Partial Loss in Reward Models
detailed and corrected derivations

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1 Partial incremental work loss in an SMP environment

Let $\{Z(t), t \geq 0\}$ be a semi-Markov process (SMP) on state space $R = \{1, 2, ..., N\}$ with kernel $Q(\cdot) = [Q_{ij}(\cdot)]$. Throughout this paper we assume that $Q(t)$ is the canonical kernel of $Z(t)$, i.e. $Q_{ii} = 0, \forall i$. Suppose that whenever the SMP is in state i, reward is accumulated at rate r_i. When the SMP undergoes a transition from state i, a fraction $1 - \mathcal{A}_i$ of the reward obtained during the last sojour in state i is lost and a fraction $\mathcal{A}_i$ of the reward obtained during the last sojourn remains. $\mathcal{A}_i$ is a r.v. over $(0, 1)$ with distribution $A_i(\dot{})$. $B(t)$ denotes the amount of material in the warehouse at time t. Let T_n be the time of the n^{th} transition in the SMP. Then the dynamics of the right continuous process $\{B(t), t \geq 0\}$ can be described as follows:

$$\frac{dB(t)}{dt} = r_{Z(t)} \qquad \text{for} \quad T_n < t < T_{n+1} \tag{1}$$

$$B(T_n) = B(T_{n-1}) + \mathcal{A}_{Z(T_n^-)}[B(T_n^-) - B(T_{n-1})] \tag{2}$$

1.1 Limiting distribution of $B(t)$

By (2) $B(T_n)$ is an non-decreasing serie.

Suppose $Z(t)$ is an ergodic process with steady state distribution $\underline{\pi} = \{pi_i\}$. Both $lim_{n\to\infty} B(T_n)$ and $lim_{t\to\infty} B(t)$ go to infinity if

$$\sum_i \pi_i r_i E[\mathcal{A}_i] > 0$$

1.2 Accumulated reward up to time t

Define

$$P_i(t, w) = Pr(B(t) \leq w \mid Z(0) = i).$$

<u>*Detailed derivation of the accumulated reward with state dependent deterministic lost ratio:*</u>
(i.e., the $1 - \alpha_i$ portion of the reward accumulated during the last sojourn in state i is lost when state i is left):

Conditioning on H, the sojourn time in sate i, we have:

$$P_i(t, w| \, H = \tau) = \begin{cases} U_w(w - r_i t) & \text{if} : \tau > t \\ \displaystyle\sum_{k \in R} \frac{dQ_{ik}(\tau)}{dQ_i(\tau)} \cdot P_k(t - \tau, w - \alpha_i \tau r_i) & \text{if} : \tau < t \end{cases} \tag{3}$$

Taking the Laplace-Stieltjes transform with respect to w $(\to v)$

$$P_i^{\sim}(t,v|\,H=\tau) \;=\; \begin{cases} e^{-vr_it} & \text{if} : \tau > t \\ \sum_{k\in R} \dfrac{dQ_{ik}(\tau)}{dQ_i(\tau)} \cdot e^{-vr_i\alpha_i\tau} \cdot P_k^{\sim}(t-\tau,v) & \text{if} : \tau < t \end{cases} \tag{4}$$

Unconditioning with respect to H (by $Q_i(t)$),

$$P_i^{\sim}(t,v) \;=\; e^{-vr_it}(1-Q_i(t)) + \sum_{k\in R} \int_{\tau=0}^{t} e^{-vr_i\alpha_i\tau} \cdot P_k^{\sim}(t-\tau,v)\; dQ_{ik}(\tau) \tag{5}$$

Taking the Laplace transform with respect to t ($\to s$) results:

$$\begin{aligned} &P_i^{*\sim}(s,v) \;=\; \int_{t=0}^{\infty} e^{-st} P_i^{\sim}(t,v)dt \;= \\ &\int_{t=0}^{\infty} e^{-st}e^{-vr_it}(1-Q_i(t))\; dt + \sum_{k\in R} \int_{t=0}^{\infty} e^{-st} \int_{\tau=0}^{t} e^{-vr_i\alpha_i\tau} \cdot P_k^{\sim}(t-\tau,v)\; dQ_{ik}(\tau)\; dt \;= \\ &\frac{1}{s+vr_i} - Q_i^*(s+vr_i) + \sum_{k\in R} \int_{\tau=0}^{\infty} e^{-s\tau}e^{-vr_i\alpha_i\tau} \int_{t=\tau}^{\infty} e^{-s(t-\tau)} \cdot P_k^{\sim}(t-\tau,v)\; dt\; dQ_{ik}(\tau) \;= \\ &\frac{1}{s+vr_i} - \frac{Q_i^{\sim}(s+vr_i)}{s+vr_i} + \sum_{k\in R} \int_{\tau=0}^{\infty} e^{-(s+vr_i\alpha_i)\tau}\; dQ_{ik}(\tau)\; P_k^{*\sim}(s,v) \;= \\ &\frac{1-Q_i^{\sim}(s+vr_i)}{s+vr_i} + \sum_{k\in R} Q_{ik}^{\sim}(s+vr_i\alpha_i) \cdot P_k^{*\sim}(s,v) \end{aligned} \tag{6}$$

