

Evaluation of completion time of PH work requirement*

András Pfening, Miklós Telek

Department of Telecommunications
Technical University of Budapest, 1521 Budapest, Hungary

Abstract

The common way of the numerical evaluation of task completion time in Stochastic Reward Models (SRM) requires an inverse transformation according to the work variable, and an integration according to the distribution of the work requirement, since only transform domain description of the task completion time is available, when the structure state process is a semi-Markov process (SMP). An effective algorithm is introduced for the case when the work requirement is a Phase type (PH) random variable.

Keywords: *Markov Reward Models, Phase type distribution, Completion time.*

1 Introduction

While stochastic reward processes have been studied since a long time [8, 5], the interest is only recently focused on using stochastic reward models (SRM) as modeling tools in performance/reliability evaluation. The descriptive power and the flexibility of stochastic models can be significantly increased by assigning a reward variable to each structure state. Examples for the interpretation of reward rates are execution rates of tasks in computing systems, number of active processors, throughput, etc. The different interpretation possibilities of the structure-state process and of the associated reward structure initiated various applications [9].

From the viewpoint of reliability analysis, one of the most important interpretations is the accumulation of the stress of real systems in different states. If the reward rates are restricted to be binary variables, the most important measures of the classical reliability theory [1] can be viewed as a particular case of the *SRM*.

Kulkarni et al. [7] derived the closed form Laplace transform equations of the completion time for the case when the underlying stochastic process $Z(t)$, referred to as *structure state process*, is a Semi-Markov Process (*SMP*). We refer to this case as Semi-Markov Reward Process (*SMRP*).

Various numerical techniques have been investigated in recent years for the evaluation of reward models defined over a CTMC, but the numerical analysis of SMRPs is still an open challenge. In this paper we introduce an effective algorithm for the case when the work requirement is a Phase type (PH) random variable. Bobbio and Trivedi [3] studied this problem when the structure state process is a CTMC, but we provide a different approach for SMRPs.

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2 Reward Semi-Markov Process

The modeling framework of SRMs consists in describing the behaviour of the system configuration in time by means of a stochastic process, and by associating to each state of the structure-state process a non-negative real constant representing the effective working capacity or performance level or cost or stress of the system in that state. The real variable associated to each structure-state is called the *reward rate* [5].

Let Ω be the set of the states of the structure state process and $Z(t)$ ($t \geq 0$) be the semi-Markov process defined over Ω and \underline{r} the corresponding reward vector. The initial state probability vector is $P(0)$ ($Pr\{Z(0) = i\} = P_i(0)$).

Let $\mathbf{Q}(t) = [Q_{ij}(t)]$ be the kernel of the semi-Markov process $Z(t)$. We denote by H the time duration until the first embedded time point in the semi-Markov process starting from state i at time 0 ($Z(0) = i$). The generic element (for $i, j \in \Omega$)

$$Q_{ij}(t) = Pr\{H \leq t, Z(H^+) = j | Z(0) = i\}$$

is the possibly defective distribution of H supposed that a transition from state i to state j took place at the embedded time point. If diagonal elements in $\mathbf{Q}(t)$ are nonzero the next embedded time point can be considered as a transition from state k to state k . The distribution of H is:

$$Q_i(t) = \sum_{j \in \Omega} Q_{ij}(t) \quad (i = 1, \dots, n).$$

Let us introduce the following matrix function $\mathbf{F}(t, y)$ to describe the distribution of the completion time:

$$F_{ij}(t, y) = Pr\{Z(C(y)) = j, C(y) \leq t | Z(0) = i, \gamma = y\}, \quad (1)$$

where $C(y)$ is the completion time (random variable) of y unit of work. $F_{ij}(t, y)$ is the probability that the completion of y unit of work happens in state j before t , starting in state i at $t = 0$. The distribution of the completion time is determined by $F_{ij}(t, y)$ by the mean of the following equation:

$$C(t) = \int_{y=0}^{\infty} \sum_j \sum_i P_i(0) F_{ij}(t, y) dG_g(y) = \sum_j \sum_i P_i(0) C_{ij}(t, y). \quad (2)$$

