and assume that it is PH distributed with representation $(\alpha, \mathbf{A})$ of order n_{PH} .

Model description continued....

The density, Laplace transform (LT) and mean of the random variable σ representing the vacation time, are given by $\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Pr(\sigma < t) = \alpha e^{\mathbf{A}t}\mathbf{a}$, $\sigma^*(s) = E\left(e^{-s\sigma}\right) = \alpha(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}$ and $E(\sigma) = \alpha(-\mathbf{A})^{-1}\mathbf{e}$ respectively.

Model description continued....

The density, Laplace transform (LT) and mean of the random variable σ representing the vacation time, are given by $\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Pr(\sigma < t) = \alpha e^{\mathbf{A}t}\mathbf{a}$, $\sigma^*(s) = E\left(e^{-s\sigma}\right) = \alpha(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}$ and $E(\sigma) = \alpha(-\mathbf{A})^{-1}\mathbf{e}$ respectively.
Denote by

$X(t)\#$ **fluid level**

$\varphi(t)\#$**phase** of the underlying Markov process

$N(t)\#$**queue size**

$\vartheta(t)\#$**phase** of the vacation period if the server is on vacation mode

at time t.

Model description continued....

The density, Laplace transform (LT) and mean of the random variable σ representing the vacation time, are given by $\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Pr(\sigma < t) = \alpha e^{\mathbf{A}t}\mathbf{a}$, $\sigma^*(s) = E\left(e^{-s\sigma}\right) = \alpha(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}$ and $E(\sigma) = \alpha(-\mathbf{A})^{-1}\mathbf{e}$ respectively.
Denote by

$X(t)\#$ **fluid level**

$N(t)\#$**queue size**

$\vartheta(t)\#$**phase** of the vacation period if the server is on vacation mode

at time t.

Model description continued....

The density, Laplace transform (LT) and mean of the random variable σ representing the vacation time, are given by $\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Pr(\sigma < t) = \alpha e^{\mathbf{A}t}\mathbf{a}$, $\sigma^*(s) = E\left(e^{-s\sigma}\right) = \alpha(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}$ and $E(\sigma) = \alpha(-\mathbf{A})^{-1}\mathbf{e}$ respectively. Denote by

$X(t)\#$ **fluid level**

$\vartheta(t)\#$**phase** of the vacation period if the server is on vacation mode

at time t.

Model description continued....

The density, Laplace transform (LT) and mean of the random variable σ representing the vacation time, are given by $\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Pr(\sigma < t) = \alpha e^{\mathbf{A}t}\mathbf{a}$, $\sigma^*(s) = E\left(e^{-s\sigma}\right) = \alpha(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}$ and $E(\sigma) = \alpha(-\mathbf{A})^{-1}\mathbf{e}$ respectively. Denote by

$X(t)\#$ **fluid level**

$\varphi(t)\#$**phase** of the underlying Markov process

$N(t)\#$**queue size**

$\vartheta(t)\#$**phase** of the vacation period if the server is on vacation mode

at time t.

Model description continued....

The density, Laplace transform (LT) and mean of the random variable σ representing the vacation time, are given by $\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Pr(\sigma < t) = \alpha e^{\mathbf{A}t}\mathbf{a}$, $\sigma^*(s) = E\left(e^{-s\sigma}\right) = \alpha(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}$ and $E(\sigma) = \alpha(-\mathbf{A})^{-1}\mathbf{e}$ respectively. Denote by

$X(t)\#$ **fluid level**

$\varphi(t)\#$**phase** of the underlying Markov process

$N(t)\#$**queue size**

$\vartheta(t)\#$**phase** of the vacation period if the server is on vacation mode

at time t.

Analysis

Assume that the underlying process $\varphi(t)$ has state space $S_{\varphi(t)}$ and generator $Q_{\varphi(t)}$.

Analysis

Assume that the underlying process $\varphi(t)$ has state space $S_{\varphi(t)}$ and generator $Q_{\varphi(t)}$.
Let $R_{\varphi(t)} = diag(c_k)$, having order $|S_{\varphi(t)}|$.

Analysis

Assume that the underlying process $\varphi(t)$ has state space $S_{\varphi(t)}$ and generator $Q_{\varphi(t)}$.
Let $R_{\varphi(t)} = diag(c_k)$, having order $|S_{\varphi(t)}|$.
In order to simplify the analysis, we combine the independent processes $\varphi(t)$ and $N(t)$ together to obtain a CTMC $\{Y(t)\}$, given by

Analysis

Assume that the underlying process $\varphi(t)$ has state space $S_{\varphi(t)}$ and generator $Q_{\varphi(t)}$.
Let $R_{\varphi(t)} = diag(c_k)$, having order $|S_{\varphi(t)}|$.
In order to simplify the analysis, we combine the independent processes $\varphi(t)$ and $N(t)$ together to obtain a CTMC $\{Y(t)\}$, given by

$$Y(t) = (\varphi(t), N(t)),$$

which is having state space over vacation period as $S_{\varphi(t)} \times S_N$

Analysis

Assume that the underlying process $\varphi(t)$ has state space $S_{\varphi(t)}$ and generator $Q_{\varphi(t)}$.
Let $R_{\varphi(t)} = diag(c_k)$, having order $|S_{\varphi(t)}|$.
In order to simplify the analysis, we combine the independent processes $\varphi(t)$ and $N(t)$ together to obtain a CTMC $\{Y(t)\}$, given by

$$Y(t) = (\varphi(t), N(t)),$$

which is having state space over vacation period as $S_{\varphi(t)} \times S_N$

Analysis

Assume that the underlying process $\varphi(t)$ has state space $S_{\varphi(t)}$ and generator $Q_{\varphi(t)}$.
Let $R_{\varphi(t)} = diag(c_k)$, having order $|S_{\varphi(t)}|$.
In order to simplify the analysis, we combine the independent processes $\varphi(t)$ and $N(t)$ together to obtain a CTMC $\{Y(t)\}$, given by

$$Y(t) = (\varphi(t), N(t)),$$

which is having state space over vacation period as $S_{\varphi(t)} \times S_N$
and having state space over active period as $S_{\varphi(t)} \times (\{0\} \cup S_N)$ where $S_N = \{1, 2, 3, ..., N\}$.

