$M/M/1/N$ Queues with Energy Enabled Service and General Vacation Times

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We consider the performance analysis of an M/M/1/N queueing system where the server consumes energy from a battery during the service of customers. The energy supply of the battery depends on a randomly changing environment. When the battery gets empty, the server goes on vacation for a random amount of time. We model the energy level of the battery as the fluid in an infinite fluid buffer and the environment as a continuous-time Markov chain(CTMC).

The analytical framework resembles the one used in fluid vacation models with exhaustive discipline, but the properties of the considered queueing model require the extension of the available methodology, because the model evolution is different during the service and the vacation period due to the inactivity of the server during vacation.

Essentially, new results are derived to cope with the general properties of the considered model. Consequently, the results in this paper extend the analysis of fluid vacation models to a more general class than the fluid vacation models with exhaustive discipline.

The steady-state vector density of the fluid level, its Laplace transform, and the mean fluid level are derived together with the probability mass function of the queue size. A special case of the model with phase-type (PH)-distributed vacation time is also analyzed.

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1 INTRODUCTION

A conventional fluid flow model has a buffer, finite or infinite, of fluid where the rates at which the fluid flows to the buffer (inflow rates) and the rates at which the fluid flows out of the buffer (outflow rates) are governed by a continuous-time Markov chain. The classical method for the determination of the stationary density for the infinite buffer model, as in the seminal paper by Anick et al. [1], involved a spectral analysis of the system. The solution thus obtained is expressed in terms of linear combinations of exponentials of the eigenvalues of the system. However, since the eigenvalues are of both signs, the usual numerical procedures are unstable. In [2] and [3] the authors established a connection between analytical approaches to fluid flow models and Quasi Birth and Death (QBD) models by reducing the continuous state-space problem of the fluid model to the discrete state space problem of an associated QBD. Building on those results, da Silva Soares and Latouche ([4],[5]), by using matrix analytical methods, obtained a representation

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of the fluid model in terms of a QBD, which also improved the computational efficiency and stability of the numerical methods used for finding various performance measures of fluid flow models.

In line with the classical queueing models with vacation, the fluid models with vacation were introduced in [6]. Fluid vacation models with gated and exhaustive disciplines have been analyzed in [7], [8], and [9], where the systems that govern the fluid inflow process during service and vacation were assumed to be identical, the fluid outflow rate was a positive constant during service and zero during vacation, and the net fluid rate (inflow − outflow) during service was negative. These assumptions allowed the use of the descendant set approach, which is traditionally employed in discrete queuing models. The assumption of a negative fluid rate during service has been relaxed in [10], where various performance measures were introduced and evaluated based on the matrix analytic approach. This work greatly builds on the performance measures computed there.

In this work, we consider an M/M/1/N queueing model where the server consumes energy during service, which is modeled as a fluid outflow from a fluid buffer. The fluid buffer is filled by a fluid inflow process, which is governed by a continuous-time Markov chain describing the changes of the random environment. The analysis of this model requires further extension of the available solution methods because the fluid outflow rate is not constant during the service period of the fluid vacation model due to the fact that the $M/M/1/N$ queue might become idle (all previously arrived customers are served) and, in this case, the server does not consume energy. That is, we adopt the terminology that within the service period of the fluid vacation model, the server of the $M/M/1/N$ queue might have many busy-idle cycles. In addition to the model description, the new contribution of the paper starts in Section 2.5 with Theorem 2.3.Theorem 2.3, which is a key result for the evaluation of the considered queueing system, presents an identity that was not available in the previous literature and inhibited the analysis of more general vacation models.The performance measures subsequently derived, which are based on Theorem 2.3, share some similarities, but are more general than the ones derived in [10]. In this way, the results in this paper generalize the results of [10] to a more general set of vacation models.

In summary, the paper carries modeling and methodological novelty. The novelty of modeling is the consideration of the energy budget of a server which consumes energy during service and has a time-varying energy supply. As a methodological novelty, the paper relaxes the existing restriction on the fluid vacation models, that the fluid-level governing process is identical during the vacation and the service period.

The rest of this paper is organized as follows. A mathematical description of the model and its analysis are given in Section 2. Sections 2.1 to 2.4, detail the model behavior and collect the necessary analytical results from the existing literature, while Section 2.5 presents new results that are essential for the analysis of the queueing system considered in this paper. The results in the consecutive subsection are based on those provided in Section 2.5. Section 3 deals with a special case of the model, where vacation time is assumed as the PH distributed instead of the general vacation time in the model considered in Section 2. A detailed analysis of this special case using an approach which is entirely different from the one used in Section 2 is given in Section 3. Finally, Section 4 shows a numerical illustration of the theoretical results which are derived in the previous sections.

1.1 Notations

We denote the $(i, j)^{th}$ element of the matrix $\mathbf{X}$ as $\mathbf{X}_{i,j}$. Likewise, $\mathbf{x}_j$ represents the j-th element of the vector $\mathbf{x}$. $\mathbf{X}^*(s)$, with $Re(s) \geq 0$, represents the matrix Laplace Transform (LT) of the matrix function $\mathbf{X}(x)$, defined as $\mathbf{X}^*(s) = \int_0^\infty e^{-sx}\mathbf{X}(x)dx$. For $k \geq 0$, $\mathbf{X}^{(k)}$ is defined as $\mathbf{X}^{(k)} = (-1)^k \frac{d^k}{ds^k}\mathbf{X}^*(s)|_{s=0}$. Similar conventions are used for vector LT and scalar LT. $\mathbb{1}$

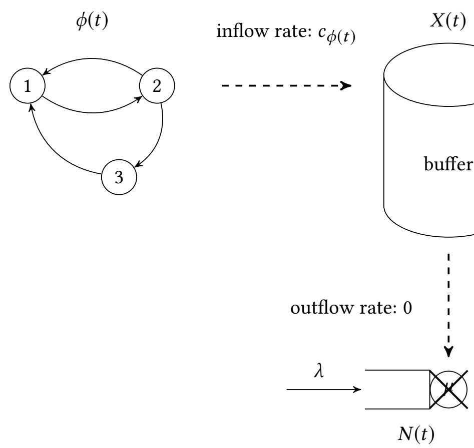

Fig. 1. System behavior during vacation period

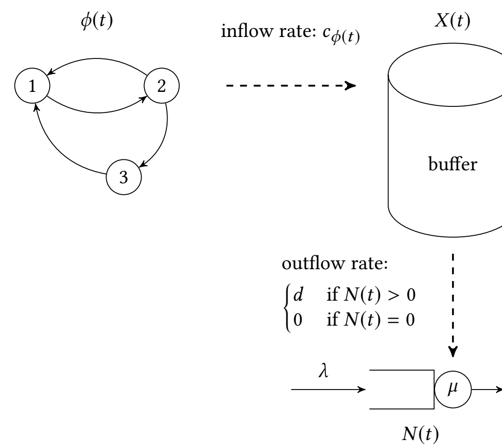

Fig. 2. System behavior during service period

denotes a column vector of ones of appropriate order. $|\bullet|$ represents the cardinality of a set when $\bullet$ is a set and it represents the matrix of absolute values of the elements when $\bullet$ is a matrix. $\mathbf{I}_n$ denotes the identity matrix of size n.

2 MODEL DESCRIPTION AND ANALYSIS

We consider a $M/M/1/N$ queue, equipped with an infinite-capacity fluid buffer, where the server consumes fluid at a constant rate (for example, in the form of energy) during customer service. Customers arrive to the $M/M/1/N$ queue at rate λ and are served at rate μ. The fluid inflow to the fluid buffer is governed by a continuous time Markov chain $\{\varphi(t)\}$, having finite state space $\mathcal{S}_\varphi$ and a generator matrix $\mathbf{Q}_\varphi$. While $\varphi(t) = k$, fluid flows into the buffer at rate c_k and during customer service, fluid flows out of the buffer at rate d, where$c_k > 0$ for $\forall k \in \mathcal{S}_\varphi$ and $d > 0$. Let $\mathbf{R}_\varphi = \text{diag}\langle c_k|_{k\in\mathcal{S}_\varphi}\rangle$.

Once server starts the service of customers it will go on continuously as long as there are customers to serve and the fluid buffer is nonempty. In each instance where the fluid buffer becomes empty, the server goes on vacation and waits for a random amount of time without serving customers to accumulate fluid. This period is called *the vacation period*. The length of the vacation period, denoted by σ, follows a general distribution with density $\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Pr(\sigma < t)$ and Laplace transform (LT) $\sigma^*(s) = E(e^{-s\sigma})$. At the end of the vacation period, the server resumes customer service that was suspended when the fluid buffer became empty. The period between consecutive vacation periods is called the *service period*. During a service period, the server is busy (servers a customer) as long as there is a customer in the queue. Figures 1 and 2 depict the behavior of the system during vacation and service periods, respectively.

The considered system is a Markov process with state variables

$X(t)$ - fluid level ($X(t) \geq 0$)

$\varphi(t)$ - *phase* of the inflow modulating Markov chain ($\varphi(t) \in \mathcal{S}_\varphi$),

$N(t)$ - number of customers in the queue ($N(t) \in \{0, 1, \ldots, N\}$),

$V(t)$ - vacation indicator ($V(t) \in \{0, 1\}$).

In order to simplify the analysis, we introduce $Y(t) = (N(t), \varphi(t))$, where $Y(t) \in \mathcal{S} = \{0, 1, \ldots, N\} \times \mathcal{S}_\varphi$. During a vacation period, the queue size is non-zero due to the presence of the customer whose service got interrupted when the fluid buffer became empty and consequently $Y(t) \neq (0, k)$ for $\forall k \in \mathcal{S}_\varphi$ when $V(t) = 1$. In contrast, it is possible that the queue is idle during the service period, when all previously arrived customers are served.

During a service period, when $V(t) = 0$, $Y(t) = (n, k)$ and $X(t) > 0$ the fluid level changes at rate

$$r_{(n,k)} = \begin{cases} c_k - d & \text{for } n > 0 \\ c_k & \text{for } n = 0. \end{cases} \tag{1}$$

During a vacation period, when $V(t) = 1$ and $Y(t) = (n, k)$ the fluid level changes at rate

$$r^v_{(n,k)} = c_k.$$

In the applied notation the subscript v refers to the vacation period, but to simplify the notation we omit the service period-related subscripts from the notations associated with the service period.

2.1 System description during vacation period

During the vacation period, the number of customers in the queue increases by one if a new arrival occurs and $N(t) < N$ and the level of the fluid increases at a rate c_k when $\varphi(t) = k$. That is the generator matrix and the fluid rate matrix are $\hat{\mathbf{Q}}_v = \mathbf{Q}^v_N \oplus \mathbf{Q}_\varphi$ and $\hat{\mathbf{R}}_v = \text{diag}\langle r^v_{(n,k)} \rangle = \mathbf{I}_{N+1} \otimes \mathbf{R}_\varphi$, where $\otimes$ and $\oplus$ stand for Kronecker product and summation, respectively, and$\mathbf{Q}^v_N$ of size $N + 1 \times N + 1$ is

$$\mathbf{Q}^v_N = \begin{bmatrix} -\lambda & \lambda & & & \\ & -\lambda & \lambda & & \\ & & \ddots & \ddots & \\ & & & -\lambda & \lambda \\ & & & & 0 \end{bmatrix}. \tag{2}$$

We note that $N(t) > 0$ during the vacation period, but the introduced matrices have compatible size with the matrices of the service period.

2.2 System behavior during vacation period

Let $X_v(t)$ be the fluid accumulated in the buffer until time t during a vacation, and let $\hat{\mathbf{A}}(t, x)$ the transition density matrix with elements

$$\hat{\mathbf{A}}_{i,j}(t, x) = \frac{\partial}{\partial x} \Pr\left(X_v(t) < x, Y_v(t) = j \mid X_v(0) = 0, Y_v(0) = i\right) \quad \text{for} \quad i, j \in \mathcal{S}.$$

The differential equations describing the evolution of $\hat{\mathbf{A}}(t, x)$ are (see [8])

$$\frac{\partial}{\partial t}\hat{\mathbf{A}}(t, x) + \frac{\partial}{\partial x}\hat{\mathbf{A}}(t, x)\hat{\mathbf{R}}_v = \hat{\mathbf{A}}(t, x)\hat{\mathbf{Q}}_v, \tag{3}$$

with the initial conditions

$$\hat{\mathbf{A}}(0, x) = \delta(0, x)\mathbf{I}_{|\mathcal{S}|} \text{ and } \hat{\mathbf{A}}(t, 0) = \mathbf{0}, \quad t > 0. \tag{4}$$

Here $\delta(0,x)$ denotes Kronecker delta function. The Laplace transform of differential equation (3) yields

$$\hat{\mathbf{A}}^{**}(s,z) = \int_{t=0}^{\infty}\int_{x=0}^{\infty} \hat{\mathbf{A}}(t,x)e^{-st}e^{-zx}\,\mathrm{d}x\,\mathrm{d}t = (s\mathbf{I}_{|\mathcal{S}|} + z\hat{\mathbf{R}}_v - \hat{\mathbf{Q}}_v)^{-1} = \hat{\mathbf{R}}_v^{-1}(s\hat{\mathbf{R}}_v^{-1} + z\mathbf{I}_{|\mathcal{S}|} - \hat{\mathbf{Q}}_v\hat{\mathbf{R}}_v^{-1})^{-1}. \tag{5}$$

The LT of $\hat{\mathbf{A}}(t,x)$ with respect to x is matrix exponential, and is given by

$$\hat{\mathbf{A}}^{*}(t,z) = \int_{x=0}^{\infty} \hat{\mathbf{A}}(t,x)e^{-zx}\,\mathrm{d}x = e^{(\hat{\mathbf{Q}}_v - z\hat{\mathbf{R}}_v)t} \tag{6}$$

and the LT of $\hat{\mathbf{A}}(t,x)$ with respect to t is also matrix exponential

$$\hat{\mathbf{A}}^{*}(s,x) = \int_{t=0}^{\infty} \hat{\mathbf{A}}(t,x)e^{-st}\,\mathrm{d}t = \hat{\mathbf{R}}_v^{-1}e^{(\hat{\mathbf{Q}}_v\hat{\mathbf{R}}_v^{-1} - s\hat{\mathbf{R}}_v^{-1})x}. \tag{7}$$