The derivation of $F_{ij}(t, y)$ based on the kernel matrix $Q_{ij}(t)$ can be inferred from [7, 2]:

$$F_{ij}^{\sim*}(s, w) = \delta_{ij} \frac{r_i [1 - Q_i^{\sim}(s + w r_i)]}{s + w r_i} + \sum_{k \in \Omega} Q_{ik}^{\sim}(s + w r_i) F_{kj}^{\sim*}(s, w). \quad (3)$$

2.1 Evaluation of the completion time

The evaluation of the completion time requires the following steps to be performed:

- Derivation of the matrix function $F_{ij}^{\sim*}(s, w)$ in double transform domain according to Equation 3.
- Evaluation of the LST transform $F_{ij}^{\sim}(s, y)$ by symbolic inverse Laplace transformation with respect to the work requirement variable w .

- Evaluation of the LST transform of the completion time $\mathbf{C}^\sim(s)$ by unconditioning the results of the previous step with respect to the distribution of the work requirement $G(y)$ (see Equation 2).
- Time domain solution obtained by a numerical inversion of $\mathbf{C}^\sim(s)$, for example by resorting to the Jagerman's method [6].

Due to the required symbolic and numerical steps, the procedure outlined in the previous points is effective only for small values of cardinality of the structure state process. But when the work requirement ($G(y)$) is a *PH* random variable, steps 2 and 3, can be evaluated by an effective computational method.

2.2 PH distributed work requirement

Let us define a Phase type (*PH*) distribution as a probability distribution with rational Laplace transform whose probability density function can be expressed as:

$$g(y) = \sum_{p=1}^n \sum_{r=0}^{m-1} c_{pr} y^r e^{-\lambda_p y}, \quad (4)$$

where n is the number of distinct eigenvalues (λ_p), m is the maximum of the eigenvalue multiplicities, and c_{pr} is a constant coefficient.

An efficient computational procedure can be envisaged for handling the Laplace inverse transformation with respect to w and the integration with respect to $g(y)$, suppose that the work requirement is a *PH* random variable.

Theorem 1 *The distribution of the completion time of a PH the work requirement with the probability density function $g(y)$ (4), can be evaluated as follows:*

$$C_{ij}^\sim(s) = \sum_{p=1}^n \sum_{r=0}^{m-1} (-1)^r c_{pr} \left. \frac{d^r F_{ij}^{\sim*}(s, w)}{dw^r} \right|_{w=\lambda_p} \quad (5)$$

where the derivative of order $r = 0$ simply means the substitution of the value $w = \lambda_p$ in the r.h.s.

Proof: When γ is a *PH* r.v. Equation 2 becomes:

$$\begin{aligned} C_{ij}^\sim(s) &= \int_{y=0}^{\infty} F_{ij}^\sim(s, y) dG(y) = \int_{y=0}^{\infty} g(y) F_{ij}^\sim(s, y) dy = \\ &= \sum_{p=1}^n \sum_{r=0}^{m-1} c_{pr} \int_{y=0}^{\infty} y^r e^{-\lambda_p y} F_{ij}^\sim(s, y) dy = \sum_{p=1}^n \sum_{r=0}^{m-1} (-1)^r c_{pr} \int_{y=0}^{\infty} \frac{d^r}{d\lambda_p^r} e^{-\lambda_p y} F_{ij}^\sim(s, y) dy = \\ &= \sum_{p=1}^n \sum_{r=0}^{m-1} (-1)^r c_{pr} \frac{d^r}{d\lambda_p^r} \int_{y=0}^{\infty} e^{-\lambda_p y} F_{ij}^\sim(s, y) dy = \sum_{p=1}^n \sum_{r=0}^{m-1} (-1)^r c_{pr} \frac{d^r F_{ij}^{\sim*}(s, \lambda_p)}{d\lambda_p^r} \end{aligned} \quad (6)$$

from which the theorem (Equation 5) follows. \square

This approach is very effective when the multiplicity of the eigenvalues is equal to 1, since the inverse Laplace transformation and integration in (6) reduces to a simple substitution; otherwise the symbolic derivation is required.