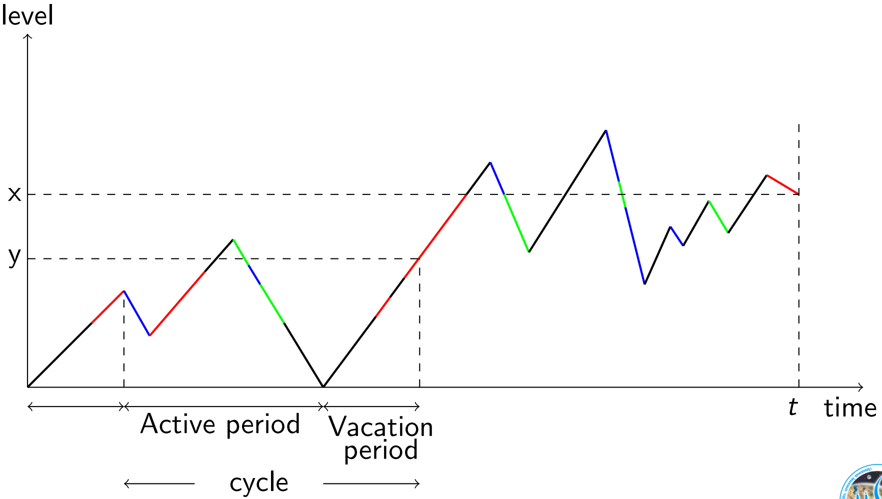
Analysis

Assume that the underlying process $\varphi(t)$ has state space $S_{\varphi(t)}$ and generator $Q_{\varphi(t)}$.
Let $R_{\varphi(t)} = diag(c_k)$, having order $|S_{\varphi(t)}|$.
In order to simplify the analysis, we combine the independent processes $\varphi(t)$ and $N(t)$ together to obtain a CTMC $\{Y(t)\}$, given by

$$Y(t) = (\varphi(t), N(t)),$$

which is having state space over vacation period as $S_{\varphi(t)} \times S_N$ and having state space over active period as $S_{\varphi(t)} \times (\{0\} \cup S_N)$ where $S_N = \{1, 2, 3, ..., N\}$.

Possible evolution of the fluid level



Analysis

Identify each of the two component state vector (k, n) corresponding to active period with a state variable j and arrange them in lexicographical order to get the set of states S.

Analysis

Identify each of the two component state vector (k, n) corresponding to active period with a state variable j and arrange them in lexicographical order to get the set of states S.
During active period the fluid is consumed at rate $d > 0$ if there is at least one customer in the system.

Analysis

Identify each of the two component state vector (k, n) corresponding to active period with a state variable j and arrange them in lexicographical order to get the set of states S.
During active period the fluid is consumed at rate $d > 0$ if there is at least one customer in the system.
Thus the rate at which the fluid level in the buffer changes during active period when $Y(t) = (k, n) \cong j$ is given by

Analysis

Identify each of the two component state vector (k, n) corresponding to active period with a state variable j and arrange them in lexicographical order to get the set of states S.
During active period the fluid is consumed at rate $d > 0$ if there is at least one customer in the system.
Thus the rate at which the fluid level in the buffer changes during active period when $Y(t) = (k, n) \cong j$ is given by

$$r_j = \begin{cases} c_k - d & n > 0 \\ c_k & n = 0 \end{cases}$$

Analysis

Identify each of the two component state vector (k, n) corresponding to active period with a state variable j and arrange them in lexicographical order to get the set of states S.

During active period the fluid is consumed at rate $d > 0$ if there is at least one customer in the system.

Thus the rate at which the fluid level in the buffer changes during active period when $Y(t) = (k, n) \cong j$ is given by

$$r_j = \begin{cases} c_k - d & n > 0 \\ c_k & n = 0 \end{cases}$$

Analysis

Now we partition S into two sets namely,
$S_+ = \{i \in S, r_i > 0\}$ and $S_- = \{i \in S, r_i < 0\}$ according to the signs assumed by r_i and arrange the states $(k, n) \cong j$ in S as $\{S_+, S_-\}$.

Analysis

Now we partition S into two sets namely,
$S_+ = \{i \in S, r_i > 0\}$ and $S_- = \{i \in S, r_i < 0\}$ according to the signs assumed by r_i and arrange the states $(k, n) \cong j$ in S as $\{S_+, S_-\}$.
Suppose that corresponding to vacation period also we are keeping the same order of arrangement of states, as mentioned above, for $Y(t)$.