2.3 System description during service period

During the service period, the system behavior is more complicated because both the fluid level and the number of customers can increase and decrease. The size $(N+1)\times(N+1)$ generator matrix characterizing the number of customers in the queue is

$$\mathbf{Q}_N = \begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -\lambda-\mu & \lambda & & \\ & \ddots & \ddots & \ddots & \\ & & \mu & -\lambda-\mu & \lambda \\ & & & \mu & -\mu \end{bmatrix}, \tag{8}$$

the generator matrix of $Y(t)$ is $\hat{\mathbf{Q}} = \mathbf{Q}_N \oplus \mathbf{Q}_\varphi$ and the associated fluid rate matrix is

$$\hat{\mathbf{R}} = \mathrm{diag}\langle r_{(n,k)}\rangle = \mathbf{I}_{N+1}\otimes\mathbf{R}_\varphi - d(\mathbf{I}_{N+1} - \mathbf{e}_1\mathbf{e}_1^T)\otimes\mathbf{I}_{|\mathcal{S}_\varphi|},$$

where the $r_{(n,k)}$ elements are defined in (1) and $\mathbf{e}_i = [0,\ldots,0,1,0,\ldots,0]^T$ denotes the i-th unit column vector whose only non-zero element is the ith element. For later use we define $\hat{\mathbf{R}}_\delta = (\mathbf{I}_{N+1} - \mathbf{e}_1\mathbf{e}_1^T)\otimes\mathbf{I}_{|\mathcal{S}_\varphi|}$ and note that

$$\hat{\mathbf{R}}_v - \hat{\mathbf{R}} = \left(\mathbf{I}_{N+1}\otimes\mathbf{R}_\varphi\right) - \left(\mathbf{I}_{N+1}\otimes\mathbf{R}_\varphi - d(\mathbf{I}_{N+1} - \mathbf{e}_1\mathbf{e}_1^T)\otimes\mathbf{I}_{|\mathcal{S}_\varphi|}\right) = d\,\hat{\mathbf{R}}_\delta. \tag{9}$$

Assumption 1. *To avoid the introduction of additional notational complexity, we assume that $c_k \neq d$ for all $k \in \mathcal{S}_\varphi$.*

That is, the fluid rate is either positive or negative. Based on this assumption, we partition $\mathcal{S} = \{0,1,\ldots,N\}\times\mathcal{S}_\varphi$ into two disjoint subsets $\mathcal{S}^+ = \{(n,k) : r_{(n,k)} > 0\}$ and $\mathcal{S}^- = \{(n,k) : r_{(n,k)} < 0\}$. With the help of the permutation matrix $\mathbf{P}$, we order the states in $\mathcal{S}$ so that the indices of the states in $\mathcal{S}^+$ are less than the indices of the states in $\mathcal{S}^-$. That is

$$\mathbf{Q} = \mathbf{P}\hat{\mathbf{Q}}\mathbf{P}^T = \begin{bmatrix} \mathbf{Q}^{++} & \mathbf{Q}^{+-} \\ \mathbf{Q}^{-+} & \mathbf{Q}^{--} \end{bmatrix}, \qquad \mathbf{R} = \mathbf{P}\hat{\mathbf{R}}\mathbf{P}^T = \begin{bmatrix} \mathbf{R}^{+} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{-} \end{bmatrix}. \tag{10}$$

We apply the same permutation also to the matrices of the vacation period

$$\mathbf{Q}_v = \mathbf{P}\hat{\mathbf{Q}}_v\mathbf{P}^T, \quad \mathbf{R}_v = \mathbf{P}\hat{\mathbf{R}}_v\mathbf{P}^T, \quad \mathbf{A}(t,x) = \mathbf{P}\hat{\mathbf{A}}(t,x)\mathbf{P}^T, \quad \mathbf{A}^*(t,z) = \mathbf{P}\hat{\mathbf{A}}^*(t,z)\mathbf{P}^T, \quad \mathbf{R}_\delta = \mathbf{P}\hat{\mathbf{R}}_\delta\mathbf{P}^T. \tag{11}$$

Additionally, we define a scaled version of the fluid model with characterizing matrices

$$\tilde{\mathbf{Q}} = |\mathbf{R}|^{-1}\mathbf{Q}, \quad \tilde{\mathbf{R}} = |\mathbf{R}|^{-1}\mathbf{R} = \begin{bmatrix} \mathbf{I}^{+} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}^{-} \end{bmatrix}, \tag{12}$$

where $|\mathbf{R}|$ refers to the element-wise absolute values of matrix $\mathbf{R}$, $\mathbf{I}^{+}$ and $\mathbf{I}^{-}$ are the identity matrices of size $|\mathcal{S}^{+}|$ and $|\mathcal{S}^{-}|$, respectively.

2.4 System behavior during service period

For $i \in \mathcal{S}$ the joint distribution of the fluid level and the state of the Markov process $Y(t)$ at time t is given by

$$\boldsymbol{\pi}_i(x,t) = \frac{\partial}{\partial x}\Pr\left(X(t) \le x, Y(t) = i\right).$$

2.4.1 Condition of stability. Let $\boldsymbol{\pi}_\varphi$ and $\boldsymbol{\pi}_N$ be the steady-state probability vectors of the Markov processes with generators $\mathbf{Q}_\varphi$ and $\mathbf{Q}_N$, respectively. That is, $\boldsymbol{\pi}_\varphi \mathbf{Q}_\varphi = \mathbf{0}$, $\boldsymbol{\pi}_\varphi \mathbb{1} = 1$ and $\boldsymbol{\pi}_N \mathbf{Q}_N = \mathbf{0}$, $\boldsymbol{\pi}_N \mathbb{1} = 1$. The condition of stability of the fluid vacation model is

$$(\boldsymbol{\pi}_N \otimes \boldsymbol{\pi}_\varphi)\hat{\mathbf{R}}\mathbb{1} < 0. \tag{13}$$

In this paper, we assume the fluid vacation model to be stable.

2.4.2 Fundamental matrices $\boldsymbol{\Psi}, \mathbf{K}, \mathbf{U}$. In order to analyze the system behavior during service, we need some system characteristics of the finite buffer Markov fluid model, which we provide here for completeness. The derivation of these quantities requires fundamental matrices of Markov fluid models $\boldsymbol{\Psi}$, $\mathbf{K}$, and $\mathbf{U}$, which were introduced and studied in detail in [4] and [5], as well as similar matrices associated with the level-reversed process.

Matrix $\boldsymbol{\Psi}$. The return probability matrix is defined as

$$\Psi_{ij} = \Pr[\gamma^{(0)} < \infty, Y(\gamma^{(0)}) = j \mid X(0) = 0, Y(0) = i], \quad \text{for} \quad i \in \mathcal{S}^{+}, j \in \mathcal{S}^{-},$$

where $\gamma^{(x)} = \inf\{t > 0; X(t) = x\}$ is the first time when fluid level is x. For a stable Markov fluid model $\boldsymbol{\Psi}\mathbb{1} = \mathbb{1}$. Matrix $\boldsymbol{\Psi}$ is the minimal non-negative solution to the non-symmetric algebraic Riccati equation (NARE)

$$\boldsymbol{\Psi}\tilde{\mathbf{Q}}^{-+}\boldsymbol{\Psi} + \boldsymbol{\Psi}\tilde{\mathbf{Q}}^{--} + \tilde{\mathbf{Q}}^{++}\boldsymbol{\Psi} + \tilde{\mathbf{Q}}^{+-} = \mathbf{0}, \tag{14}$$

where $\tilde{\mathbf{Q}}^{++}$, $\tilde{\mathbf{Q}}^{+-}$, $\tilde{\mathbf{Q}}^{-+}$ and $\tilde{\mathbf{Q}}^{--}$ are defined by (10) and (12).

Matrix $\mathbf{K}$. The (i,j) entry of the matrix $e^{\mathbf{K}x}$, is the expected number of crossings of the fluid level x in phase $j \in \mathcal{S}^{+}$ starting from level 0 in phase $i \in \mathcal{S}^{+}$, before returning to level 0. If the Markov fluid model is stable, then all eigenvalues of matrix $\mathbf{K}$ have negative real parts (hence it is nonsingular). The matrix $\mathbf{K}$ can be expressed as

$$\mathbf{K} = \tilde{\mathbf{Q}}^{++} + \boldsymbol{\Psi}\tilde{\mathbf{Q}}^{-+}.$$

Starting from level 0 the expected number of level crossings at level x for both positive and negative states is given by matrix $\mathbf{N}(x)$, as

$$\mathbf{N}(x) = e^{\mathbf{K}x}\begin{bmatrix}\mathbf{I}^{+} & \boldsymbol{\Psi}\end{bmatrix}. \tag{15}$$

Matrix $\mathbf{U}$. The "downward record" matrix is the generator of a CTMC, which characterizes the evolution of $Y(t)$ while it visits $\mathcal{S}^{-}$. If the Markov fluid model is stable, then matrix $\mathbf{U}$ is a proper generator matrix such that $\mathbf{U}\mathbb{1} = \mathbf{0}$ and consequently, $e^{\mathbf{U}x}\mathbb{1} = \mathbb{1}$. The matrix $\mathbf{U}$ can be expressed as

$$\mathbf{U} = \tilde{\mathbf{Q}}^{--} + \tilde{\mathbf{Q}}^{-+}\boldsymbol{\Psi}.$$

${e^{\mathbf{U}x}}_{ij}$ is the probability that the background process is in state $j \in \mathcal{S}^-$ when level 0 is hit for the first time, starting from phase $i \in \mathcal{S}^-$ and level $x > 0$. That is

$${e^{\mathbf{U}x}}_{ij} = \Pr[Y(\gamma^{(0)}) = j \mid X(0) = x, Y(0) = i], \quad x > 0.$$

Even though the present model has an infinite buffer capacity, its analysis during the service period requires the use of finite buffer model results. Therefore, in the next section we discuss the results related to the finite buffer model.

2.4.3 Markov fluid models with finite buffer. To analyze the behavior of Markov fluid models with finite buffer, it is necessary to consider the level-reversed process. The fundamental matrices corresponding to the level-reversed process denoted as $\hat{\mathbf{\Psi}}, \hat{\mathbf{K}}$ and $\hat{\mathbf{U}}$ are derived by swapping the roles of states in $\mathcal{S}^+$ and $\mathcal{S}^-$.

Hence, we have that matrix $\hat{\mathbf{\Psi}}$ is the solution to NARE

$$\hat{\mathbf{\Psi}}\tilde{\mathbf{Q}}^{+-}\hat{\mathbf{\Psi}} + \hat{\mathbf{\Psi}}\tilde{\mathbf{Q}}^{++} + \tilde{\mathbf{Q}}^{--}\hat{\mathbf{\Psi}} + \tilde{\mathbf{Q}}^{-+} = \mathbf{0}.$$

Matrices $\hat{\mathbf{K}}$ and $\hat{\mathbf{U}}$ are obtained by

$$\hat{\mathbf{K}} = \tilde{\mathbf{Q}}^{--} + \hat{\mathbf{\Psi}}\tilde{\mathbf{Q}}^{+-}, \quad \hat{\mathbf{U}} = \tilde{\mathbf{Q}}^{++} + \tilde{\mathbf{Q}}^{+-}\hat{\mathbf{\Psi}}.$$

If the Markov fluid model is stable then

- the level-reversed process (with infinite buffer) is a transient process and its fluid level increases to infinity. Consequently, the fluid level does not return to 0 with probability one, that is $\hat{\mathbf{\Psi}}\mathbb{1} \leq \mathbb{1}$,
- zero is an eigenvalue of $\hat{\mathbf{K}}$,
- $\hat{\mathbf{U}}$ is a transient generator, that is $\hat{\mathbf{U}}\mathbb{1} \leq 0$ and $e^{\hat{\mathbf{U}}x}\mathbb{1} \leq \mathbb{1}$.

The characterizing matrices of the level-forward and the level-reversed processes satisfy many important relations which were used for the analysis of fluid vacation models in [10].

Lemma 2.1. *[10] The fundamental matrices of the fluid queues satisfy*

$$(\mathbf{I}^- - \hat{\mathbf{\Psi}}\mathbf{\Psi})^{-1}\hat{\mathbf{K}} = \mathbf{U}(\mathbf{I}^- - \hat{\mathbf{\Psi}}\mathbf{\Psi})^{-1}, \tag{16}$$

$$(\mathbf{I}^+ - \mathbf{\Psi}\hat{\mathbf{\Psi}})^{-1}\mathbf{K} = \hat{\mathbf{U}}(\mathbf{I}^+ - \mathbf{\Psi}\hat{\mathbf{\Psi}})^{-1}, \tag{17}$$

$$\mathbf{Q}\begin{bmatrix}\mathbf{\Psi}\\ \mathbf{I}^-\end{bmatrix} = -\mathbf{R}\begin{bmatrix}\mathbf{\Psi}\\ \mathbf{I}^-\end{bmatrix}\mathbf{U}, \tag{18}$$

$$\mathbf{Q}\begin{bmatrix}\mathbf{I}^+\\ \hat{\mathbf{\Psi}}\end{bmatrix} = +\mathbf{R}\begin{bmatrix}\mathbf{I}^+\\ \hat{\mathbf{\Psi}}\end{bmatrix}\hat{\mathbf{U}}, \tag{19}$$

$$\mathbf{Q}\begin{bmatrix}\mathbf{\Psi}\\ \mathbf{I}^-\end{bmatrix}(\mathbf{I}^- - \hat{\mathbf{\Psi}}\mathbf{\Psi})^{-1} = -\mathbf{R}\begin{bmatrix}\mathbf{\Psi}\\ \mathbf{I}^-\end{bmatrix}(\mathbf{I}^- - \hat{\mathbf{\Psi}}\mathbf{\Psi})^{-1}\hat{\mathbf{K}}, \tag{20}$$

$$\mathbf{Q}\begin{bmatrix}\mathbf{I}^+\\ \hat{\mathbf{\Psi}}\end{bmatrix}(\mathbf{I}^+ - \mathbf{\Psi}\hat{\mathbf{\Psi}})^{-1} = +\mathbf{R}\begin{bmatrix}\mathbf{I}^+\\ \hat{\mathbf{\Psi}}\end{bmatrix}(\mathbf{I}^+ - \mathbf{\Psi}\hat{\mathbf{\Psi}})^{-1}\mathbf{K}, \tag{21}$$

and for $i \geq 0$

$$\mathbf{R}\left(\begin{bmatrix}\mathbf{I}^+\\ \hat{\mathbf{\Psi}}\end{bmatrix}(\mathbf{I}^+ - \mathbf{\Psi}\hat{\mathbf{\Psi}})^{-1}\mathbf{K}^i\begin{bmatrix}\mathbf{I}^+ & \mathbf{\Psi}\end{bmatrix} - \begin{bmatrix}\mathbf{\Psi}\\ \mathbf{I}^-\end{bmatrix}(\mathbf{I}^- - \hat{\mathbf{\Psi}}\mathbf{\Psi})^{-1}(-\hat{\mathbf{K}})^i\begin{bmatrix}\hat{\mathbf{\Psi}} & \mathbf{I}^-\end{bmatrix}\right)|\mathbf{R}|^{-1} = \left(\mathbf{Q}\mathbf{R}^{-1}\right)^i. \tag{22}$$

Lemma 2.1 is proved in [10]. Equations (16)-(21) contain a set of algebraic matrix identities, which can be used to eliminate a matrix multiplication form the right with $\mathbf{K}$, $\hat{\mathbf{K}}$, $\mathbf{U}$ and $\hat{\mathbf{U}}$. Equation (22), whose main application is in (23), is also based on the (16)-(21) matrix identities.