The class of *PH* distributions arises from the time to absorption in *CTMCs* with at least one absorbing state. When the work requirement is a *PH* random variable and the structure state process is a *CTMC*, the completion time problem can be alternatively evaluated by expanding the state space taking into account all the possible stages of the *PH* distribution. A completely automated tool that implements the state space expansion technique is in [4]. In this paper we devote our attention to the case when the structure state process is a *SMP*, and the method of the state space expansion can not be applied.

3 Example: Series System with Repair

In this section we illustrate the application of Theorem 1 through a numerical example of a series system with repair.

3.1 System Description

Consider a series system of two machines, **a** and **b**, with constant failure rates λ_a and λ_b , respectively. If any of the machines fails, both machines are switched off, and the faulty machine is repaired with a generally distributed random repair time, according to distribution functions $G_a(t)$ or $G_b(t)$. We assume that no machine can fail while the system is down, and that the two machines are independent.

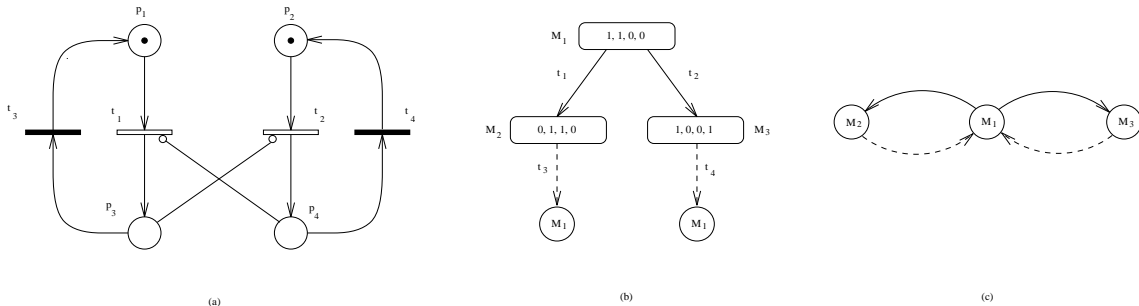


Figure 1: Series System with Repair

The system behaviour is described on Figure 1(a) by a Stochastic Petri Net. Place p_1 contains a token, when machine **a** is in up state. Transition t_1 represents the failure process of machine **a**. When a failure happens, a token is placed to p_3 , and the repair is immediately started. Transition t_3 represents the repair procedure. The firing time of t_1 is exponentially distributed with parameter λ_a , while the firing time of t_3 is generally distributed, according to $G_a(t)$. The same description can be applied to machine **b**, with the appropriate indices. The inhibitor arcs represent the restriction that no machine can fail when the system is down, i.e. when there is a token in place p_3 or p_4 .

The reachability graph of the Petri net can be seen on Figure 1(b). Each marking is a 4-tuple counting the number of tokens in places p_1 to p_4 . Solid arcs represent transitions firing according to exponential distribution functions, while dashed arcs represent transitions firing according to general distributions.

3.2 Evaluation of the Completion Time

Since the only up state of the system is when both p_1 and p_2 contain a token, the reward rate vector is $\underline{r} = \{1, 0, 0\}$. Let us suppose that the system starts from state 1 at time $t = 0$, i.e. $P(0) = \{1, 0, 0\}$. The $Q^\sim(s)$ matrix can be written as

$$Q^\sim(s) = \begin{bmatrix} 0 & \frac{\lambda_a}{s + \lambda_a + \lambda_b} & \frac{\lambda_b}{s + \lambda_a + \lambda_b} \\ G_a^\sim(s) & 0 & 0 \\ G_b^\sim(s) & 0 & 0 \end{bmatrix}. \quad (7)$$