Analysis

Now we partition S into two sets namely,
$S_+ = \{i \in S, r_i > 0\}$ and $S_- = \{i \in S, r_i < 0\}$ according to the signs assumed by r_i and arrange the states $(k, n) \cong j$ in S as $\{S_+, S_-\}$.
Suppose that corresponding to vacation period also we are keeping the same order of arrangement of states, as mentioned above, for $Y(t)$.
In line with the above order of arrangement, let the states in $S_{\varphi(t)}$ be arranged as $\{S^+_{\varphi(t)}, S^-_{\varphi(t)}\}$ and matrix $R_{\varphi(t)}$ be partitioned as

$$R_{\varphi(t)} = \begin{bmatrix} \mathbf{R}^+_{\varphi(\mathbf{t})} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^-_{\varphi(\mathbf{t})} \end{bmatrix}$$

Analysis

Now we partition S into two sets namely,
$S_+ = \{i \in S, r_i > 0\}$ and $S_- = \{i \in S, r_i < 0\}$ according to the signs assumed by r_i and arrange the states $(k, n) \cong j$ in S as $\{S_+, S_-\}$.
Suppose that corresponding to vacation period also we are keeping the same order of arrangement of states, as mentioned above, for $Y(t)$.
In line with the above order of arrangement, let the states in $S_{\varphi(t)}$ be arranged as $\{S^+_{\varphi(t)}, S^-_{\varphi(t)}\}$ and matrix $R_{\varphi(t)}$ be partitioned as

$$R_{\varphi(t)} = \begin{bmatrix} \mathbf{R}^+_{\varphi(\mathbf{t})} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^-_{\varphi(\mathbf{t})} \end{bmatrix}$$

The generator matrix of $Y(t)$ over vacation period corresponding to state space $V = S_{\varphi(t)} \times S_N = \{S^+_{\varphi(t)}, S^-_{\varphi(t)}\} \times S_N$ is $\mathbf{Q}_v = \mathbf{Q}_{\varphi(t)} \oplus \mathbf{Q}^v_{N(t)}$

Analysis

Now we partition S into two sets namely,
$S_+ = \{i \in S, r_i > 0\}$ and $S_- = \{i \in S, r_i < 0\}$ according to the signs assumed by r_i and arrange the states $(k, n) \cong j$ in S as $\{S_+, S_-\}$.
Suppose that corresponding to vacation period also we are keeping the same order of arrangement of states, as mentioned above, for $Y(t)$.
In line with the above order of arrangement, let the states in $S_{\varphi(t)}$ be arranged as $\{S^+_{\varphi(t)}, S^-_{\varphi(t)}\}$ and matrix $R_{\varphi(t)}$ be partitioned as

$$R_{\varphi(t)} = \begin{bmatrix} \mathbf{R}^+_{\varphi(\mathbf{t})} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^-_{\varphi(\mathbf{t})} \end{bmatrix}$$

The generator matrix of $Y(t)$ over vacation period corresponding to state space $V = S_{\varphi(t)} \times S_N = \{S^+_{\varphi(t)}, S^-_{\varphi(t)}\} \times S_N$ is $\mathbf{Q}_v = \mathbf{Q}_{\varphi(t)} \oplus \mathbf{Q}^v_{N(t)}$
and the same over active period corresponding to state space
$S_* = S_{\varphi(t)} \times \{\{0\}, S_N\} = \{S^+_{\varphi(t)}, S^-_{\varphi(t)}\} \times \{\{0\}, S_N\}$ is $\mathbf{Q}_s = \mathbf{Q}_{\varphi(t)} \oplus \mathbf{Q}^s_{N(t)}$

Analysis

Now we partition S into two sets namely,
$S_+ = \{i \in S, r_i > 0\}$ and $S_- = \{i \in S, r_i < 0\}$ according to the signs assumed by r_i and arrange the states $(k, n) \cong j$ in S as $\{S_+, S_-\}$.
Suppose that corresponding to vacation period also we are keeping the same order of arrangement of states, as mentioned above, for $Y(t)$.
In line with the above order of arrangement, let the states in $S_{\varphi(t)}$ be arranged as $\{S^+_{\varphi(t)}, S^-_{\varphi(t)}\}$ and matrix $R_{\varphi(t)}$ be partitioned as

$$R_{\varphi(t)} = \begin{bmatrix} \mathbf{R}^+_{\varphi(\mathbf{t})} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^-_{\varphi(\mathbf{t})} \end{bmatrix}$$

The generator matrix of $Y(t)$ over vacation period corresponding to state space $V = S_{\varphi(t)} \times S_N = \{S^+_{\varphi(t)}, S^-_{\varphi(t)}\} \times S_N$ is $\mathbf{Q}_v = \mathbf{Q}_{\varphi(t)} \oplus \mathbf{Q}^v_{N(t)}$ and the same over active period corresponding to state space $S_* = S_{\varphi(t)} \times \{\{0\}, S_N\} = \{S^+_{\varphi(t)}, S^-_{\varphi(t)}\} \times \{\{0\}, S_N\}$ is $\mathbf{Q}_s = \mathbf{Q}_{\varphi(t)} \oplus \mathbf{Q}^s_{N(t)}$ where $\mathbf{Q}_{\varphi(t)}$ is the generator of $\varphi(t)$ corresponding to state space $\{S^+_{\varphi(t)}, S^-_{\varphi(t)}\}$ and $\mathbf{Q}^v_{N(t)}$ and $\mathbf{Q}^s_{N(t)}$ are generators of $N(t)$ over vacation and active period respectively.