Multiplying (22) by $1/s^i$, summing up from $i = 0$ to ∞ and multiplying it with $\mathbf{R}^{-1}$ from the left gives

$$\left(\begin{bmatrix}\mathbf{I}^+\\ \hat{\mathbf{\Psi}}\end{bmatrix}(\mathbf{I}^+ - \mathbf{\Psi}\hat{\mathbf{\Psi}})^{-1}(s\mathbf{I}^+ - \mathbf{K})^{-1}\begin{bmatrix}\mathbf{I}^+ & \mathbf{\Psi}\end{bmatrix} - \begin{bmatrix}\mathbf{\Psi}\\ \mathbf{I}^-\end{bmatrix}(\mathbf{I}^- - \hat{\mathbf{\Psi}}\mathbf{\Psi})^{-1}(s\mathbf{I}^- + \hat{\mathbf{K}})^{-1}\begin{bmatrix}\hat{\mathbf{\Psi}} & \mathbf{I}^-\end{bmatrix}\right)|\mathbf{R}|^{-1}$$
$$= (s\mathbf{R} - \mathbf{Q})^{-1}. \tag{23}$$

Based on the matrix relations satisfied by the fundamental matrices, the following result is derived for the following level crossing measure in [10]. For $i, j \in \mathcal{S}$, let $\mathbf{M}_{i,j}(x, y)$ be the expected number of crossings of level y in state j starting from state i and fluid level x before the fluid buffer becomes empty for the first time. That is

$$\mathbf{M}_{i,j}(x,y) = \lim_{\Delta\to 0}\frac{1}{\Delta}E\left(\int_{t=0}^{\infty}\mathcal{I}\{t < \gamma^{(0)}, Y(t) = j, X(t) \in (y, y + \Delta r_j)\}\,\mathrm{d}t \mid Y(0) = i, X(0) = x\right), \tag{24}$$

where $\mathcal{I}\{\bullet\}$ is the indicator of event $\bullet$.

Theorem 2.2 ([10]). *The density of the fluid level during the active period given that the initial fluid level is x is obtained by*

$$\mathbf{M}(x,y) = -\begin{bmatrix}\mathbf{\Psi}\\ \mathbf{I}^-\end{bmatrix}(\mathbf{I}^- - \hat{\mathbf{\Psi}}\mathbf{\Psi})^{-1}e^{\hat{\mathbf{K}}x}\hat{\mathbf{\Psi}}e^{\mathbf{K}y}\begin{bmatrix}\mathbf{I}^+ & \mathbf{\Psi}\end{bmatrix} + \begin{bmatrix}\mathbf{\Psi}\\ \mathbf{I}^-\end{bmatrix}(\mathbf{I}^- - \hat{\mathbf{\Psi}}\mathbf{\Psi})^{-1}e^{\hat{\mathbf{K}}(x-y)}\begin{bmatrix}\hat{\mathbf{\Psi}} & \mathbf{I}^-\end{bmatrix} \tag{25}$$

for $0 < y < x$, and

$$\mathbf{M}(x,y) = -\begin{bmatrix}\mathbf{\Psi}\\ \mathbf{I}^-\end{bmatrix}(\mathbf{I}^- - \hat{\mathbf{\Psi}}\mathbf{\Psi})^{-1}e^{\hat{\mathbf{K}}x}\hat{\mathbf{\Psi}}e^{\mathbf{K}y}\begin{bmatrix}\mathbf{I}^+ & \mathbf{\Psi}\end{bmatrix} + \begin{bmatrix}\mathbf{I}^+\\ \hat{\mathbf{\Psi}}\end{bmatrix}(\mathbf{I}^+ - \mathbf{\Psi}\hat{\mathbf{\Psi}})^{-1}e^{\mathbf{K}(y-x)}\begin{bmatrix}\mathbf{I}^+ & \mathbf{\Psi}\end{bmatrix} \tag{26}$$

for $y > x$.

The consideration of the various cases when the process starts from fluid level x and stays at level y before getting idle results in Theorem 2.2. Unfortunately, the nice intuitive meaning of the initial equations in [10] got lost by the algebraic manipulations, resulting in the relatively simple to compute expressions in (25) and (26).

2.5 The stationary distribution of the fluid level

The stationary density of the fluid level $\boldsymbol{\pi}(y)$ with components $\boldsymbol{\pi}_\ell(y) = \frac{\mathrm{d}}{\mathrm{d}y}\lim_{t\to\infty}\Pr(X(t) < y, Y(t) = \ell)$ and the stationary density of the fluid level $\mathbf{q}(y)$ with components $\mathbf{q}_k(y) = \frac{\mathrm{d}}{\mathrm{d}y}\lim_{t\to\infty}\Pr(X(t) < y, \varphi(t) = k)$ are obtained as

$$\boldsymbol{\pi}(y) = \frac{1}{c}\left(L_v(y) + L_s(y)|\mathbf{R}|^{-1}\right) \tag{27}$$

and

$$\mathbf{q}(y) = \boldsymbol{\pi}(y)\left[\mathbb{1}_{N+1} \otimes \mathbf{I}_{|\mathcal{S}_\varphi|}\right],$$

where

$$L_v(y) = \int_{t=0}^{\infty}\sigma(t)\int_{x=0}^{t}\boldsymbol{\beta}\,\mathrm{A}(x,y)\,\mathrm{d}x\,\mathrm{d}t, \quad L_s(y) = \int_{t=0}^{\infty}\sigma(t)\int_{x=0}^{\infty}\boldsymbol{\beta}\,\mathrm{A}(t,x)\mathbf{M}(x,y)\,\mathrm{d}x\,\mathrm{d}t, \tag{28}$$

c is a normalizing constant, $\boldsymbol{\beta} = \boldsymbol{\beta}^- \begin{bmatrix} \mathbf{0} & \mathbf{I}^- \end{bmatrix}$ and $\boldsymbol{\beta}^-$ is the stationary distribution of the phase in $\mathcal{S}^-$ at the end of the service period. Note that the expressions for $\boldsymbol{\pi}(y)$ and $\mathbf{q}(y)$ are derived based on the fact that the density of the fluid level y and the expected number of crossings of level y in a stationary cycle are proportional and are related by the inverse of fluid rates (see [11]). Below we compute the unknowns of (27) and (28), $\boldsymbol{\beta}^-$ and c.

The computation of the overall fluid level distribution, $\mathbf{q}(y)$, is based on the state dependent measure $\boldsymbol{\pi}(y)$, which describe the fluid level at different system states. The system states belong to two subsets, vacation and service states, and the computation of the fluid-level distribution follows a different pattern for those states. $\frac{1}{c}L_v(y)$ and $\frac{1}{c}L_s(y)|\mathbf{R}|^{-1}$ describe the state-dependent fluid level in the vacation and service states, respectively. With the help of these quantities, one can gain practical information e.g., on the average fluid (energy) level in the system, the amount of fluid (energy) at the end of the vacation (purely charging) period, etc.

In order to derive the probability vector $\boldsymbol{\beta}^-$ appearing in equation (28), we need the following fundamental result.

Theorem 2.3. *Suppose the eigenvalues of matrix* $\mathbf{U}$ *have non-positive real parts. Then for any vector* $\boldsymbol{\beta}$ *of size* $|\mathcal{S}|$, *any matrix* Θ *of size* $|\mathcal{S}| \times |\mathcal{S}^-|$ *and matrix* $\mathbf{U}$ *of size* $|\mathcal{S}^-| \times |\mathcal{S}^-|$ ($|\mathcal{S}^-| \geq 1$), *we have*

$$\left(\int_{x=0}^{\infty} \boldsymbol{\beta}\mathbf{A}(t,x)\Theta e^{\mathbf{U}x} \, \mathrm{d}x\right)^T = (\boldsymbol{\beta} \otimes \mathbf{I}^-)\, e^{(\mathbf{Q}_v \otimes \mathbf{I}^- + \mathbf{R}_v \otimes \mathbf{U}^T)t} \, \mathrm{vec}(\Theta^T) \tag{29}$$

where vec() *is the column stacking vector operator.*

Proof. Multiplying both sides of (3) by $\Theta e^{\mathbf{U}x}$ and integrating from 0 to ∞, we get

$$\frac{\partial}{\partial t}\int_{x=0}^{\infty} \boldsymbol{\beta}\mathbf{A}(t,x)\Theta e^{\mathbf{U}x} \, \mathrm{d}x + \int_{x=0}^{\infty} \left(\frac{\partial}{\partial x}\boldsymbol{\beta}\mathbf{A}(t,x)\mathbf{R}_v\right)\Theta e^{\mathbf{U}x} \, \mathrm{d}x = \int_{x=0}^{\infty} \boldsymbol{\beta}\mathbf{A}(t,x)\mathbf{Q}_v\Theta e^{\mathbf{U}x} \, \mathrm{d}x. \tag{30}$$

The integration of the second term by parts leads to

$$\frac{\partial}{\partial t}\int_{x=0}^{\infty} \boldsymbol{\beta}\mathbf{A}(t,x)\Theta e^{\mathbf{U}x} \, \mathrm{d}x - \int_{x=0}^{\infty} \boldsymbol{\beta}\mathbf{A}(t,x)\mathbf{R}_v\Theta\mathbf{U}e^{\mathbf{U}x} \, \mathrm{d}x = \int_{x=0}^{\infty} \boldsymbol{\beta}\mathbf{A}(t,x)\mathbf{Q}_v\Theta e^{\mathbf{U}x} \, \mathrm{d}x \tag{31}$$

and

$$\frac{\partial}{\partial t}\int_{x=0}^{\infty} \boldsymbol{\beta}\mathbf{A}(t,x)\Theta e^{\mathbf{U}x} \, \mathrm{d}x = \int_{x=0}^{\infty} \boldsymbol{\beta}\mathbf{A}(t,x)\underbrace{(\mathbf{Q}_v\Theta + \mathbf{R}_v\Theta\mathbf{U})}_{\triangleq\hat{\Theta}} e^{\mathbf{U}x} \, \mathrm{d}x. \tag{32}$$

By taking transpose of both sides of (32) we get

$$\frac{\partial}{\partial t}\int_{x=0}^{\infty} e^{\mathbf{U}^T x}\Theta^T\mathbf{A}^T(t,x)\boldsymbol{\beta}^T \, \mathrm{d}x = \int_{x=0}^{\infty} e^{\mathbf{U}^T x}\hat{\Theta}^T\mathbf{A}^T(t,x)\boldsymbol{\beta}^T \, \mathrm{d}x.$$

Now using the column stacking vector operator relation $\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A})\,\mathrm{vec}(\mathbf{B})$, we have

$$\frac{\partial}{\partial t}\int_{x=0}^{\infty} \left(\boldsymbol{\beta}\mathbf{A}(t,x) \otimes e^{\mathbf{U}^T x}\right) \mathrm{d}x \, \mathrm{vec}(\Theta^T) = \int_{x=0}^{\infty} \left(\boldsymbol{\beta}\mathbf{A}(t,x) \otimes e^{\mathbf{U}^T x}\right) \mathrm{d}x \, \mathrm{vec}(\hat{\Theta}^T).$$