Since the procedure starts in state 1, only the first column of matrix $\mathbf{F}^{\sim*}(s, w)$ plays role in the evaluation of the completion time. Furthermore, since the reward vector in our example is $\underline{r} = \{1, 0, 0\}$, $F_{11}^{\sim*}(s, w)$ is the only entry of $\mathbf{F}^{\sim*}(s, w)$ that has an affect on $C^\sim(s)$:

$$C^\sim(s) = \int_{y=0}^{\infty} F_{11}^\sim(s, y) dG_g(y), \quad (8)$$

where

$$F_{11}^{\sim*}(s, w) = \frac{1}{s + w + \lambda_a(1 - G_a^\sim(s)) + \lambda_b(1 - G_b^\sim(s))}. \quad (9)$$

There are two possibilities to derive $C^\sim(s)$:

- Symbolic inverse Laplace transformation of $F_{11}^{\sim*}(s, w)$, and integration according to Equation 8.
- Application of Theorem 1 where phase type approximation of the work requirement is applied when it is not a PH random variable.

In the latter case, the multiple eigenvalues of the PH random variable causes an other computationally intensive step, i.e. the symbolic evaluation of the first or higher order derivatives of $C_{ij}^{\sim*}(s, w)$ according to w . An alternative solution to avoid this time consuming method is to approximate the random work requirement with a PH random variable, whose eigenvalues are distinct.

In the following numerical example we introduce all of these cases.

3.3 Numerical Results

In the studied numerical example we analyze the case of constant work requirement. The repair times were also chosen to be deterministic values. The values of the different parameters were as follows: the failure rates are $\lambda_a = \lambda_b = 1$, the deterministic repair times are $\mu_a = \mu_b = 5$, and the deterministic work requirement is $w = 3$.

The best k th order PH approximation of the deterministic work requirement is the Erlang(k) structure, where k is the number of phases. However, this model results equal eigenvalues for the approximating PH distribution, i.e. $n = k$ and $m = 1$ in Theorem 1, so the evaluation of the order n derivative is necessary. The calculation can be simplified if we enforce the eigenvalues to be different in the approximating structure. The PH approximation of the deterministic work requirement with different eigenvalues was obtained by slightly modifying the intensities of the Erlang(k) structure and by maintaining the mean value to remain the same.

The constant work requirement was approximated by phase type distributions of order 2, 4 and 8 (Figure 2). The figures below show the distribution of the completion time derived by the

exact calculation method, by approximating the work requirement by Erlang(k) distributions and by modified Erlang distributions, where the intensities were separated by 1% and 10%, in percent of the original Erlang(k) intensity.

Figures 3,4,5 show, as it is expected, that the higher the order of the PH approximation is, the more accurate the approximation of the completion time distribution is. More interesting conclusion is that the separation of the eigenvalues of the approximating PH distribution made no significant affect on the resulting curves, i.e. the accuracy of the proposed effective algorithm is tolerable.

The $C^{\sim}(s)$ Laplace-Stieltjes transforms derived by either of the above discussed ways were transformed to time domain using the Jagerman method [6]. We experienced serious numerical problems using the Jagerman method, when the intensity difference was small, 1% (Figures 6,7,8). The reason could follow from the fact, that when the intensity difference is small, the coefficients are higher of orders of magnitude, resulting in numerical problems. The results could be smoothed by choosing proper parameter values for the Jagerman method; however the scope of this paper is not enough to discuss all the experiences.

The Laplace-Stieltjes transform can be used to get the moments of the completion time distributions. This way the numerical problems caused by the Jagerman method are avoided. The first three moments of the resulting probability density functions are included in Table 1 below. The first moments are the same for all the distributions, while the higher the degree of the PH approximation is, the closer are higher moments of the distributions. The separation of the eigenvalues does not result in significant difference of the higher order moments.