Analysis

Now we partition S into two sets namely,
$S_+ = \{i \in S, r_i > 0\}$ and $S_- = \{i \in S, r_i < 0\}$ according to the signs assumed by r_i and arrange the states $(k, n) \cong j$ in S as $\{S_+, S_-\}$.
Suppose that corresponding to vacation period also we are keeping the same order of arrangement of states, as mentioned above, for $Y(t)$.
In line with the above order of arrangement, let the states in $S_{\varphi(t)}$ be arranged as $\{S^+_{\varphi(t)}, S^-_{\varphi(t)}\}$ and matrix $R_{\varphi(t)}$ be partitioned as

$$R_{\varphi(t)} = \begin{bmatrix} \mathbf{R}^+_{\varphi(\mathbf{t})} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^-_{\varphi(\mathbf{t})} \end{bmatrix}$$

The generator matrix of $Y(t)$ over vacation period corresponding to state space $V = S_{\varphi(t)} \times S_N = \{S^+_{\varphi(t)}, S^-_{\varphi(t)}\} \times S_N$ is $\mathbf{Q}_v = \mathbf{Q}_{\varphi(t)} \oplus \mathbf{Q}^v_{N(t)}$ and the same over active period corresponding to state space $S_* = S_{\varphi(t)} \times \{\{0\}, S_N\} = \{S^+_{\varphi(t)}, S^-_{\varphi(t)}\} \times \{\{0\}, S_N\}$ is $\mathbf{Q}_s = \mathbf{Q}_{\varphi(t)} \oplus \mathbf{Q}^s_{N(t)}$ where $\mathbf{Q}_{\varphi(t)}$ is the generator of $\varphi(t)$ corresponding to state space $\{S^+_{\varphi(t)}, S^-_{\varphi(t)}\}$ and $\mathbf{Q}^v_{N(t)}$ and $\mathbf{Q}^s_{N(t)}$ are generators of $N(t)$ over vacation and active period respectively.

Analysis

Then generator matrix Q and fluid rate matrix R of $(Y(t), \vartheta(t))$ are given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_v \oplus \mathbf{A} & \mathbf{I} \otimes \mathbf{a} \\ \mathbf{0} & \mathbf{Q_s} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_{\varphi(t)} \otimes \mathbf{I}_N \otimes \mathbf{I}_{n_{PH}} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\varphi(t)} \otimes \mathbf{I}_{(N+1)} - \mathbf{I} \otimes \mathbf{D_{N(t)}} \end{bmatrix}$$

where

$$\mathbf{D_{N(t)}} = diag(d_n) \text{ is of order } N+1 \text{ with } d_n = \begin{cases} 0 & \text{if} \quad n = 1 \\ d & \text{if} \quad 1 < n \leq N+1 \end{cases}$$

Now let us arrange the state space of the process $(Y(t), \vartheta(t))$ as

Analysis

Then generator matrix Q and fluid rate matrix R of $(Y(t), \vartheta(t))$ are given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_v \oplus \mathbf{A} & \mathbf{I} \otimes \mathbf{a} \\ \mathbf{0} & \mathbf{Q_s} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_{\varphi(t)} \otimes \mathbf{I}_N \otimes \mathbf{I}_{n_{PH}} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\varphi(t)} \otimes \mathbf{I}_{(N+1)} - \mathbf{I} \otimes \mathbf{D_{N(t)}} \end{bmatrix}$$

where

$$\mathbf{D_{N(t)}} = diag(d_n) \text{ is of order } N+1 \text{ with } d_n = \begin{cases} 0 & \text{if } \ n = 1 \\ d & \text{if } \ 1 < n \leq N+1 \end{cases}$$

Now let us arrange the state space of the process $(Y(t), \vartheta(t))$ as
$\Theta = \{S^+_{\varphi(t)} \times S_N \times S_{PH}, S^-_{\varphi(t)} \times S_N \times S_{PH}, S_{\varphi(t)} \times \{0\}, S^+_{\varphi(t)} \times S_N, S^-_{\varphi(t)} \times S_N\}$

Analysis

Then generator matrix Q and fluid rate matrix R of $(Y(t), \vartheta(t))$ are given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_v \oplus \mathbf{A} & \mathbf{I} \otimes \mathbf{a} \\ \mathbf{0} & \mathbf{Q_s} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_{\varphi(t)} \otimes \mathbf{I}_N \otimes \mathbf{I}_{n_{PH}} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\varphi(t)} \otimes \mathbf{I}_{(N+1)} - \mathbf{I} \otimes \mathbf{D_{N(t)}} \end{bmatrix}$$

where

$$\mathbf{D_{N(t)}} = diag(d_n) \text{ is of order } N+1 \text{ with } d_n = \begin{cases} 0 & \text{if } \; n = 1 \\ d & \text{if } \; 1 < n \leq N+1 \end{cases}$$

Now let us arrange the state space of the process $(Y(t), \vartheta(t))$ as
$\Theta = \{S^+_{\varphi(t)} \times S_N \times S_{PH}, S^-_{\varphi(t)} \times S_N \times S_{PH}, S_{\varphi(t)} \times \{0\}, S^+_{\varphi(t)} \times S_N, S^-_{\varphi(t)} \times S_N\}$
where S_{PH} stands for the set of transient states of PH vacation variable.

Analysis

Then generator matrix Q and fluid rate matrix R of $(Y(t), \vartheta(t))$ are given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_v \oplus \mathbf{A} & \mathbf{I} \otimes \mathbf{a} \\ \mathbf{0} & \mathbf{Q_s} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_{\varphi(t)} \otimes \mathbf{I}_N \otimes \mathbf{I}_{n_{PH}} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\varphi(t)} \otimes \mathbf{I}_{(N+1)} - \mathbf{I} \otimes \mathbf{D_{N(t)}} \end{bmatrix}$$

where

$$\mathbf{D_{N(t)}} = diag(d_n) \text{ is of order } N+1 \text{ with } d_n = \begin{cases} 0 & \text{if } \; n = 1 \\ d & \text{if } \; 1 < n \leq N+1 \end{cases}$$

Now let us arrange the state space of the process $(Y(t), \vartheta(t))$ as
$\Theta = \{S^+_{\varphi(t)} \times S_N \times S_{PH}, S^-_{\varphi(t)} \times S_N \times S_{PH}, S_{\varphi(t)} \times \{0\}, S^+_{\varphi(t)} \times S_N, S^-_{\varphi(t)} \times S_N\}$
where S_{PH} stands for the set of transient states of PH vacation variable.
Here note that the first two blocks of states correspond to vacation period and the rest to active period.