Finally, by using the relation $\mathbf{AC} \otimes \mathbf{BD} = (\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D})$ the above equation becomes

$$(\boldsymbol{\beta} \otimes \mathbf{I}^-)\,\frac{\partial}{\partial t}\underbrace{\int_{x=0}^{\infty} \left(\mathbf{A}(t,x) \otimes e^{\mathbf{U}^T x}\right) \mathrm{d}x}_{\triangleq\bar{\Theta}(t)} \, \mathrm{vec}(\Theta^T) = (\boldsymbol{\beta} \otimes \mathbf{I}^-)\underbrace{\int_{x=0}^{\infty} \left(\mathbf{A}(t,x) \otimes e^{\mathbf{U}^T x}\right) \mathrm{d}x}_{=\bar{\Theta}(t)} \, \mathrm{vec}(\hat{\Theta}^T), \tag{33}$$

whose solution is $\bar{\mathbf{\Theta}}(t) = e^{\mathbf{Z}t}$ with $\mathbf{Z} = \mathbf{Q}_v \otimes \mathbf{I}^- + \mathbf{R}_v \otimes \mathbf{U}^T$, since $\bar{\mathbf{\Theta}}(0) = \mathbf{I}$ by definition. Substituting this solution into the left-hand side of (33) gives

$$\begin{aligned}
(\boldsymbol{\beta} \otimes \mathbf{I}^-)\, e^{\mathbf{Z}t}\mathbf{Z}\,\mathrm{vec}(\mathbf{\Theta}^T) &= (\boldsymbol{\beta} \otimes \mathbf{I}^-)\, e^{\mathbf{Z}t} \left(\mathbf{Q}_v \otimes \mathbf{I}^- + \mathbf{R}_v \otimes \mathbf{U}^T\right)\ \mathrm{vec}(\mathbf{\Theta}^T) \\
&= (\boldsymbol{\beta} \otimes \mathbf{I}^-)\, e^{\mathbf{Z}t} \left(\ \mathrm{vec}(\mathbf{I}^-\mathbf{\Theta}^T{\mathbf{Q}_\mathbf{v}}^T) +\ \mathrm{vec}(\mathbf{U}^T\mathbf{\Theta}^T{\mathbf{R}_\mathbf{v}}^T)\right) = (\boldsymbol{\beta} \otimes \mathbf{I}^-)\, e^{\mathbf{Z}t}\ \mathrm{vec}(\hat{\mathbf{\Theta}}^T),
\end{aligned}$$

which is the right hand side of (33). Using $\bar{\mathbf{\Theta}}(t) = e^{\mathbf{Z}t}$, we have

$$\begin{aligned}
\left(\int_{x=0}^{\infty} \boldsymbol{\beta}\mathbf{A}(t,x)\mathbf{\Theta}e^{\mathbf{U}x}\ \mathrm{d}x\right)^T &= \int_{x=0}^{\infty} e^{\mathbf{U}^T x}\mathbf{\Theta}^T\mathbf{A}^T(t,x)\boldsymbol{\beta}^T\ \mathrm{d}x \\
= (\boldsymbol{\beta} \otimes \mathbf{I}^-) \int_{x=0}^{\infty} &\left(\mathbf{A}(t,x) \otimes e^{\mathbf{U}^T x}\right)\ \mathrm{d}x\ \mathrm{vec}(\mathbf{\Theta}^T) = (\boldsymbol{\beta} \otimes \mathbf{I}^-)\, \bar{\mathbf{\Theta}}(t)\ \mathrm{vec}(\mathbf{\Theta}^T) \\
= (\boldsymbol{\beta} \otimes \mathbf{I}^-)\, &e^{(\mathbf{Q}_v \otimes \mathbf{I}^- + \mathbf{R}_v \otimes \mathbf{U}^T)t}\ \mathrm{vec}(\mathbf{\Theta}^T),
\end{aligned}$$

which completes the proof. □

A straightforward consequence of Theorem 2.3 is the following corollary.

Corollary 2.4. *Under the conditions of Theorem 2.3, we have*

$$\left(\int_{t=0}^{\infty} \sigma(t) \int_{x=0}^{\infty} \boldsymbol{\beta}\mathbf{A}(t,x)\mathbf{\Theta}e^{\mathbf{U}x}\ \mathrm{d}x\ \mathrm{d}t\right)^T = (\boldsymbol{\beta} \otimes \mathbf{I}^-)\, \sigma^*(-\mathbf{Q}_v \otimes \mathbf{I}^- - \mathbf{R}_v \otimes \mathbf{U}^T)\ \mathrm{vec}(\mathbf{\Theta}^T). \tag{34}$$

where $\sigma^*(X) = \int_0^\infty \sigma(u)e^{-\mathbf{X}u}\,\mathrm{d}u$.

We define a matrix $\mathbf{W}$ of size $|\mathcal{S}| \times |\mathcal{S}^-|$ such that its ith row is $\left((\mathbf{e}_i \otimes \mathbf{I}^-)\, \sigma^*(-\mathbf{Q}_v \otimes \mathbf{I}^- - \mathbf{R}_v \otimes \mathbf{U}^T)\ \mathrm{vec}(\mathbf{\Theta}^T)\right)^T$, where $\mathbf{\Theta} = \begin{bmatrix}\mathbf{\Psi} \\ \mathbf{I}^-\end{bmatrix}$.

Then we have the following theorem.

Theorem 2.5. *The stationary distribution of the phase at the end of the service period is given by the solution of the linear equation*

$$\boldsymbol{\beta}^- \begin{bmatrix}0 & \mathbf{I}^-\end{bmatrix} \mathbf{W} = \boldsymbol{\beta}^- \tag{35}$$

with the normalizing condition $\boldsymbol{\beta}^-\mathbb{1} = 1$.

Proof. If the phase probability distribution at the end of the service period is $\boldsymbol{\beta}^-$ then the phase probability distribution at the beginning of a vacation period is $\boldsymbol{\beta} = \boldsymbol{\beta}^- \begin{bmatrix}\mathbf{0} & \mathbf{I}^-\end{bmatrix}$. Considering the amount of fluid accumulated during the vacation, we get the following relation

$$\boldsymbol{\beta} \int_{u=0}^{\infty} \sigma(u) \int_{x=0}^{\infty} \mathbf{A}(u,x)\mathbf{\Theta}e^{\mathbf{U}x}\ \mathrm{d}x\ \mathrm{d}u = \boldsymbol{\beta}^-, \tag{36}$$

where $\mathbf{\Theta} = \begin{bmatrix}\mathbf{\Psi} \\ \mathbf{I}^-\end{bmatrix}$. Then by using Corollary 2.4 and the definition of $\mathbf{W}$ the right hand side of (36) can be written as

$$\boldsymbol{\beta} \int_{u=0}^{\infty} \sigma(u) \int_{x=0}^{\infty} \mathbf{A}(u,x)\mathbf{\Theta}e^{\mathbf{U}x}\ \mathrm{d}x\ \mathrm{d}u\ = \left((\boldsymbol{\beta} \otimes \mathbf{I}^-)\, \sigma^*(-\mathbf{Q}_v \otimes \mathbf{I}^- - \mathbf{R}_v \otimes \mathbf{U}^T)\ \mathrm{vec}(\mathbf{\Theta}^T)\right)^T = \boldsymbol{\beta}\mathbf{W}.$$

□

Corollary 2.6. *For* $\boldsymbol{\beta} = \boldsymbol{\beta}^- \begin{bmatrix} 0 & \mathbf{I}^- \end{bmatrix}$ *we have*

$$\boldsymbol{\beta} \int_{u=0}^{\infty} \sigma(u) \int_{x=0}^{\infty} \mathbf{A}(u,x) \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{I}^- \end{bmatrix} (\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi})^{-1} e^{\hat{\mathbf{K}}x} \,\mathrm{d}x \,\mathrm{d}u = \boldsymbol{\beta}^- (\mathbf{I} - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi})^{-1}. \tag{37}$$

Proof. Applying (16) in equation (36) we get

$$\begin{aligned}
&\boldsymbol{\beta}^- \begin{bmatrix} 0 & \mathbf{I}^- \end{bmatrix} \int_{u=0}^{\infty} \sigma(u) \int_{x=0}^{\infty} \mathbf{A}(u,x) \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{I} \end{bmatrix} (\mathbf{I} - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi})^{-1} e^{\hat{\mathbf{K}}x} \,\mathrm{d}x \,\mathrm{d}u \\
&= \boldsymbol{\beta}^- \begin{bmatrix} 0 & \mathbf{I}^- \end{bmatrix} \int_{u=0}^{\infty} \sigma(u) \int_{x=0}^{\infty} \mathbf{A}(u,x) \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{I} \end{bmatrix} e^{\mathbf{U}x} \,\mathrm{d}x \,\mathrm{d}u (\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi})^{-1} = \boldsymbol{\beta}^- (\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi})^{-1}.
\end{aligned}$$

□

2.6 Fluid density level during vacation period

To compute the fluid level during the vacation period in Laplace transform domain, we need to integrate according to the lengths of the vacation period (characterized by $\sigma(u)$) and the fluid level (y) due to Laplace transformation.

$$\begin{aligned}
L_v^*(s) = \int_{y=0}^{\infty} e^{-sy} L_v(y) \,\mathrm{d}y &= \int_{u=0}^{\infty} \sigma(u) \boldsymbol{\beta} \int_{x=0}^{u} \int_{y=0}^{\infty} e^{-sy} \mathbf{A}(x,y) \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}u \\
&= \boldsymbol{\beta} \int_{u=0}^{\infty} \sigma(u) \int_{x=0}^{u} e^{(\mathbf{Q}_v - s\mathbf{R}_v)x} \,\mathrm{d}x \,\mathrm{d}u \\
&= \boldsymbol{\beta} \int_{u=0}^{\infty} \sigma(u) \left(e^{(\mathbf{Q}_v - s\mathbf{R}_v)u} - \mathbf{I}_{|\mathcal{S}|} \right) (\mathbf{Q}_v - s\mathbf{R}_v)^{-1} \,\mathrm{d}u \\
&= \boldsymbol{\beta} \left(\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|} \right) (\mathbf{Q}_v - s\mathbf{R}_v)^{-1}.
\end{aligned} \tag{38}$$

Equation (38) already indicates that the analysis relies on integral descriptions that need to be expressed in a closed form. One of the main difficulties in this process is the lack of a closed-form expression for $\mathbf{A}(x,y)$. Unfortunately, it is given only by the partial differential equation (3). Section 2.5 (Theorem 2.3 - Corollary 2.6) collects algebraic results which are used to obtain the closed-form equations based on (3).

2.7 Expected number of level crossing during service period

To compute the fluid level during the service period in Laplace transform domain, we need to integrate according to three variables: the lengths of the vacation period (characterized by $\sigma(u)$), the fluid level at the end of the vacation period (x), and the considered fluid level (y).

$L_s(y)$ is obtained by substituting (25) and (26) into the definition (27). For the LT of $L_s(y)$ we have

$$
\begin{aligned}
L_s^*(s) &= \int_{y=0}^{\infty} e^{-ys} L_s(y)\,\mathrm{d}y = \int_{u=0}^{\infty} \sigma(u) \int_{y=0}^{\infty} e^{-sy} \int_{x=0}^{\infty} \boldsymbol{\beta}\,\mathbf{A}(u,x)\mathbf{M}(x,y)\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}u \\
&= \underbrace{-\int_{u=0}^{\infty} \sigma(u) \int_{y=0}^{\infty} e^{-sy} \int_{x=0}^{\infty} \boldsymbol{\beta}\,\mathbf{A}(u,x) \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{I}^- \end{bmatrix} (\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi})^{-1} e^{\hat{\mathbf{K}}x} \hat{\boldsymbol{\Psi}} e^{\mathbf{K}y} \begin{bmatrix} \mathbf{I}^+ & \boldsymbol{\Psi} \end{bmatrix}\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}u}_{\triangleq L_1^*(s)} \\
&\quad \underbrace{+\int_{u=0}^{\infty} \sigma(u) \int_{y=0}^{\infty} e^{-sy} \int_{x=0}^{y} \boldsymbol{\beta}\,\mathbf{A}(u,x) \begin{bmatrix} \mathbf{I}^+ \\ \hat{\boldsymbol{\Psi}} \end{bmatrix} (\mathbf{I}^+ - \boldsymbol{\Psi}\hat{\boldsymbol{\Psi}})^{-1} e^{\mathbf{K}(y-x)} \begin{bmatrix} \mathbf{I}^+ & \boldsymbol{\Psi} \end{bmatrix}\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}u}_{\triangleq L_2^*(s)} \\
&\quad \underbrace{+\int_{u=0}^{\infty} \sigma(u) \int_{y=0}^{\infty} e^{-sy} \int_{x=y}^{\infty} \boldsymbol{\beta}\,\mathbf{A}(u,x) \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{I}^- \end{bmatrix} (\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi})^{-1} e^{\hat{\mathbf{K}}(x-y)} \begin{bmatrix} \hat{\boldsymbol{\Psi}} & \mathbf{I}^- \end{bmatrix}\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}u,}_{\triangleq L_3^*(s)} \qquad (39)
\end{aligned}
$$

Both, in (25) and (26), $\mathbf{M}(x,y)$ is composed of two terms. The identity of the first terms in (25) and (26), makes it possible to handle those terms in a unique way for $x \in (0,\infty)$ in $L_1^*(s)$, while $L_2^*(s)$ results from the second term of (26) and $L_3^*(s)$ results from the second term of (25).

Next, we use the algebraic results from Section 2.5 to express $L_1^*(s)$, $L_2^*(s)$ and $L_3^*(s)$ in close form. Using corollary 2.6 and $\boldsymbol{\beta}^- = \boldsymbol{\beta} \begin{bmatrix} \mathbf{0} \\ \mathbf{I}^- \end{bmatrix}$ the first term is expressed as

$$
\begin{aligned}
L_1^*(s) &= \int_{u=0}^{\infty} \sigma(u) \int_{x=0}^{\infty} \boldsymbol{\beta}\mathbf{A}(u,x) \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{I}^- \end{bmatrix} (\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi})^{-1} e^{\hat{\mathbf{K}}x} \hat{\boldsymbol{\Psi}} \int_{y=0}^{\infty} e^{-sy} e^{\mathbf{K}y} \begin{bmatrix} \mathbf{I}^+ & \boldsymbol{\Psi} \end{bmatrix} dy\, dx\, du \\
&= \boldsymbol{\beta} \int_{u=0}^{\infty} \sigma(u) \int_{x=0}^{\infty} \mathbf{A}(u,x) \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{I}^- \end{bmatrix} (\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi})^{-1} e^{\hat{\mathbf{K}}x} \hat{\boldsymbol{\Psi}} (s\mathbf{I}^+ - \mathbf{K})^{-1} \begin{bmatrix} \mathbf{I}^+ & \boldsymbol{\Psi} \end{bmatrix} dx\, du \\
&= \boldsymbol{\beta}^- (\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi})^{-1} \hat{\boldsymbol{\Psi}} (s\mathbf{I}^+ - \mathbf{K})^{-1} \begin{bmatrix} \mathbf{I}^+ & \boldsymbol{\Psi} \end{bmatrix} \\
&= \boldsymbol{\beta}^- \hat{\boldsymbol{\Psi}} (\mathbf{I}^+ - \boldsymbol{\Psi}\hat{\boldsymbol{\Psi}})^{-1} (s\mathbf{I}^+ - \mathbf{K})^{-1} \begin{bmatrix} \mathbf{I}^+ & \boldsymbol{\Psi} \end{bmatrix}. \qquad (40)
\end{aligned}
$$

where we used (37) in the last step.