Theorem 1 *With random loss potion, $\mathcal{A}_i$, the following double transform domain equation holds for $P_i(t,w)$:*

$$P_i^{*\sim}(s,v) \;=\; \frac{1-Q_i^{\sim}(s+vr_i)}{s+vr_i} + \sum_{k\in R} \int_{\tau=0}^{\infty} e^{s\tau} A_i^{\sim}(vr_i\tau)\; dQ_{ik}(\tau) \cdot P_k^{*\sim}(s,v) \tag{7}$$

where $Q_i(t) = \sum_{j\in R} Q_{ij}(t)$.

Proof **1** *Conditioning on H, the sojourn time in sate i, we have:*

$$P_i(t,w|\,H=\tau) \;=\; \begin{cases} U_w(w-r_it) & \text{if} : \tau > t \\ \sum_{k\in R} \dfrac{dQ_{ik}(\tau)}{dQ_i(\tau)} \cdot \int_{\alpha=0}^{1} P_k(t-\tau,w-\alpha\tau r_i)dA_i(\alpha) & \text{if} : \tau < t \end{cases} \tag{8}$$

Taking the Laplace-Stieltjes transform with respect to w ($\to v$)

$$P_i^{\sim}(t,v|\,H=\tau) \;=\; \begin{cases} e^{-vr_it} & \text{if} : \tau > t \\ \sum_{k\in R} \dfrac{dQ_{ik}(\tau)}{dQ_i(\tau)} \cdot A_i^{\sim}(vr_i\tau) \cdot P_k^{\sim}(t-\tau,v) & \text{if} : \tau < t \end{cases} \tag{9}$$

Unconditioning with respect to H (by $Q_i(t)$),

$$P_i^{\sim}(t,v) \;=\; e^{-vr_it}(1-Q_i(t)) + \sum_{k\in R} \int_{\tau=0}^{t} A_i^{\sim}(vr_i\tau)\;\cdot\; P_k^{\sim}(t-\tau,v)\; dQ_{ik}(\tau) \tag{10}$$

Taking the Laplace transform with respect to t ($\rightarrow s$) results in the theorem. $\square$

Comment 2:

In a CTMC environment with generator $B=[b_{ij}]$ ($b_i=-b_{ii}$)

$$P_i^{*\sim}(s,v) \;=\; \frac{1}{s+vr_i+b_i} + \sum_{k\in R,k\neq i} \frac{b_{ik}}{s+vr_i\alpha_i+b_i}\;\cdot\; P_k^{*\sim}(s,v) \tag{11}$$

Whose solution, in matrix form, is:

$$P^{*\sim}(s,v) = (sI+vR_\alpha - B)^{-1} D_1(s,v) \tag{12}$$

where the diagonal matrices are defined as R_α=diag$<r_i\alpha_i>$ and $D_1(s,v)$=diag$<\frac{s+vr_i\alpha_i+b_i}{s+vr_i+b_i}>$.

1.3 Completion time

Let us define the completion time (r.v.) and its conditional distribution as follow

$$C(w) = min[t : B(t)\geq w]$$

and

$$C_i(t,w) = Pr(C(w)\leq t | Z(0)=i).$$

Detailed derivation of the completion time with state dependent deterministic lost ratio:
(i.e., the $1-\alpha_i$ portion of the reward accumulated during the last sojourn in state i is lost when state i is left)

Conditioning on the sojourn time in state i (H), we have:

$$C_i(t,w|\,H=h) \;=\; \begin{cases} U\left(t-\dfrac{w}{r_i}\right) & \text{if}: h\,r_i \geq w \\[2ex] \displaystyle\sum_{k\in R} \frac{dQ_{ik}(h)}{dQ_i(h)}\;\cdot\; C_k(t-h,w-hr_i\alpha_i) & \text{if}: h\,r_i < w \end{cases} \tag{13}$$

Taking the Laplace-Stieltjes transform with respect to t results:

$$C_i^{\sim}(s,w|\,H=h) \;=\; \begin{cases} e^{-s\frac{w}{r_i}} & \text{if}: h\,r_i \geq w \\[2ex] \displaystyle\sum_{k\in R} \frac{dQ_{ik}(h)}{dQ_i(h)}\;\cdot\; e^{-sh}\;\cdot\; C_k^{\sim}(s,w-hr_i\alpha_i) & \text{if}: h\,r_i < w \end{cases} \tag{14}$$