Analysis

Then generator matrix Q and fluid rate matrix R of $(Y(t), \vartheta(t))$ are given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_v \oplus \mathbf{A} & \mathbf{I} \otimes \mathbf{a} \\ \mathbf{0} & \mathbf{Q_s} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_{\varphi(t)} \otimes \mathbf{I}_N \otimes \mathbf{I}_{n_{PH}} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\varphi(t)} \otimes \mathbf{I}_{(N+1)} - \mathbf{I} \otimes \mathbf{D_{N(t)}} \end{bmatrix}$$

where

$$\mathbf{D_{N(t)}} = diag(d_n) \text{ is of order } N+1 \text{ with } d_n = \begin{cases} 0 & \text{if } \; n = 1 \\ d & \text{if } \; 1 < n \leq N+1 \end{cases}$$

Now let us arrange the state space of the process $(Y(t), \vartheta(t))$ as
$\Theta = \{S^+_{\varphi(t)} \times S_N \times S_{PH}, S^-_{\varphi(t)} \times S_N \times S_{PH}, S_{\varphi(t)} \times \{0\}, S^+_{\varphi(t)} \times S_N, S^-_{\varphi(t)} \times S_N\}$
where S_{PH} stands for the set of transient states of PH vacation variable.
Here note that the first two blocks of states correspond to vacation period and the rest to active period.
Technically, this arrangement can be achieved by using a transformation with an appropriate permutation matrix **P**.

Analysis

Then generator matrix Q and fluid rate matrix R of $(Y(t), \vartheta(t))$ are given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_v \oplus \mathbf{A} & \mathbf{I} \otimes \mathbf{a} \\ \mathbf{0} & \mathbf{Q_s} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_{\varphi(t)} \otimes \mathbf{I}_N \otimes \mathbf{I}_{n_{PH}} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\varphi(t)} \otimes \mathbf{I}_{(N+1)} - \mathbf{I} \otimes \mathbf{D_{N(t)}} \end{bmatrix}$$

where

$$\mathbf{D_{N(t)}} = diag(d_n) \text{ is of order } N+1 \text{ with } d_n = \begin{cases} 0 & \text{if } \ n = 1 \\ d & \text{if } \ 1 < n \leq N+1 \end{cases}$$

Now let us arrange the state space of the process $(Y(t), \vartheta(t))$ as
$\Theta = \{S^+_{\varphi(t)} \times S_N \times S_{PH}, S^-_{\varphi(t)} \times S_N \times S_{PH}, S_{\varphi(t)} \times \{0\}, S^+_{\varphi(t)} \times S_N, S^-_{\varphi(t)} \times S_N\}$
where S_{PH} stands for the set of transient states of PH vacation variable.
Here note that the first two blocks of states correspond to vacation period and the rest to active period.
Technically, this arrangement can be achieved by using a transformation with an appropriate permutation matrix **P**.

Under this transformation, the matrices R and Q assume the forms

$$\mathbf{C} = \mathbf{P}\mathbf{R}\mathbf{P}^{-1} = \left[\begin{array}{c|c} \mathbf{C}^{+} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{C}^{-} \end{array}\right] =$$

$$\left[\begin{array}{ccccc|c} \mathbf{R}^{+}_{\varphi(t)} \otimes \mathbf{I}_N \otimes \mathbf{I}_{n_{PH}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{-}_{\varphi(t)} \otimes \mathbf{I}_N \otimes \mathbf{I}_{n_{PH}} & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_{\varphi(t)} & \mathbf{0} & & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{R}^{+}_{\varphi(t)} - d\mathbf{I}) \otimes \mathbf{I}_N & & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & & (\mathbf{R}^{-}_{\varphi(t)} - d\mathbf{I}) \otimes \mathbf{I}_N \end{array}\right]$$

and

$$\mathbf{T} = \mathbf{P}\mathbf{Q}\mathbf{P}^{-1} = \left[\begin{array}{c|c} \mathbf{T}^{++} & \mathbf{T}^{+-} \\ \hline \mathbf{T}^{-+} & \mathbf{T}^{--} \end{array}\right].$$

Analysis

The first four diagonal matrix blocks of **C** exhibit positive fluid rates, while the last block negative rates.

Analysis

The first four diagonal matrix blocks of **C** exhibit positive fluid rates, while the last block negative rates.

This way the first four matrix blocks of **C** and **T** are associated with the states in Θ^+ and the last one with the states in Θ^-. Consequently, $|\Theta^+| = |S^+_{\varphi(t)}| \cdot N \cdot n_{PH} + |S^-_{\varphi(t)}| \cdot N \cdot n_{PH} + |S_{\varphi(t)}| + |S^+_{\varphi(t)}| \cdot N$ and

Analysis

The first four diagonal matrix blocks of **C** exhibit positive fluid rates, while the last block negative rates.