$$
\begin{aligned}
L_2^*(s) &= \int_{u=0}^{\infty} \sigma(u) \int_{y=0}^{\infty} e^{-sy} \int_{x=0}^{y} \boldsymbol{\beta}\mathbf{A}(u,x) \begin{bmatrix} \mathbf{I}^+ \\ \hat{\boldsymbol{\Psi}} \end{bmatrix} (\mathbf{I}^+ - \boldsymbol{\Psi}\hat{\boldsymbol{\Psi}})^{-1} e^{\mathbf{K}(y-x)} \begin{bmatrix} \mathbf{I}^+ & \boldsymbol{\Psi} \end{bmatrix} dx\, dy\, du \\
&= \boldsymbol{\beta} \int_{u=0}^{\infty} \sigma(u) \int_{x=0}^{\infty} e^{-sx} \mathbf{A}(u,x) \begin{bmatrix} \mathbf{I}^+ \\ \hat{\boldsymbol{\Psi}} \end{bmatrix} (\mathbf{I}^+ - \boldsymbol{\Psi}\hat{\boldsymbol{\Psi}})^{-1} (s\mathbf{I}^+ - \mathbf{K})^{-1} \begin{bmatrix} \mathbf{I}^+ & \boldsymbol{\Psi} \end{bmatrix} dx\, du \\
&= \boldsymbol{\beta} \int_{u=0}^{\infty} \sigma(u) e^{(s\mathbf{R}_v - \mathbf{Q}_v)u} \begin{bmatrix} \mathbf{I}^+ \\ \hat{\boldsymbol{\Psi}} \end{bmatrix} (\mathbf{I}^+ - \boldsymbol{\Psi}\hat{\boldsymbol{\Psi}})^{-1} (s\mathbf{I}^+ - \mathbf{K})^{-1} \begin{bmatrix} \mathbf{I}^+ & \boldsymbol{\Psi} \end{bmatrix} du \\
&= \boldsymbol{\beta}\ \sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) \begin{bmatrix} \mathbf{I}^+ \\ \hat{\boldsymbol{\Psi}} \end{bmatrix} (\mathbf{I}^+ - \boldsymbol{\Psi}\hat{\boldsymbol{\Psi}})^{-1} (s\mathbf{I}^+ - \mathbf{K})^{-1} \begin{bmatrix} \mathbf{I}^+ & \boldsymbol{\Psi} \end{bmatrix}. \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
L_3^*(s) &= \int_{u=0}^{\infty} \sigma(u) \int_{y=0}^{\infty} e^{-sy} \int_{x=y}^{\infty} \boldsymbol{\beta}\mathbf{A}(u,x) \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{I}^- \end{bmatrix} \left(\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi}\right)^{-1} e^{\hat{\mathbf{K}}(x-y)} \begin{bmatrix} \hat{\boldsymbol{\Psi}} & \mathbf{I}^- \end{bmatrix} dx\, dy\, du \\
&= \boldsymbol{\beta} \int_{u=0}^{\infty} \sigma(u) \int_{x=0}^{\infty} e^{-sx} \mathbf{A}(u,x) \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{I}^- \end{bmatrix} \left(\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi}\right)^{-1} \int_{y=0}^{x} e^{s(x-y)} e^{\hat{\mathbf{K}}(x-y)} \begin{bmatrix} \hat{\boldsymbol{\Psi}} & \mathbf{I}^- \end{bmatrix} dy\, dx\, du \\
&= \boldsymbol{\beta} \int_{u=0}^{\infty} \sigma(u) \int_{x=0}^{\infty} e^{-sx} \mathbf{A}(u,x) \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{I}^- \end{bmatrix} \left(\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi}\right)^{-1} \left(\mathbf{I}^- - e^{(s\mathbf{I}^- + \hat{\mathbf{K}})x}\right) \left(-s\mathbf{I}^- - \hat{\mathbf{K}}\right)^{-1} \begin{bmatrix} \hat{\boldsymbol{\Psi}} & \mathbf{I}^- \end{bmatrix} dx\, du \\
&= \boldsymbol{\beta} \int_{u=0}^{\infty} \sigma(u) e^{(\mathbf{Q}_v - s\mathbf{R}_v)u} du \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{I}^- \end{bmatrix} \left(\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi}\right)^{-1} \left(-s\mathbf{I}^- - \hat{\mathbf{K}}\right)^{-1} \begin{bmatrix} \hat{\boldsymbol{\Psi}} & \mathbf{I}^- \end{bmatrix} \\
&\quad - \boldsymbol{\beta}^- \left(\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi}\right)^{-1} \left(-s\mathbf{I}^- - \hat{\mathbf{K}}\right)^{-1} \begin{bmatrix} \hat{\boldsymbol{\Psi}} & \mathbf{I}^- \end{bmatrix} \\
&= \left(\boldsymbol{\beta}\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) \begin{bmatrix} \boldsymbol{\Psi} \\ \mathbf{I}^- \end{bmatrix} - \boldsymbol{\beta}^-\right) \left(\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi}\right)^{-1} \left(-s\mathbf{I}^- - \hat{\mathbf{K}}\right)^{-1} \begin{bmatrix} \hat{\boldsymbol{\Psi}} & \mathbf{I}^- \end{bmatrix},
\end{aligned} \tag{42}
$$

where we used (37) in the third step.

2.8 LT of the stationary distribution of the fluid level

In spite of the algebraic complexity of $L_v^*(s)$, $L_1^*(s)$, $L_2^*(s)$ and $L_3^*(s)$, their sum form a rather simple expression for the stationary fluid level in the following theorem.

THEOREM 2.7. *The vector Laplace transform of the stationary distribution of the fluid level is given by*

$$
\boldsymbol{\pi}^*(s) = \frac{1}{c}\boldsymbol{\beta}\left(\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)\left((\mathbf{Q}_v - s\mathbf{R}_v)^{-1} + (s\mathbf{R} - \mathbf{Q})^{-1}\right). \tag{43}
$$

PROOF. The Laplace transform of (27) and (39) gives

$$
\boldsymbol{\pi}^*(s) = \frac{1}{c}L_v^*(s) + \frac{1}{c}L_s^*(s)|\mathbf{R}|^{-1} = \frac{1}{c}L_v^*(s) + \frac{1}{c}\left(-L_1^*(s) + L_2^*(s) + L_3^*(s)\right) \cdot |\mathbf{R}|^{-1}
$$

and using equations (38), (40), (41) and (42) we obtain

$$
\begin{aligned}
\boldsymbol{\pi}^*(s) &= \frac{1}{c}\boldsymbol{\beta}\left(\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)(\mathbf{Q}_v - s\mathbf{R}_v)^{-1} \\
&+ \frac{1}{c}\left(\boldsymbol{\beta}\,\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v)\begin{bmatrix}\mathbf{I}^+\\ \hat{\boldsymbol{\Psi}}\end{bmatrix} - \boldsymbol{\beta}^-\hat{\boldsymbol{\Psi}}\right)(\mathbf{I}^+ - \boldsymbol{\Psi}\hat{\boldsymbol{\Psi}})^{-1}(s\mathbf{I}^+ - \mathbf{K})^{-1}\begin{bmatrix}\mathbf{I}^+ & \boldsymbol{\Psi}\end{bmatrix}|\mathbf{R}|^{-1} \\
&+ \frac{1}{c}\left(\boldsymbol{\beta}\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v)\begin{bmatrix}\boldsymbol{\Psi}\\ \mathbf{I}^-\end{bmatrix} - \boldsymbol{\beta}^-\right)\left(\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi}\right)^{-1}\left(-s\mathbf{I}^- - \hat{\mathbf{K}}\right)^{-1}\begin{bmatrix}\hat{\boldsymbol{\Psi}} & \mathbf{I}^-\end{bmatrix}|\mathbf{R}|^{-1} \\
&= \frac{1}{c}\boldsymbol{\beta}\left(\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)(\mathbf{Q}_v - s\mathbf{R}_v)^{-1} \\
&+ \frac{1}{c}\boldsymbol{\beta}\left(\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)\begin{bmatrix}\mathbf{I}^+\\ \hat{\boldsymbol{\Psi}}\end{bmatrix}(\mathbf{I}^+ - \boldsymbol{\Psi}\hat{\boldsymbol{\Psi}})^{-1}(s\mathbf{I}^+ - \mathbf{K})^{-1}\begin{bmatrix}\mathbf{I}^+ & \boldsymbol{\Psi}\end{bmatrix}|\mathbf{R}|^{-1} \\
&+ \frac{1}{c}\boldsymbol{\beta}\left(\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)\begin{bmatrix}\boldsymbol{\Psi}\\ \mathbf{I}^-\end{bmatrix}\left(\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi}\right)^{-1}\left(-s\mathbf{I}^- - \hat{\mathbf{K}}\right)^{-1}\begin{bmatrix}\hat{\boldsymbol{\Psi}} & \mathbf{I}^-\end{bmatrix}|\mathbf{R}|^{-1} \\
&= \frac{1}{c}\boldsymbol{\beta}\left(\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)(\mathbf{Q}_v - s\mathbf{R}_v)^{-1} \\
&+ \frac{1}{c}\boldsymbol{\beta}\left(\ \sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)\left(\begin{bmatrix}\mathbf{I}^+\\ \hat{\boldsymbol{\Psi}}\end{bmatrix}(\mathbf{I}^+ - \boldsymbol{\Psi}\hat{\boldsymbol{\Psi}})^{-1}(s\mathbf{I}^+ - \mathbf{K})^{-1}\begin{bmatrix}\mathbf{I}^+ & \boldsymbol{\Psi}\end{bmatrix}|\mathbf{R}|^{-1}\right. \\
&+ \left.\begin{bmatrix}\boldsymbol{\Psi}\\ \mathbf{I}^-\end{bmatrix}(\mathbf{I}^- - \hat{\boldsymbol{\Psi}}\boldsymbol{\Psi})^{-1}(-s\mathbf{I}^- - \hat{\mathbf{K}})^{-1}\begin{bmatrix}\hat{\boldsymbol{\Psi}} & \mathbf{I}^-\end{bmatrix}|\mathbf{R}|^{-1}\right),
\end{aligned}
$$

where we applied $\boldsymbol{\beta}^-\hat{\boldsymbol{\Psi}} = \boldsymbol{\beta}\begin{bmatrix}\mathbf{I}^+\\ \hat{\boldsymbol{\Psi}}\end{bmatrix}$ and $\boldsymbol{\beta}^- = \boldsymbol{\beta}\begin{bmatrix}\boldsymbol{\Psi}\\ \mathbf{I}^-\end{bmatrix}$ in the second step. Using (23) , $\boldsymbol{\pi}^*(s)$ can be expressed as

$$
\boldsymbol{\pi}^*(s) = \underbrace{\frac{1}{c}\boldsymbol{\beta}\left(\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)(\mathbf{Q}_v - s\mathbf{R}_v)^{-1}}_{\triangleq \boldsymbol{\pi}_v^*(s)} + \underbrace{\frac{1}{c}\boldsymbol{\beta}\left(\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)(s\mathbf{R} - \mathbf{Q})^{-1}}_{\triangleq \boldsymbol{\pi}_s^*(s)}.
$$

□

Having the Laplace transform description of the stationary fluid level distribution many practically interesting measures can be computed. Some of them can be obtained via numerical inverse Laplace transformation, while some others can be obtained symbolically at the $s \longrightarrow 0$ limit. In the following, we present some symbolic results obtained from the Laplace transform description using the $s \longrightarrow 0$ limit.

Lemma 2.8. *The stationary distribution of $Y(t)$, with $\boldsymbol{\pi}_j = \lim_{t\to\infty} \Pr(Y(t) = j)$, is*

$$
\begin{aligned}
\boldsymbol{\pi} &= \int_{y=0}^{\infty} \boldsymbol{\pi}(y)\,\mathrm{d}y = \lim_{s\longrightarrow 0} \boldsymbol{\pi}^*(s) \\
&= \frac{1}{c}\left(\boldsymbol{\beta}\left(\sigma^*(-\mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)(\mathbf{Q}_v - \mathbb{1}\boldsymbol{\pi}_v)^{-1} + E(\sigma)\boldsymbol{\pi}_v\right)\left(-d\rho\mathbf{R}_\delta\mathbb{1}\boldsymbol{\pi}_s + (\mathbf{Q} - \mathbf{Q}_v)(\mathbf{Q} - \rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\right)
\end{aligned}
$$

where $\boldsymbol{\pi}_v$ is the solution of $\boldsymbol{\pi}_v\mathbf{Q}_v = \mathbf{0}$ with $\boldsymbol{\pi}_v\mathbb{1} = 1$, $\boldsymbol{\pi}_s$ is the solution of $\boldsymbol{\pi}_s\mathbf{Q} = \mathbf{0}$ with $\boldsymbol{\pi}_s\mathbb{1} = 1$ and $\rho = \frac{1}{\boldsymbol{\pi}_s\mathbf{R}\mathbb{1}}$.