	Exact calc.	Erlang(2)			Erlang(4)			Erlang(8)		
		Intensity diff.			Intensity diff.			Intensity diff.		
		0%	1%	10%	0%	1%	10%	0%	1%	10%
m_1	33	33	33	33	33	33	33	33	33	33
m_2	1239	1784	1784	1785	1511	1511	1515	1405	1375	1382
m_3	51537	130836	130839	131123	86694	86700	87247	72234	68003	68996

Table 1: First three moments of the distributions derived in different ways

4 Conclusion

A computational method for the evaluation of the completion time of PH work requirement is proposed. The benefit of the proposed method is more significant, when the PH work requirement has different eigenvalues. We studied the accuracy-computational complexity trade off, when the completion time of a non- PH work requirement is evaluated by approximation.

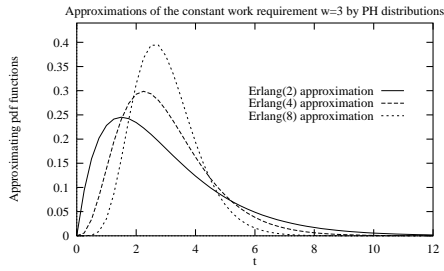


Figure 2: Approximations of the constant work requirement $w = 3$ by PH distributions

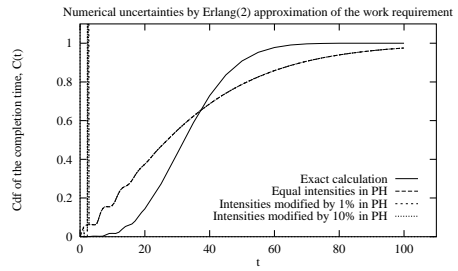


Figure 6: Numerical uncertainties at Erlang(2) approximation of the work requirement

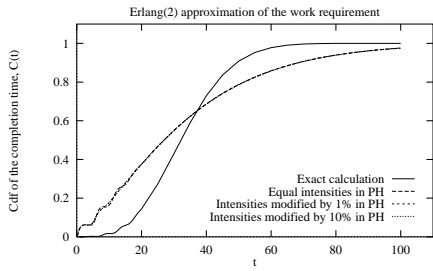


Figure 3: Erlang(2) approximation of the work requirement

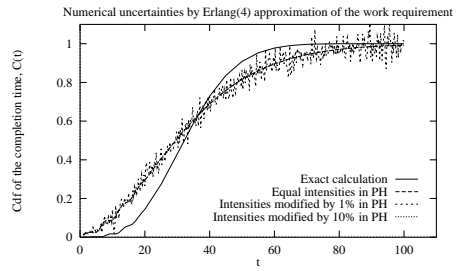


Figure 7: Numerical uncertainties at Erlang(4) approximation of the work requirement

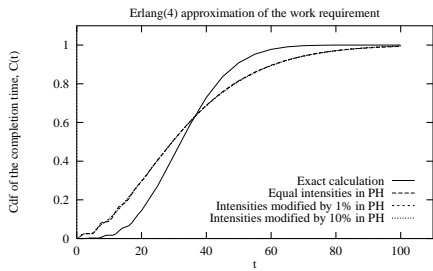


Figure 4: Erlang(4) approximation of the work requirement

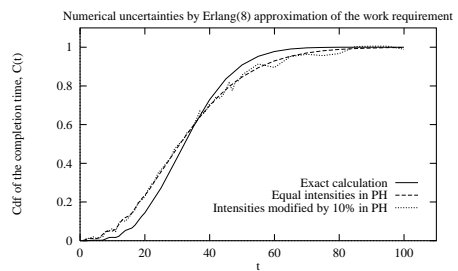


Figure 8: Numerical uncertainties at Erlang(8) approximation of the work requirement

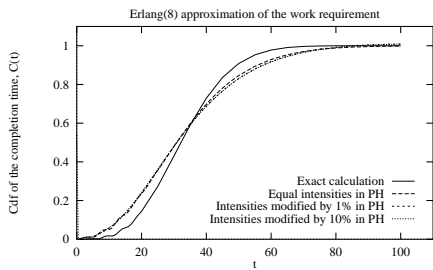


Figure 5: Erlang(8) approximation of the work requirement

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