This way the first four matrix blocks of **C** and **T** are associated with the states in Θ^+ and the last one with the states in Θ^-. Consequently, $|\Theta^+| = |S^+_{\varphi(t)}| \cdot N \cdot n_{PH} + |S^-_{\varphi(t)}| \cdot N \cdot n_{PH} + |S_{\varphi(t)}| + |S^+_{\varphi(t)}| \cdot N$ and $|\Theta^-| = |S^-_{\varphi(t)}| \cdot N$.

Stability Condition

Let $\bar{\pi}_{\varphi(t)}$ and $\bar{\pi}_{N(t)}$ be the steady state probability vectors of the Markov processes $\varphi(t)$ and $N(t)$ respectively. That is,

$$\bar{\pi}_{\varphi(t)}\mathbf{Q}_{\varphi(t)} = \mathbf{0} \quad , \quad \bar{\pi}_{\varphi(t)}\mathbf{e} = 1$$

and

$$\bar{\pi}_{N(t)}\mathbf{Q}^{s}_{N(t)} = \mathbf{0} \quad , \quad \bar{\pi}_{N(t)}\mathbf{e} = 1.$$

Mean fluid inflow rate

$$\lambda_{in} = \bar{\pi}_{\varphi(t)}\mathbf{R}_{\varphi(t)}\mathbf{e}$$

Effective mean fluid outflow rate

$$d_{out} = \bar{\pi}_{N(t)}\mathbf{D}_{N(t)}\mathbf{e}.$$

Stability condition: $\rho = \frac{\lambda_{in}}{d_{out}} < 1.$

Mean cycle length and fluid density in steady state regime

Theorem 1

Stationary distribution of $Y(t) = (\varphi(t), N(t))$ *at a vacation start epoch is given by* $\mathbf{m} = [\mathbf{m}^+, \mathbf{m}^-] = [\mathbf{0}, \mathbf{m}^-]$ *where* $\mathbf{m}^-$ *satisfies*

$$\mathbf{m}^- = \mathbf{m}^- \left[\begin{array}{cccc} \mathbf{0} & \mathbf{I}_{|S^-_{\varphi(t)}|} \otimes \mathbf{I}_N \otimes \alpha & \mathbf{0} & \mathbf{0} \end{array} \right] \boldsymbol{\Psi}$$

with normalizing condition $\mathbf{m}^- \mathbf{e}_{|S^-_{\varphi(t)}| \cdot N} = 1.$

Here matrix $\boldsymbol{\Psi} = (\Psi_{ij})$ is given by

$$\Psi_{ij} = \Pr[Z(\gamma) = j \mid X(0) = 0, Z(0) = i], \quad \text{for} \quad i \in \Theta^+, j \in \Theta^-$$

where $\gamma = inf\{t > 0; X(t) = 0\}$ is the first passage time to level 0 and Ψ is the solution of the Ricatti equation

$$\boldsymbol{\Psi}\left(-\mathbf{C}^-\right)^{-1}\mathbf{T}_{-+}\boldsymbol{\Psi} + \boldsymbol{\Psi}\left(-\mathbf{C}^-\right)^{-1}\mathbf{T}_{--} + \left(\mathbf{C}^+\right)^{-1}\mathbf{T}_{++}\boldsymbol{\Psi} + \left(\mathbf{C}^+\right)^{-1}\mathbf{T}_{+-} = \mathbf{0}.$$

Theorem 2

The steady state mean cycle time $\mathbf{c}$ *and the vector density* $\mathbf{q}(.)$ *of fluid level at an arbitrary epoch are given by*

$$\mathbf{c} = \mathbf{m}^{-} \begin{bmatrix} \mathbf{0} & \mathbf{I}_{|S^{-}_{\varphi(t)}|} \otimes \mathbf{I}_N \otimes \alpha & \mathbf{0} & \mathbf{0} \end{bmatrix} (-\mathbf{K})^{-1} \begin{bmatrix} \mathbf{I}_{|\Theta^{+}|} & \Psi \end{bmatrix} |\mathbf{C}|^{-1} \mathbf{e}_{|\Theta|} \text{ and}$$

$$\mathbf{q}(x) = \frac{1}{c} \quad \mathbf{m}^{-} \begin{bmatrix} \mathbf{0} & \mathbf{I}_{|S^{-}_{\varphi(t)}|} \otimes \mathbf{I}_N \otimes \alpha & \mathbf{0} & \mathbf{0} \end{bmatrix} e^{\mathbf{K}x} \begin{bmatrix} \mathbf{I}_{|\Theta^{+}|} & \Psi \end{bmatrix} |\mathbf{C}|^{-1} \mathbf{P} \begin{bmatrix} \mathbf{I}_{|S_{\varphi(t)}|} \otimes \mathbf{e}_N \otimes \mathbf{e}_{n_{PH}} \\ \mathbf{I}_{|S_{\varphi(t)}|} \otimes \mathbf{e}_{N+1} \end{bmatrix}.$$

Proof:

$$c = \lim_{\ell\to\infty} \sum_{i\in\Theta} E\left(\gamma \mid Z\left(t^m(\ell)\right) = i, X\left(t^m(\ell)\right) = 0\right) \Pr\left(Z\left(t^m(\ell)\right) = i\right)$$

$$= \lim_{\ell\to\infty} \sum_{i\in\Theta} \int_{t=t^m(\ell)}^{\infty} \Pr\left(\gamma > t - t^m(\ell) \mid Z\left(t^m(\ell)\right) = i, X\left(t^m(\ell)\right) = 0\right) dt \Pr\left(Z\left(t^m(\ell)\right) = i\right)$$

$$= \mathbf{m}^- \begin{bmatrix} \mathbf{0} & \mathbf{I}_{|S^-_{\varphi(t)}|} \otimes \mathbf{I}_N \otimes \alpha & \mathbf{0} & \mathbf{0} \end{bmatrix} \int_{x=0}^{\infty} \mathbf{G}(x)\mathbf{e}$$

$$= \mathbf{m}^- \begin{bmatrix} \mathbf{0} & \mathbf{I}_{|S^-_{\varphi(t)}|} \otimes \mathbf{I}_N \otimes \alpha & \mathbf{0} & \mathbf{0} \end{bmatrix} (-\mathbf{K})^{-1} \begin{bmatrix} \mathbf{I}_{|\Theta^+|} & \mathbf{\Psi} \end{bmatrix} |\mathbf{C}|^{-1} \mathbf{e}_{|\Theta|}.$$