Proof. We compute $\boldsymbol{\pi}$ from $\boldsymbol{\pi} = \lim_{s\to 0} \boldsymbol{\pi}^*(s)$. To obtain the limit as s tends to 0, we rewrite (43) as follows

$$\begin{aligned}\boldsymbol{\pi}^*(s) &= \frac{1}{c}\boldsymbol{\beta}\left(\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)(\mathbf{Q}_v - s\mathbf{R}_v)^{-1}\left(\mathbf{I}_{|\mathcal{S}|} + (\mathbf{Q}_v - s\mathbf{R}_v)(s\mathbf{R}-\mathbf{Q})^{-1}\right)\\ &= \underbrace{\frac{1}{c}\boldsymbol{\beta}\left(\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)(\mathbf{Q}_v - s\mathbf{R}_v)^{-1}}_{=\boldsymbol{\pi}_v^*(s)}\underbrace{\left(s(\mathbf{R}-\mathbf{R}_v)(s\mathbf{R}-\mathbf{Q})^{-1} - (\mathbf{Q}-\mathbf{Q}_v)(s\mathbf{R}-\mathbf{Q})^{-1}\right)}_{=\boldsymbol{\pi}_2^*(s)}.\end{aligned}$$

According to [10]

$$\begin{aligned}\lim_{s\to 0}\boldsymbol{\pi}_v^*(s) &= \lim_{s\to 0}\frac{1}{c}\left(\beta\int_{x=0}^{\infty}\sigma(x)\sum_{k=1}^{\infty}\frac{x^k(\mathbf{Q}_v - s\mathbf{R}_v)^{k-1}}{k!}\,\mathrm{d}x\right)\\ &= \frac{1}{c}\left(\beta\int_{x=0}^{\infty}\sigma(x)\sum_{k=1}^{\infty}\frac{x^k\lim_{s\to 0}(\mathbf{Q}_v - s\mathbf{R}_v)^{k-1}}{k!}(\mathbf{Q}_v - \mathbb{1}\boldsymbol{\pi}_v)(\mathbf{Q}_v - \mathbb{1}\boldsymbol{\pi}_v)^{-1}\,\mathrm{d}x\right)\\ &= \frac{1}{c}\left(\beta\int_{x=0}^{\infty}\sigma(x)(e^{\mathbf{Q}_v x} - \mathbf{I} - x\mathbb{1}\boldsymbol{\pi}_v)\,\mathrm{d}x\,(\mathbf{Q}_v - \mathbb{1}\boldsymbol{\pi}_v)^{-1}\right)\\ &= \frac{1}{c}\left(\beta\left(\sigma^*(-\mathbf{Q}_v) - \mathbf{I}\right)(\mathbf{Q}_v - \mathbb{1}\boldsymbol{\pi}_v)^{-1} + \boldsymbol{\pi}_v E(\sigma)\right).\end{aligned}\tag{44}$$

For $\lim_{s\to 0}\boldsymbol{\pi}_2^*(s)$, we use [10, eq. (69)], which states that

$$\begin{aligned}(s\mathbf{R}-\mathbf{Q})^{-1} &= \left(s\mathbf{I} - \mathbf{R}^{-1}\mathbf{Q}\right)^{-1}\mathbf{R}^{-1} \overset{\text{eq. (69)}}{=} \left((s\mathbf{I} - \mathbf{R}^{-1}\mathbf{Q} + \mathbb{1}\tilde{\boldsymbol{\pi}}_s)^{-1} + \frac{1}{s(s+1)}\mathbb{1}\tilde{\boldsymbol{\pi}}_s\right)\mathbf{R}^{-1}\\ &= (s\mathbf{I} - \mathbf{R}^{-1}\mathbf{Q} + \mathbb{1}\tilde{\boldsymbol{\pi}}_s)^{-1}\mathbf{R}^{-1} + \frac{1}{s(s+1)}\mathbb{1}\tilde{\boldsymbol{\pi}}_s\mathbf{R}^{-1}\\ &= (s\mathbf{R} - \mathbf{Q} + \mathbf{R}\mathbb{1}\tilde{\boldsymbol{\pi}}_s)^{-1} + \frac{1}{s(s+1)}\mathbb{1}\tilde{\boldsymbol{\pi}}_s\mathbf{R}^{-1},\end{aligned}$$

where the first term is non-singular at $s = 0$ and $\tilde{\boldsymbol{\pi}}_s = \frac{1}{\boldsymbol{\pi}_s\mathbf{R}\mathbb{1}}\boldsymbol{\pi}_s\mathbf{R}$. We note that $\tilde{\boldsymbol{\pi}}_s$ satisfies $\tilde{\boldsymbol{\pi}}_s\mathbf{R}^{-1}\mathbf{Q} = \mathbf{0}$ and $\tilde{\boldsymbol{\pi}}_s\mathbb{1} = 1$. Using $\rho = \frac{1}{\boldsymbol{\pi}_s\mathbf{R}\mathbb{1}}$, it can be written as

$$(s\mathbf{R}-\mathbf{Q})^{-1} = (s\mathbf{R} - \mathbf{Q} + \rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1} + \frac{\rho}{s(s+1)}\mathbb{1}\boldsymbol{\pi}_s,\tag{45}$$

from which, we write

$$\begin{aligned}\lim_{s\to 0}\boldsymbol{\pi}_2^*(s) &= \lim_{s\to 0}\Bigg(s\underbrace{(\mathbf{R}-\mathbf{R}_v)}_{=-d\,\mathbf{R}_\delta}\left((s\mathbf{R} - \mathbf{Q} + \rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1} + \frac{\rho}{s(s+1)}\mathbb{1}\boldsymbol{\pi}_s\right)\\ &\quad - (\mathbf{Q}-\mathbf{Q}_v)\left((s\mathbf{R} - \mathbf{Q} + \rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1} + \frac{\rho}{s(s+1)}\mathbb{1}\boldsymbol{\pi}_s\right)\Bigg)\\ &= -d\,\rho\,\mathbf{R}_\delta\mathbb{1}\boldsymbol{\pi}_s + (\mathbf{Q}-\mathbf{Q}_v)(\mathbf{Q} - \rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1},\end{aligned}\tag{46}$$

where we used (9) and the fact that $(\mathbf{Q}-\mathbf{Q}_v)\mathbb{1} = 0$. The lemma comes from $\boldsymbol{\pi} = \lim_{s\to 0}\boldsymbol{\pi}^*(s) = \lim_{s\to 0}\boldsymbol{\pi}_v^*(s)\lim_{s\to 0}\boldsymbol{\pi}_2^*(s)$. □

Corollary 2.9. *The normalization constant is obtained as*

$$c = \left(\boldsymbol{\beta}\left(\sigma^*(-\mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)(\mathbf{Q}_v - \mathbb{1}\boldsymbol{\pi}_v)^{-1} + E(\sigma)\boldsymbol{\pi}_v\right)\left(-d\,\rho\,\mathbf{R}_\delta\mathbb{1}\boldsymbol{\pi}_s + (\mathbf{Q}-\mathbf{Q}_v)(\mathbf{Q} - \rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\right)\mathbb{1}.\tag{47}$$

Proof. The normalization constant can be computed from $\lim_{s\to 0}\boldsymbol{\pi}^*(s)\mathbb{1} = 1$. □

We also note that

$$\boldsymbol{\pi}_v^*(0)\mathbb{1} = \frac{1}{c}\boldsymbol{\beta}\underbrace{\left(\sigma^*(-\mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)\underbrace{(\mathbf{Q}_v - \mathbb{1}\boldsymbol{\pi}_v)^{-1}\mathbb{1}}_{=\mathbb{1}}}_{=\mathbf{0}} + E(\sigma)\underbrace{\boldsymbol{\pi}_v\mathbb{1}}_{=1} = \frac{E(\sigma)}{c}.$$

Lemma 2.10. *The state dependent mean fluid level, defined as $\hat{\boldsymbol{\pi}}_j = \lim_{t\to\infty} E(X(t)\,\mathcal{I}\{Y(t)=j\})$, is*

$$\hat{\boldsymbol{\pi}} = -\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\pi}^*(s) = -\left(\boldsymbol{\pi}_v^*(0)\hat{\boldsymbol{\pi}}_2^*(0) + \hat{\boldsymbol{\pi}}_v^*(0)\boldsymbol{\pi}_2^*(0)\right),$$

where $\boldsymbol{\pi}_v^*(0) = \lim_{s\to 0}\boldsymbol{\pi}_v^*(s)$ and $\boldsymbol{\pi}_2^*(0) = \lim_{s\to 0}\boldsymbol{\pi}_2^*(s)$ are provided in (44) *and* (46)*, respectively, and*

$$\hat{\boldsymbol{\pi}}_2^*(0) = (\mathbf{R}-\mathbf{R}_v)\left((-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1} - \rho\mathbb{1}\pi_s\right) + (\mathbf{Q}-\mathbf{Q}_v)(-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\mathbf{R}(-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}, \tag{48}$$

$$\begin{aligned}\hat{\boldsymbol{\pi}}_v^*(0) = {}& \frac{1}{c}\boldsymbol{\beta}\begin{bmatrix}\mathbf{I}_{|\mathcal{S}|} & \mathbf{0}\end{bmatrix}\left(\sigma^*(-\mathbf{M}) - \mathbf{I}_{2|\mathcal{S}|} - E(\sigma)\mathbf{M}\right)\begin{bmatrix}-(\mathbf{Q}_v-\mathbb{1}\pi_v)^{-1}\mathbf{R}_v(\mathbf{Q}_v-\mathbb{1}\pi_v)^{-1}\\ (\mathbf{Q}_v-\mathbb{1}\pi_v)^{-1}\end{bmatrix}\\ &+\frac{1}{c}\boldsymbol{\beta}\left(\left(\sigma^*(-\mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|} - E(\sigma)\mathbf{Q}_v\right)(\mathbf{Q}_v-\mathbb{1}\pi_v)^{-2}\mathbf{R}_v\mathbb{1}\pi_v + \frac{E(\sigma^2)}{2}\mathbb{1}\pi_v\mathbf{R}_v\mathbb{1}\pi_v\right)(\mathbf{Q}_v-\mathbb{1}\pi_v)^{-1},\end{aligned} \tag{49}$$

where matrix $\mathbf{M}$ of size $2|\mathcal{S}|\times 2|\mathcal{S}|$ is defined as $\mathbf{M} = \begin{bmatrix}\mathbf{Q}_v & -\mathbf{R}_v\\ & \mathbf{Q}_v\end{bmatrix}$.

Proof. For $\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\pi}^*(s)$, we have

$$\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\pi}^*(s) = \underbrace{\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\pi}_v^*(s)}_{=\hat{\boldsymbol{\pi}}_v^*(0)}\boldsymbol{\pi}_2^*(0) + \boldsymbol{\pi}_v^*(0)\underbrace{\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\pi}_2^*(s)}_{=\hat{\boldsymbol{\pi}}_2^*(0)}.$$

First we focus on $\hat{\boldsymbol{\pi}}_v^*(0)$.

$$\begin{aligned}
\hat{\boldsymbol{\pi}}_v^*(0) &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{c}\boldsymbol{\beta}\left(\sigma^*(s\mathbf{R}_v - \mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|}\right)(\mathbf{Q}_v - s\mathbf{R}_v)^{-1}\bigg|_{s=0} \\
&= \frac{1}{c}\boldsymbol{\beta}\int_{x=0}^{\infty}\sigma(x)\sum_{k=1}^{\infty}\frac{x^k}{k!}\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{Q}_v - s\mathbf{R}_v)^{k-1}\bigg|_{s=0}\mathrm{d}x \\
&= \frac{1}{c}\boldsymbol{\beta}\int_{x=0}^{\infty}\sigma(x)\sum_{k=2}^{\infty}\frac{x^k}{k!}\sum_{i=0}^{k-2}\mathbf{Q}_v^i(-\mathbf{R}_v)\mathbf{Q}_v^{k-2-i}\,\mathrm{d}x \\
&= \frac{1}{c}\boldsymbol{\beta}\int_{x=0}^{\infty}\sigma(x)\sum_{k=2}^{\infty}\frac{x^k}{k!}\begin{bmatrix}\mathbf{I}_{|\mathcal{S}|} & \mathbf{0}\end{bmatrix}\mathbf{M}^{k-1}(\mathbf{M}-\hat{\mathbf{M}})(\mathbf{M}-\hat{\mathbf{M}})^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{I}_{|\mathcal{S}|}\end{bmatrix}\mathrm{d}x \\
&= \frac{1}{c}\boldsymbol{\beta}\begin{bmatrix}\mathbf{I}_{|\mathcal{S}|} & \mathbf{0}\end{bmatrix}\int_{x=0}^{\infty}\sigma(x)\sum_{k=2}^{\infty}\frac{x^k}{k!}\mathbf{M}^{k-1}(\mathbf{M}-\hat{\mathbf{M}})\,\mathrm{d}x(\mathbf{M}-\hat{\mathbf{M}})^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{I}_{|\mathcal{S}|}\end{bmatrix} \\
&= \frac{1}{c}\boldsymbol{\beta}\begin{bmatrix}\mathbf{I}_{|\mathcal{S}|} & \mathbf{0}\end{bmatrix}\left(\left(\sigma^*(-\mathbf{M}) - \mathbf{I}_{2|\mathcal{S}|} - E(\sigma)\mathbf{M}\right) - \underbrace{\int_{x=0}^{\infty}\sigma(x)\sum_{k=2}^{\infty}\frac{x^k}{k!}\mathbf{M}^{k-1}\hat{\mathbf{M}}\,\mathrm{d}x}_{\mathbf{M}^{k-1}\hat{\mathbf{M}}\text{ term with } k>1}\right)(\mathbf{M}-\hat{\mathbf{M}})^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{I}_{|\mathcal{S}|}\end{bmatrix} \qquad (50)
\end{aligned}$$

where $\hat{\mathbf{M}} = \begin{bmatrix}\mathbb{1}\boldsymbol{\pi}_v & \\ & \mathbb{1}\boldsymbol{\pi}_v\end{bmatrix}$. When $k > 0$, for $\mathbf{M}^k\hat{\mathbf{M}}$ we have

$$\mathbf{M}^k\hat{\mathbf{M}} = \begin{bmatrix}\mathbf{Q}_v^k\mathbb{1}\boldsymbol{\pi}_v & -\sum_{j=0}^{k-1}\mathbf{Q}_v^{k-j-1}\mathbf{R}_v\mathbf{Q}_v^j\mathbb{1}\boldsymbol{\pi}_v \\ & \mathbf{Q}_v^k\mathbb{1}\boldsymbol{\pi}_v\end{bmatrix} = \begin{bmatrix}\mathbf{0} & -\mathbf{Q}_v^{k-1}\mathbf{R}_v\mathbb{1}\boldsymbol{\pi}_v \\ & \mathbf{0}\end{bmatrix},$$

from which

$$\begin{aligned}
&-\int_{x=0}^{\infty}\sigma(x)\sum_{k=2}^{\infty}\frac{x^k}{k!}\mathbf{M}^{k-1}\hat{\mathbf{M}}\,\mathrm{d}x = \begin{bmatrix}\mathbf{I}_{|\mathcal{S}|}\\ \mathbf{0}\end{bmatrix}\int_{x=0}^{\infty}\sigma(x)\sum_{k=2}^{\infty}\frac{x^k}{k!}\mathbf{Q}_v^{k-2}(\mathbf{Q}_v - \mathbb{1}\pi_v)^2(\mathbf{Q}_v - \mathbb{1}\pi_v)^{-2}\mathbf{R}_v\mathbb{1}\pi_v\,\mathrm{d}x\begin{bmatrix}\mathbf{0} & \mathbf{I}_{|\mathcal{S}|}\end{bmatrix} \\
&= \begin{bmatrix}\mathbf{I}_{|\mathcal{S}|}\\ \mathbf{0}\end{bmatrix}\left(\left(\sigma^*(-\mathbf{Q}_v) - \mathbf{I}_{|\mathcal{S}|} - E(\sigma)\mathbf{Q}_v\right)(\mathbf{Q}_v - \mathbb{1}\pi_v)^{-2}\mathbf{R}_v\mathbb{1}\pi_v + \frac{E(\sigma^2)}{2}\mathbb{1}\pi_v\mathbf{R}_v\mathbb{1}\pi_v\right)\begin{bmatrix}\mathbf{0} & \mathbf{I}_{|\mathcal{S}|}\end{bmatrix}. \qquad (51)
\end{aligned}$$

Using additionally $(\mathbf{M}-\hat{\mathbf{M}})^{-1} = \begin{bmatrix}(\mathbf{Q}_v - \mathbb{1}\pi_v)^{-1} & -(\mathbf{Q}_v - \mathbb{1}\pi_v)^{-1}\mathbf{R}_v(\mathbf{Q}_v - \mathbb{1}\pi_v)^{-1} \\ & (\mathbf{Q}_v - \mathbb{1}\pi_v)^{-1}\end{bmatrix}$, (50) simplifies to (49).