Unconditioning with respect to H, yields

$$\begin{aligned} C_i^{\sim}(s,w) \;=\; &\int_{h=\frac{w}{r_i}}^{\infty} e^{-s\frac{w}{r_i}} dQ_i(h) + \sum_{k\in R}\int_{h=0}^{\frac{w}{r_i}} e^{-sh}\; C_k^{\sim}(s,w-hr_i\alpha_i)\; dQ_{ik}(h) \;= \\ &e^{-s\frac{w}{r_i}}\left[1-Q_i\left(\frac{w}{r_i}\right)\right] + \sum_{k\in R}\int_{h=0}^{\frac{w}{r_i}} e^{-sh}\; C_k^{\sim}(s,w-hr_i\alpha_i)\; dQ_{ik}(h) \end{aligned} \tag{15}$$

Now we try to take the Laplace transform with respect to w:

$$C_i^{\sim *}(s,v) \;=\; \int_{w=0}^{\infty} e^{-wv}\; C_i^{\sim}(s,w)\; dw \;=$$

$$\int_{w=0}^{\infty} e^{-wv}\; e^{-s\frac{w}{r_i}} \left[1 - Q_i\left(\frac{w}{r_i}\right)\right]\; dw + \sum_{k\in R} \int_{w=0}^{\infty} e^{-wv} \int_{h=0}^{\frac{w}{r_i}} e^{-sh}\; C_k^{\sim}(s, w - hr_i\alpha_i)\; dQ_{ik}(h)\; dw \;=$$

$$\frac{r_i\;[1 \;-\; Q_i^{\sim}(s \;+\; v\,r_i)\,]}{s \;+\; v\,r_i} \;+\; \sum_{k\in R} \int_{h=0}^{\infty} e^{-sh} e^{-hvr_i\alpha_i} \int_{w=hr_i}^{\infty} e^{-v(w-hr_i\alpha_i)}\; C_k^{\sim}(s, w - hr_i\alpha_i)\; dw\; dQ_{ik}(h) \tag{16}$$

Unfortunately the inner integral with respect to w is not a complete Laplace transform.

$$\begin{aligned} C_i^{\sim *}(s,v) \;=\; & \frac{r_i\;[1 \;-\; Q_i^{\sim}(s \;+\; v\,r_i)\,]}{s \;+\; v\,r_i} + \\ & \sum_{k\in R} \int_{h=0}^{\infty} e^{-sh} e^{-hvr_i\alpha_i} \int_{w=hr_i\alpha_i}^{\infty} e^{-v(w-hr_i\alpha_i)}\; C_k^{\sim}(s, w - hr_i\alpha_i)\; dw\; dQ_{ik}(h) - \\ & \sum_{k\in R} \int_{h=0}^{\infty} e^{-sh} e^{-hvr_i\alpha_i} \int_{w=hr_i\alpha_i}^{hr_i} e^{-v(w-hr_i\alpha_i)}\; C_k^{\sim}(s, w - hr_i\alpha_i)\; dw\; dQ_{ik}(h) \;= \\ & \frac{r_i\;[1 \;-\; Q_i^{\sim}(s \;+\; v\,r_i)\,]}{s \;+\; v\,r_i} \;+\; \sum_{k\in R} Q_{ik}^{\sim}(s \;+\; v\,r_i\,\alpha_i)\, C_k^{\sim *}(s,v) - \\ & \sum_{k\in R} \int_{h=0}^{\infty} e^{-sh} e^{-hvr_i\alpha_i} \int_{w=hr_i\alpha_i}^{hr_i} e^{-v(w-hr_i\alpha_i)}\; C_k^{\sim}(s, w - hr_i\alpha_i)\; dw\; dQ_{ik}(h) \end{aligned} \tag{17}$$

Hence the theorem, that we provided in our conference paper, is not correct!!!

Theorem 2

$$C_i^{\sim *}(s,v) \underbrace{\neq}_{!!!!!} \frac{r_i\;[1 \;-\; Q_i^{\sim}(s \;+\; v\,r_i)\,]}{s \;+\; v\,r_i} \;+\; \sum_{k\in R} Q_{ik}^{\sim}(s \;+\; v\,r_i\,\alpha_i)\, C_k^{\sim *}(s,v) \tag{18}$$

One of the most embarrassing facts in this work that we were aware of the different 'behaviour' of the accumulated reward and the completion time, as it was noted in an early version of our paper (below), but we did not recognize what this difference exactly means with respect to the completion time of the partial incremental loss model.

Comment 3: In this partial loss case

$$Pr(B(t) \leq w \mid Z(0) = i) \neq Pr(C(w) > t \mid Z(0) = i)$$