Here $\mathbf{G}(x) = [\mathbf{G}_{i,j}(x)]$, where

$$\mathbf{G}_{i,j}(x) = \frac{d}{dx} \int_{t=0}^{\infty} \Pr(\gamma > t, Z(t) = j, X(t) < x \mid Z(0) = i, X(0) = 0) dt,$$

of size $|\Theta^+| \times |\Theta|$ can be computed as

$$\mathbf{G}(x) = e^{\mathbf{K}x} \begin{bmatrix} \mathbf{I} & \mathbf{\Psi} \end{bmatrix} |\mathbf{C}|^{-1},$$

where $\mathbf{K}$, of size $|\Theta^+| \times |\Theta^+|$, is given by $\mathbf{K} = \mathbf{\Psi}\left(-\mathbf{C}^-\right)^{-1} \mathbf{T}_{-+} + \left(\mathbf{C}^+\right)^{-1} \mathbf{T}_{++}$.

The vector density of fluid level at arbitrary epoch is the normalized fluid level during the stationary cycle.So

$$\mathbf{q}(x) = \frac{1}{c} \quad \mathbf{m}^{-} \begin{bmatrix} \mathbf{0} & \mathbf{I}_{|S^{-}_{\varphi(t)}|} \otimes \mathbf{I}_N \otimes \alpha & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{G}(x) \begin{bmatrix} \mathbf{I}_{|S_{\varphi(t)}|} \otimes \mathbf{e}_N \otimes \mathbf{e}_{n_{PH}} \\ \mathbf{I}_{|S_{\varphi(t)}|} \otimes \mathbf{e}_{N+1} \end{bmatrix}$$

Theorem 3

The vector Laplace transform of the stationary fluid level,

$$\mathbf{q}^{*}(v) = \frac{1}{c}\mathbf{m}^{-} \begin{bmatrix} \mathbf{0} & \mathbf{I}_{|S^{-}_{\varphi(t)}|} \otimes \mathbf{I}_N \otimes \alpha & \mathbf{0} & \mathbf{0} \end{bmatrix} (v\mathbf{I}_{|\Theta^{+}|} - \mathbf{K})^{-1} \begin{bmatrix} \mathbf{I}_{|\Theta^{+}|} & \Psi \end{bmatrix} |\mathbf{C}|^{-1}\mathbf{P} \begin{bmatrix} \mathbf{I}_{|S_{\varphi(t)}|} \otimes \mathbf{e}_N \otimes \mathbf{e}_{n_{PH}} \\ \mathbf{I}_{|S_{\varphi(t)}|} \otimes \mathbf{e}_{N+1} \end{bmatrix}.$$

Mean fluid level

Theorem 4

The n^{th} order stationary moment of the fluid level is given by

$$\mathbf{q^{(n)}} = \frac{n!}{c}\mathbf{m}^{-} \begin{bmatrix} \mathbf{0} & \mathbf{I}_{|S^{-}_{\varphi(t)}|} \otimes \mathbf{I}_N \otimes \alpha & \mathbf{0} & \mathbf{0} \end{bmatrix} (-\mathbf{K})^{-1-n} \begin{bmatrix} \mathbf{I}_{|\Theta^{+}|} & \Psi \end{bmatrix} |\mathbf{C}|^{-1}\mathbf{P} \begin{bmatrix} \mathbf{I}_{|S_{\varphi(t)}|} \otimes \mathbf{e}_N \otimes \mathbf{e}_{n_{PH}} \\ \mathbf{I}_{|S_{\varphi(t)}|} \otimes \mathbf{e}_{N+1} \end{bmatrix}$$

In particular, the stationary mean fluid level

$$\mathbf{q^{(1)}} = \frac{1}{c} m^{-}\mathbf{W}(-\mathbf{K})^{-2} \begin{bmatrix} \mathbf{I}_{|\Theta^{+}|} & \mathbf{\Psi} \end{bmatrix} |\mathbf{C}|^{-1}\mathbf{M}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{|S^{-}_{\varphi(t)}|} \otimes \mathbf{I}_N \otimes \alpha & \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ and } \mathbf{M} = \mathbf{P} \begin{bmatrix} \mathbf{I}_{|S_{\varphi(t)}|} \otimes \mathbf{e}_N \otimes \mathbf{e}_{n_{PH}} \\ \mathbf{I}_{|S_{\varphi(t)}|} \otimes \mathbf{e}_{N+1} \end{bmatrix}$$

Queue size distribution

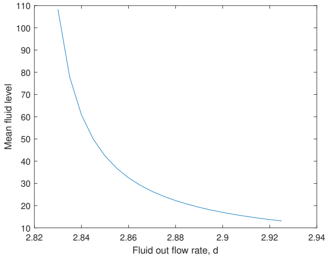
Theorem 5

The steady state distribution of queue size $p = (p_0, p_1, p_2, \ldots, p_N)$, *where* $p_n = \lim_{t\to\infty} p_n(t) = \lim_{t\to\infty} \Pr\left[N(t) = n\right]$, *is given by*