For $\hat{\boldsymbol{\pi}}_2^*(0)$, we note that

$$(s\mathbf{R} - \mathbf{Q} + \rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})\,\mathbb{1} = (s+1)\mathbf{R}\mathbb{1} \text{ and } \pi_s\,(s\mathbf{R} - \mathbf{Q} + \rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R}) = (s+1)\pi_s\mathbf{R},$$

implies

$$(s\mathbf{R} - \mathbf{Q} + \rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\,\mathbf{R}\mathbb{1} = \frac{1}{s+1}\mathbb{1} \text{ and } \pi_s\mathbf{R}\,(s\mathbf{R} - \mathbf{Q} + \rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1} = \pi_s\frac{1}{s+1}.$$

Using this and (45), for the first term of $\boldsymbol{\pi}_2^*(s)$ we write

$$\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}s(s\mathbf{R}-\mathbf{Q})^{-1} = \lim_{s\to 0}(s\mathbf{R}-\mathbf{Q})^{-1} - s\left((s\mathbf{R}-\mathbf{Q})^{-1}\mathbf{R}\,(s\mathbf{R}-\mathbf{Q})^{-1}\right)$$
$$= \lim_{s\to 0}\left((s\mathbf{R}-\mathbf{Q})^{-1}\left(\mathbf{I}-s\mathbf{R}(s\mathbf{R}-\mathbf{Q})^{-1}\right)\right)$$
$$= \lim_{s\to 0}\left((s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}+\frac{\rho}{s(s+1)}\mathbb{1}\pi_s\right)\left(\mathbf{I}-s\mathbf{R}\,(s\mathbf{R}-\mathbf{Q}-\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}-\frac{\rho s}{s(s+1)}\mathbf{R}\mathbb{1}\pi_s\right)$$
$$= \lim_{s\to 0}(s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\left(\mathbf{I}-s\mathbf{R}\,(s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\right)-\frac{\rho s}{s(s+1)}(s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\mathbf{R}\mathbb{1}\pi_s$$
$$+\frac{\rho}{s(s+1)}\mathbb{1}\pi_s\left(\mathbf{I}-s\mathbf{R}\,(s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\right)-\frac{\rho}{s(s+1)}\mathbb{1}\pi_s\frac{\rho s}{s(s+1)}\mathbf{R}\mathbb{1}\pi_s$$
$$= \lim_{s\to 0}(s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\left(\mathbf{I}-s\mathbf{R}\,(s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\right)-\frac{\rho s}{s(s+1)^2}\mathbb{1}\pi_s$$
$$+\frac{\rho}{s(s+1)}\mathbb{1}\pi_s-\frac{\rho s}{s(s+1)^2}\mathbb{1}\pi_s-\frac{\rho s}{s^2(s+1)^2}\mathbb{1}\pi_s$$
$$= (-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}-\rho\mathbb{1}\pi_s$$

therefore

$$\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}s(\mathbf{R}-\mathbf{R}_v)(s\mathbf{R}-\mathbf{Q})^{-1} = (\mathbf{R}-\mathbf{R}_v)\left((-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}-\rho\mathbb{1}\pi_s\right). \tag{52}$$

Applying (45) again, for the second term of $\boldsymbol{\pi}_2^*(s)$ we have

$$\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}(\mathbf{Q}-\mathbf{Q}_v)(s\mathbf{R}-\mathbf{Q})^{-1} = -\lim_{s\to 0}(\mathbf{Q}-\mathbf{Q}_v)\left((s\mathbf{R}-\mathbf{Q})^{-1}\mathbf{R}\,(s\mathbf{R}-\mathbf{Q})^{-1}\right)$$
$$= -\lim_{s\to 0}(\mathbf{Q}-\mathbf{Q}_v)\left((s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}+\frac{\rho}{s(s+1)}\mathbb{1}\pi_s\right)\mathbf{R}\left((s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}+\frac{\rho}{s(s+1)}\mathbb{1}\pi_s\right)$$
$$= -\lim_{s\to 0}(\mathbf{Q}-\mathbf{Q}_v)\left((s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\mathbf{R}\,(s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}+\frac{\rho}{s(s+1)}(s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\mathbf{R}\mathbb{1}\pi_s\right.$$
$$\left.+\frac{\rho}{s(s+1)}\mathbb{1}\pi_s\mathbf{R}\,(s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}+\frac{\rho}{s(s+1)}\mathbb{1}\pi_s\mathbf{R}\frac{\rho}{s(s+1)}\mathbb{1}\pi_s\right)$$
$$= -\lim_{s\to 0}(\mathbf{Q}-\mathbf{Q}_v)\left((s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\mathbf{R}\,(s\mathbf{R}-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}+\frac{\rho}{s(s+1)^2}\mathbb{1}\pi_s\right.$$
$$\left.+\frac{\rho}{s(s+1)^2}\mathbb{1}\pi_s+\frac{\rho}{s^2(s+1)^2}\mathbb{1}\pi_s\right)$$
$$= (\mathbf{Q}_v-\mathbf{Q})\,(-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}\mathbf{R}\,(-\mathbf{Q}+\rho\mathbf{R}\mathbb{1}\boldsymbol{\pi}_s\mathbf{R})^{-1}. \tag{53}$$

(52) and (53) result in (48). □

The stationary distribution of $Y(t)$ also allows us to compute the stationary distribution of the queue size.

Lemma 2.11. *The steady-state distribution of queue size, with* $\mathbf{p}_n = \lim_{t\to\infty}\Pr(N(t)=n)$, *is*

$$\mathbf{p} = \boldsymbol{\pi}\mathbf{P}\left[\mathbf{I}_{N+1}\otimes\mathbb{1}_{|\mathcal{S}_\varphi|}\right]$$

where $\mathbf{P}$ *is the permutation matrix defined in* (10).

2.9 Summary of the analysis method

Input parameters:

- M/M/1/N queue: λ, μ, N,
- fluid input process: $\mathbf{Q}_\varphi$, $\mathbf{R}_\varphi$,
- fluid output rate: d,
- vacation time distribution: $\sigma(x)$.

Steps of the analysis procedure:

(1) check the stability condition according to (13),
(2) compute $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{Q}_v$, $\mathbf{R}_v$, $\hat{\mathbf{Q}}$ according to (10), (11) and (12),
(3) compute $\mathbf{\Psi}, \mathbf{K}, \mathbf{U}$ according to Section 2.4.2 and $\hat{\mathbf{\Psi}}, \hat{\mathbf{K}}, \hat{\mathbf{U}}$ for the level reversed processes,
(4) compute $\mathbf{W}$ and $\boldsymbol{\beta}^-$ according to Theorem 2.5,
(5) compute the normalization constant c from Corollary 2.9,
(6) compute $\boldsymbol{\pi}$ and $\hat{\boldsymbol{\pi}}$ from Lemma 2.8 and 2.10,
(7) finally, compute the queue length distribution from Lemma 2.11.

The computationally most expensive step is to compute $\sigma^*(\mathbf{Q}_v \otimes \mathbf{I}^- + \mathbf{R}_v \otimes \mathbf{U}^T)$ in Step 4.

3 PHASE-TYPE DISTRIBUTED VACATION TIME

In this section, we revisit the same model studied in the previous section, but with the restriction that the server's vacation time follows a phase-type (PH) distribution, in contrast to the generally distributed vacation time in the earlier model. The imposition of this constraint on the fluid vacation model with PH-distributed vacation time allows us to enrich the background Markov chain such that it also describes the vacation-service cycle of the vacation model, and this way we can apply standard fluid queue results to evaluate the system behavior.

3.1 Model description

We will consider the same $M/M/1/N$ queue with energy-enabled service as before, but instead of general vacation time, we assume that vacation time follows a phase-type (PH) distribution with representation $(\boldsymbol{\alpha}, \mathbf{A})$ of size n_{PH}. In this case, the density, the Laplace transform, and the mean vacation time are $\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Pr(\sigma < t) = \boldsymbol{\alpha} e^{\mathbf{A}t}\mathbf{a}$, $\sigma^*(s) = E(e^{-s\sigma}) = \boldsymbol{\alpha}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}$, and $E(\sigma) = \boldsymbol{\alpha}(-\mathbf{A})^{-1}\mathbb{1}$, respectively, where $\mathbf{a} = -\mathbf{A}\mathbb{1}$. The rest of the model behavior is identical to the one studied in the previous section.

3.2 Analysis

In this model, in addition to the state variables $X(t)$, $\varphi(t)$ and $N(t)$ defined in the general vacation setup, we need another state variable that represents the phase of the vacation period. So, here we define $\vartheta(t)$ as the phase of the vacation period at time t. The considered system is a Markov process with state variables

$X(t)$ - fluid level ($X(t) \geq 0$),

$\varphi(t)$ - *phase* of the inflow modulating Markov chain ($\varphi(t) \in \mathcal{S}_\varphi$),

$N(t)$ - number of customers in the queue ($N(t) \in \{0, 1, \ldots, N\}$),

$\vartheta(t)$ - *phase* of the service–vacation period ($\vartheta(t) \in \{1, \ldots, n_{PH} + 1\}$),

where $\vartheta(t) = n_{PH} + 1$ indicates the service period.

Similarly to the previous section, where we combined the discrete variables $\varphi(t)$ and $N(t)$ together, in this section we combine the three discrete variables together and introduce $Y_{PH}(t) = (\vartheta(t), N(t), \varphi(t))$, where $Y_{PH}(t) \in \mathcal{S}_{PH} =$

$\{1, \ldots, n_{PH}+1\} \times \{0, \ldots, N\} \times \mathcal{S}_\varphi$. The $(X(t), Y_{PH}(t))$ process is a Markov fluid queue with special behavior at an empty buffer. The characterizing matrices of the process when the buffer is non-empty are

$$\hat{\mathbf{Q}}_{PH} = \begin{bmatrix} \mathbf{A} \oplus \hat{\mathbf{Q}}_v & \mathbf{a} \otimes \mathbf{I}_{|\mathcal{S}|} \\ \mathbf{0} & \hat{\mathbf{Q}}_s \end{bmatrix}, \text{ and } \hat{\mathbf{R}}_{PH} = \begin{bmatrix} \mathbf{I}_{n_{PH}} \otimes \hat{\mathbf{R}}_v & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{R}} \end{bmatrix}.$$

When the buffer becomes empty, the process immediately performs a jump to one of the states associated with the vacation. In order to describe this behavior with the available Markov fluid queue tools, we introduce a special model behavior when the buffer is empty, which is characterized by matrices

$$\hat{\mathbf{Q}}^0_{PH} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{\alpha} \otimes \mathbf{I}_{|\mathcal{S}|} & -\mathbf{I}_{|\mathcal{S}|} \end{bmatrix}, \text{ and } \hat{\mathbf{R}}^0_{PH} = \hat{\mathbf{R}}_{PH}.$$

$\mathcal{S}_{PH}$ can be decomposed according to the sign of the fluid rates. $\mathcal{S}^+_{PH}$ contains the states with positive fluid rates and $\mathcal{S}^-_{PH}$ those with negative fluid rates. With the help of the permutation matrix $\mathbf{P}_{PH}$, we order the states in $\mathcal{S}_{PH}$ so that the indices of the states in $\mathcal{S}^+_{PH}$ are less than the indices of the states in $\mathcal{S}^-_{PH}$. That is

$$\mathbf{Q}_{PH} = \mathbf{P}_{PH}\hat{\mathbf{Q}}_{PH}\mathbf{P}^T_{PH} = \begin{bmatrix} \mathbf{Q}^{++}_{PH} & \mathbf{Q}^{+-}_{PH} \\ \mathbf{Q}^{-+}_{PH} & \mathbf{Q}^{--}_{PH} \end{bmatrix}, \qquad \mathbf{R}_{PH} = \mathbf{P}_{PH}\hat{\mathbf{R}}_{PH}\mathbf{P}^T_{PH} = \begin{bmatrix} \mathbf{R}^+_{PH} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^-_{PH} \end{bmatrix}, \tag{54}$$

and $\mathbf{Q}^0_{PH} = \mathbf{P}_{PH}\hat{\mathbf{Q}}^0_{PH}\mathbf{P}^T_{PH}$.

The fundamental matrices of the Markov fluid queue characterized by $\mathbf{Q}_{PH}$ and $\mathbf{R}_{PH}$ are $\boldsymbol{\Psi}_{PH}, \mathbf{K}_{PH}, \mathbf{U}_{PH}$. They are computed in the same way as $\boldsymbol{\Psi}, \mathbf{K}, \mathbf{U}$ are computed from $\mathbf{Q}$ and $\mathbf{R}$ in the previous section.

3.3 Stationary solution of the fluid model with special behavior at empty buffer

Let $(\boldsymbol{\pi}_{PH}(x))_i = \frac{\mathrm{d}}{\mathrm{d}x} \lim_{t\to\infty} \Pr(X(t) < x, Y_{PH}(t) = i)$ be the stationary density that the buffer level is x and the discrete state is i. $\boldsymbol{\pi}_{PH}(x)$ can be computed as

$$\boldsymbol{\pi}_{PH}(x) = \boldsymbol{\ell}^-_{PH}\mathbf{Q}^{0-+}_{PH} e^{\mathbf{K}_{PH}x} \begin{bmatrix} \mathbf{I}_{|\mathcal{S}^+_{PH}|} & \boldsymbol{\Psi}_{PH} \end{bmatrix} |\mathbf{R}_{PH}|^{-1},$$

where $\boldsymbol{\ell}^-_{PH}$ is the solution of

$$\boldsymbol{\ell}^-_{PH}\left(\mathbf{Q}^{0--}_{PH} + \mathbf{Q}^{0-+}_{PH}\boldsymbol{\Psi}_{PH}\right) = \mathbf{0}, \tag{55}$$

with normalizing condition

$$1 = \int_{x=0}^{\infty} \boldsymbol{\pi}_{PH}(x)\mathbb{1}\,\mathrm{d}x = \boldsymbol{\ell}^-_{PH}\mathbf{Q}^{0-+}_{PH}(-\mathbf{K}_{PH})^{-1} \begin{bmatrix} \mathbf{I}_{|\mathcal{S}^+_{PH}|} & \boldsymbol{\Psi}_{PH} \end{bmatrix} |\mathbf{R}_{PH}|^{-1}\mathbb{1}.$$

From the fluid density, we can compute the stationary distribution of $Y_{PH}(t)$, $(\boldsymbol{\pi}_{PH})_i = \lim_{t\to\infty}\Pr(Y_{PH}(t) = i)$, as

$$\boldsymbol{\pi}_{PH} = \int_{x=0}^{\infty} \boldsymbol{\pi}_{PH}(x)\,\mathrm{d}x = \boldsymbol{\ell}^-_{PH}\mathbf{Q}^{0-+}_{PH}(-\mathbf{K}_{PH})^{-1} \begin{bmatrix} \mathbf{I}_{|\mathcal{S}^+_{PH}|} & \boldsymbol{\Psi}_{PH} \end{bmatrix} |\mathbf{R}_{PH}|^{-1}, \tag{56}$$

and the mean fluid level, $\int_{x=0}^{\infty} x(\boldsymbol{\pi}_{PH}(x))_i dx = \lim_{t\to\infty} E(X(t)\,\mathcal{I}\{Y_{PH}(t) = i\})$, as

$$\hat{\boldsymbol{\pi}}_{PH} = \int_{x=0}^{\infty} x\ \boldsymbol{\pi}_{PH}(x)dx = \boldsymbol{\ell}^-_{PH}\mathbf{Q}^{0-+}_{PH}(-\mathbf{K}_{PH})^{-2} \begin{bmatrix} \mathbf{I}_{|\mathcal{S}^+_{PH}|} & \boldsymbol{\Psi}_{PH} \end{bmatrix} |\mathbf{R}_{PH}|^{-1}. \tag{57}$$

3.4 Stationary distribution of the queue size

The stationary distribution of $Y_{PH}(t)$ also allows us to compute the stationary distribution of the queue size.

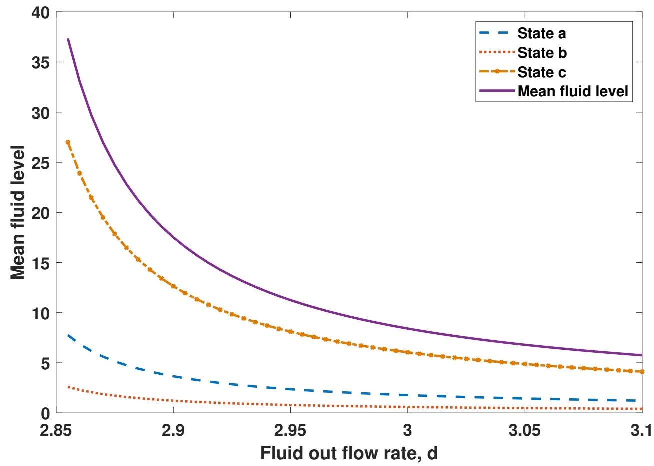

Fig. 3. State dependent and overall mean fluid level as a function of fluid outflow rate

Lemma 3.1. *The steady-state distribution of queue size, with* $\mathbf{p}_n = \lim_{t\to\infty} \Pr(N(t) = n)$, *is*

$$\mathbf{p}_{PH} = \boldsymbol{\pi}_{PH}\mathbf{P}_{PH}\left[\mathbb{1}_{n_{PH}+1} \otimes \mathbf{I}_{N+1} \otimes \mathbb{1}_{|\mathcal{S}_\varphi|}\right],$$

where $\mathbf{P}_{PH}$ *is the permutation matrix defined in* (54).

3.5 Steps of the numerical analysis

Input parameters:

- M/M/1/N queue: λ, μ, N,
- fluid input process: $\mathbf{Q}_\varphi$, $\mathbf{R}_\varphi$,
- fluid output rate: d,
- representation of the PH distributed vacation time: $(\boldsymbol{\alpha}, \mathbf{A})$.

Steps of the analysis procedure:

(1) check the stability condition according to (13),
(2) compute $\mathbf{Q}_{PH}$ and $\mathbf{R}_{PH}$ according to (54),
(3) compute $\boldsymbol{\Psi}_{PH}, \mathbf{K}_{PH}, \mathbf{U}_{PH}$ based on $\mathbf{Q}_{PH}$ and $\mathbf{R}_{PH}$ according to Section 2.4.2,
(4) Compute ℓ^-_{PH} based on (55)
(5) compute $\boldsymbol{\pi}_{PH}$ and $\hat{\boldsymbol{\pi}}_{PH}$ from (56) and (57),
(6) finally, compute the queue length distribution from Lemma 3.1.

The computationally most expensive step is to compute $\boldsymbol{\Psi}_{PH}$ in Step 3.

4 NUMERICAL EXAMPLE

In this section, we illustrate the practical computation of the analytical outcomes discussed in Sections 2 and 3. We specifically calculate the steady-state mean fluid level and the distribution of queue sizes for our model. The considered M/M/1/N queueing model with energy consuming server and server vacation is as follows. The buffer size of the queue is such that $N = 2$, that is $N(t) \in \{0, 1, 2\}$. The arrival and the service rates are $\lambda = 2$ and $\mu = 3$. The fluid inflow is

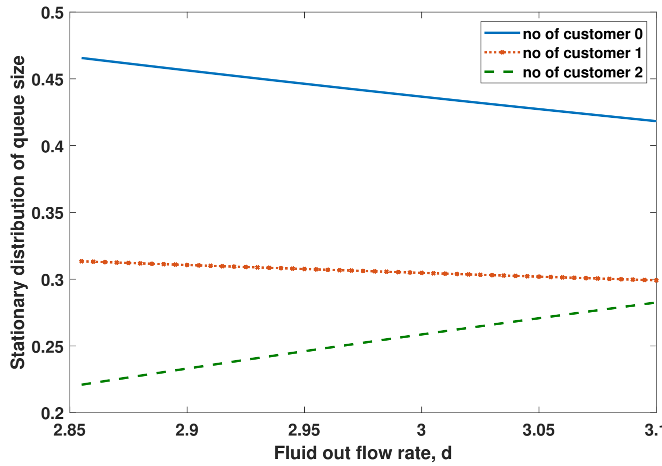

Fig. 4. Stationary queue size distribution versus fluid consumption rate

described by

$$\mathbf{Q}_\varphi = \begin{bmatrix} -8 & 4 & 4 \\ 3 & -12 & 9 \\ 2 & 0 & -2 \end{bmatrix}, \quad \mathbf{R}_\varphi = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\mathcal{S}_\varphi = \{a, b, c\}$. When the server is busy and the fluid outflow rate is greater than 3, $d > 3$, then fluid rates are negative in all states (i.e., $r_k = c_k - d < 0$ for all $k \in \mathcal{S}_\varphi$). According to Assumption 1, $d \neq \{1, 2, 3\}$ should hold to avoid zero fluid rate.

When $2 < d < 3$ and the server is busy, the fluid rate in state a is positive, $3 - d$, and the fluid rates are negative in states b and c, $2 - d$ and $1 - d$, respectively. According to (13), the fluid buffer is stable when $d > 2.81724$.

First we investigate the queue behavior when the vacation time is PH distributed with representation $(\boldsymbol{\alpha}, \mathbf{A})$, where $\boldsymbol{\alpha} = \{0.25, 0.75\}$ and $\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -4 \end{bmatrix}$. Figure 3 depicts the overall mean fluid level and the fluid inflow modulating state dependent mean fluid level as a function of d. We do not compute the performance indices at $d = 3$, but they are continuous also at $d = 3$ as suggested also by the figure. As expected, all of these measures decrease with increasing values of d, Figure 4 shows the variation of the stationary queue size distribution $p = (p_0, p_1, p_2)$ according to the variation in the values of d. As the fluid consumption rate increases, the chance that the fluid level becomes zero is higher, which may result in the accumulation of more number of customers in the system. So, the probability that the system is empty (p_0) is decreasing and the probability of seeing the system full (p_2) is increasing with increasing values of d, as is clear from Figure 4. It should be noted that for the above analysis, we used the theory developed in Section 3 for PH distributed vacation times.

Next, we utilized the procedure developed in Section 2 for general vacation times to examine the impact of the outflow rate on the mean fluid level. In this regard, we considered three distinct cases for the vacation time distribution: a continuous uniform distribution over the interval (0,1), a PH distribution with parameters $(\boldsymbol{\alpha}, \mathbf{A})$, where $\boldsymbol{\alpha} = \{0.25, 0.75\}$

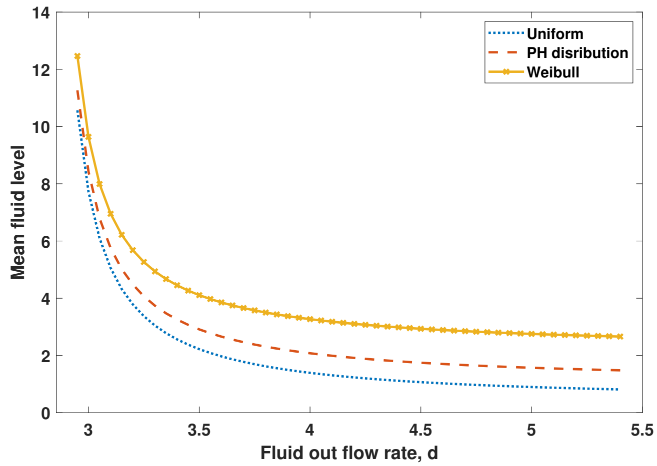

Fig. 5. Mean fluid level as the function of fluid outflow rate

and $\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -4 \end{bmatrix}$, and a Weibull distribution whose CDF is $F(x) = 1 - e^{-(x/\nu)^k}$, where the scale parameter is $\nu = 1/4$ and the shape parameter is $k = 1/2$. The mean of these distributions are $1/2$ and their squared coefficients of variation are $1/3$, 2, and 5, respectively. Figure 5 illustrates the mean fluid level versus the outflow rate for these different vacation time distributions.

As a cross-validation of the different analysis approaches, we note that the results obtained for the PH distributed vacation time using the theory developed in Section 2 precisely coincide with those results depicted in Figure 3, obtained exclusively using the theory developed for the PH case in Section 3.

The numerical computation based on the method discussed in Section 3 is significantly less expensive compared to the methods provided in Section 2, where the most resource-intensive step is the computation of matrix $\sigma^*(\mathbf{X})$ with various $\mathbf{X}$ matrices. Our implementation of the examples, coded in MATLAB, is available at https://webspn.hit.bme.hu/~telek/aa/energy_enabled_queue_matlab_code.zip

5 CONCLUSION

The paper considers a queueing system whose server consumes energy from a battery during service. When the battery becomes empty, the server goes on vacation for a random amount of time. The energy level of the battery is described by a fluid buffer, and the overall system behavior is modeled by a fluid vacation model, whose behavior is more general than the ones available in the literature because different generators govern the fluid model during vacation and service. The solution of this model requires new analytical results such as the one in Theorem 2.3.

The numerical analysis of the model with general vacation time is limited to small queue sizes due to the high computational complexity of the matrix functions.

REFERENCES

[1] D. Anick, D. Mitra, M. M. Sondhi, Stochastic theory of a data-handling system with multiple sources, Bell System technical journal 61 (8) (1982) 1871–1894.

[2] G. Latouche, V. Ramaswami, Introduction to matrix analytic methods in stochastic modeling, SIAM, 1999.
[3] V. Ramaswami, Matrix analytic methods for stochastic fluid flows, in: Proc. of ITC 16, Elsevier, 1999.
[4] A. da Silva Soares, Fluid queues: building upon the analogy with qbd processes, Ph.D. thesis, Université libre de Bruxelles (2005).
[5] A. da Silva Soares, G. Latouche, Matrix-analytic methods for fluid queues with finite buffers, Performance Evaluation 63 (4-5) (2006) 295–314.
[6] O. Czerniak, U. Yechiali, Fluid polling systems, Queueing Systems 63 (2009) 401–435.
[7] Z. Saffer, M. Telek, Fluid vacation model with markov modulated load and exhaustive discipline, in: Computer Performance Engineering: 11th European Workshop, EPEW 2014, Florence, Italy, September 11-12, 2014. Proceedings 11, Springer, 2014, pp. 59–73.
[8] Z. Saffer, M. Telek, Exhaustive fluid vacation model with markov modulated load, Performance Evaluation 98 (2016) 19–35.
[9] Z. Saffer, M. Telek, G. Horváth, Analysis of markov-modulated fluid polling systems with gated discipline, Journal of Industrial and Management Optimization 17 (2) (2021) 575–599.
[10] G. Horváth, M. Telek, Exhaustive fluid vacation model with positive fluid rate during service, Performance Evaluation 91 (2015) 286–302.
[11] N. Sonenberg, Networks of interacting stochastic fluid models, Ph.D. thesis, University of Melbourne (2017).