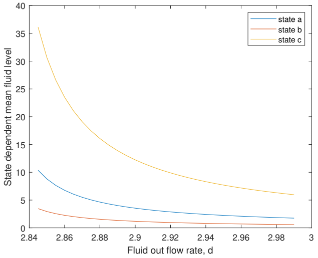
$$\mathbf{p} = \frac{1}{c}\mathbf{m}^{-}\left[\begin{array}{cccc} \mathbf{0} & \mathbf{I}_{|S^{-}_{\varphi(t)}|} \otimes \mathbf{I}_N \otimes \alpha & \mathbf{0} & \mathbf{0} \end{array}\right](-\mathbf{K})^{-1}\left[\begin{array}{cc} \mathbf{I}_{|\Theta^{+}|} & \Psi \end{array}\right]|\mathbf{C}|^{-1}$$

$$\mathbf{P}\left[\frac{\left[\mathbf{0}_{|S_{\varphi(t)}|\cdot N\cdot n_{PH}}\right]\left[\mathbf{e}_{|S_{\varphi(t)}|} \otimes \mathbf{I}_N \otimes \mathbf{e}_{n_{PH}}\right]}{\left[\mathbf{e}_{|S_{\varphi(t)}|} \otimes \mathbf{I}_{N+1}\right]}\right].$$

Numerical example



(a) Mean fluid level as the function of fluid outflow rate



(b) state dependent mean fluid level as a function of fluid outflow rate

Figure: Mean fluid level versus fluid consumption rate

Here we assume PH distributed vacation time with representation $(\alpha, \mathbf{A})$, where $\alpha = (0.2, 0.8)$ and $\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}$.

Let $\mathcal{S}_{\varphi(t)} = \{a, b, c\}, \mathbf{Q}_{\varphi(\mathbf{t})} = \begin{bmatrix} -8 & 4 & 4 \\ 3 & -12 & 9 \\ 2 & 0 & -2 \end{bmatrix}, \quad \mathbf{R}_{\varphi(\mathbf{t})} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

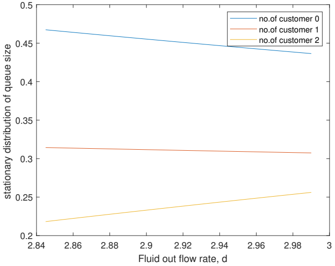


Figure: Stationary queue size distribution versus fluid consumption rate

From these model parameters we have
$\sigma = E(\widetilde{\sigma}) = 0.4, \pi_{\varphi(t)} = \{0.206897, 0.0689655, 0.724138\}$, and
$\lambda_{in} = \bar{\pi}_{\varphi(t)}\mathbf{R}_{\varphi(t)}\mathbf{e} = 1.482758$. Let $\lambda = 2$, $\mu = 3$ and $N = 2$.

When $d > 3$, all effective fluid flow rates are negative (That is, $r_i = c_k - d < 0 \ \forall i \in \mathcal{S}$) and hence $\mathcal{S}^- = \mathcal{S}_{\varphi(t)} \times \{1,2\}, \mathcal{S}^+ = S_{\varphi(t)} \times \{0\}$. When $2 < d < 3$, the effective fluid flow rate of state a namely, $3 - d$, is only positive so that $\mathcal{S}^- = \{b, c\} \times \{1,2\}$ and $\mathcal{S}^+ = (\{a, b, c\} \times (\{0\}, \{a\} \times \{1,2\})$. The system is stable when $\rho = \frac{\lambda_{in}}{d_{out}} < 1$, that is when $d_{out} > \lambda_{in} = 1.48276$. When $d = 3$ or $d = 2$, we have a state with zero effective fluid flow rate and the present results are not applicable in these cases. These cases can be addressed only by using an extended model to cover the zero fluid rate case, which is not considered here. Figure 1(a) and Figure 1(b) depict how the overall mean fluid level and the mean fluid level corresponding to each of the background process states are varying with respect to the changes in the values of d. Both these measures are decreasing with the increasing values of d, as expected.

Figure 2 exhibits the variation of stationary queue size distribution $p = (p_0, p_1, p_2)$ according to the variation in the values of d. As fluid consumption rate increases, the chance that the fluid level becomes zero is more, which may result in the accumulation of more number of customers in the system. So the probability that the system is empty (p_0) is decreasing and the probability of seeing the system full (p_2) is increasing with increasing values of d, as is clear from Figure 2.

References

[1] V.Ramaswami, Matrix analytic methods for stochastic fluid flows, Proceedings of the 16th International Teletrafic Congress, 1999

[2] N. Bean, M. O'Reilly, and P.G. Taylor. Algorithms for return probabilities for stochastic fluid flows. Comm. Statist. Stochastic Models,2005,21(1):149-184.

[3] A. da Silva Soares. Fluid Queues: Building upon the analogy with QBD processes. PhD thesis, Université Libre de Bruxelles, 2005.

[4] A. da Silva Soares, G. Latouche, Matrix-analytic methods for fluid queues with finite buffers, Perform. Eval. 63 (4–5) (2006) 295–314.

[5] G. Horváth and M. Telek. Exhaustive fluid vacation model with positive fluid rate during service. Performance Evaluation, 91:286–302, 2015.

[6] Z. Saffer and M. Telek, Exhaustive fluid vacation model with markov modulated load, Performance Evaluation, 98 (2016), 19-35.