AN EFFECTIVE APPROACH TO THE COMPLETION TIME ANALYSIS OF ON-OFF MARKOV REWARD MODELS*

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ABSTRACT

Analysis of Markov Reward Models (MRM) with preemptive resume (prs) policy usually results in a double transform expression, whose solution is based on the inverse transformations both in time and reward variable domain. This paper discusses the case when the reward rates can be either 0 *or a positive value c. These systems are called on-off MRMs. We analyze the completion time of on-off MRMs and present a symbolic expression of its moments, from which a computationally effective recursive numerical method can be obtained. The mean and the standard deviation of the completion time of a Carnegie-Mellon multiprocessor system are evaluated by the proposed method.*

1 INTRODUCTION

The properties of stochastic reward processes have been studied since a long time (McLean and Neuts, 1967; Howard, 1971). However, only recently, stochastic reward models (*SRM*) have received attention as a modeling tool in performance and reliability evaluation. Indeed, the possibility of associating a reward variable to each structure state increases the descriptive power and the flexibility of the model. Different interpretations of the structure-state process and of the associated reward structure give rise to various applications. Common assignments of the reward rates are: execution rates of tasks in computing systems, number of active processors, throughput, etc.

To point out the reliability aspects, one of the most important interpretations is the accumulation of the stress of a real system in the different states. Moreover, the most important measures of the classical reliability theory (Barlow and Proschan, 1975) can be viewed as a particular case of *SRM* obtained by constraining the reward rates to be binary variables.

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Kulkarni et al. (Kulkarni et al., 1986) derived the closed form Laplace transform equations of the completion time for the case when the underlying stochastic process $Z(t)$, referred to as the *structure state process*, is a Continuous Time Markov Chain (*CTMC*). We refer to this case as Markov Reward Model (*MRM*).

Various numerical techniques have been investigated in recent papers for the evaluation of the performability: (Meyer, 1982; Iyer et al., 1986). In this paper, we improve the results of (Begain et al., 1995) and propose a computationally effective approach not only to calculate the mean completion time of on-off *MRM*s, but to obtain its higher moments as well.

The paper is organized as follows. Section 2 provides the formal definition of *SRM*s, and introduces the studied subset of *MRM*s. In Section 3 the analysis of on-off *MRM*s is presented. Section 4 gives an application of the proposed computational approach to the completion time analysis of a Carnegie-Mellon multiprocessor system. The paper is concluded in Section 5.

2 STOCHASTIC REWARD MODELS

The adopted modeling framework consists of describing the behaviour of the system configuration in time by means of a stochastic process, and by associating a non-negative real constant to each state of the structure-state process representing the effective working capacity or performance level or cost or stress of the system in that state. The variable associated to each structure-state is called the *reward rate* (Howard, 1971).

Let the *structure-state process* $Z(t)$ $(t \geq 0)$ be a (right continuous) stochastic process defined over a discrete and finite state space Ω of cardinality n. Let f be a non-negative real-valued function defined as:

$$f[Z(t)] = r_i \geq 0 \ , \qquad \text{if } \ Z(t) = i \qquad (1)$$

$f[Z(t)]$ represents the instantaneous reward rate associated to state i.

Definition 1 *The* **accumulated reward** $B(t)$ *is a random variable which represents the accumulation of reward in time:*

$$B(t) = \int_0^t f[Z(\tau)]d\tau = \int_0^t r_{Z(\tau)}d\tau.$$

$B(t)$ is a stochastic process that depends on $Z(u)$ for $0 \leq u \leq t$. According to Definition 1 this paper restricts the attention to the class of models in which no state transition can entail to a loss of the accumulated reward. A *SRM* of this kind is called *preemptive resume* (prs) model. The distribution of the accumulated reward is defined as $B(t,w) = Pr\{B(t) < w\}$.

The complementary question concerning the reward accumulation of *SRM*s is the time needed to complete a given (possibly random) work requirement (i.e. the time to accumulate the required amount of reward).

Definition 2 *The* **completion time** *C is the random variable representing the time to accumulate a reward requirement equal to a random variable W:*

$$C = min\,[t \geq 0 \,:\, B(t) = W] \ .$$

C is the time instant at which the work accumulated by the system reaches the value W for the first time. Assume, in general, that W is a random variable with distribution $G(w)$ with support on $(0,\ \infty)$. The degenerate case, in which W is deterministic and the distribution $G(w)$ becomes the unit step function $U(w-w_d)$, can be considered as well. For a given sample of $W = w$, the completion time $C(w)$ and its *Cdf* $C(t,w)$ are defined as:

$$\begin{aligned} C(w) &= \min\,[t \geq 0 \,:\, B(t) = w] \\ C(t,w) &= Pr\,\{C(w) \leq t\} \end{aligned} \tag{2}$$

The completion time C is characterized by the following distribution:

$$\hat{C}(t) = Pr\,\{\,C \leq t\} = \int_0^\infty C(t,w)\,dG(w) \tag{3}$$

The distribution of the completion time of a *prs SRM* is closely related to the distribution of the accumulated reward by means of the following relation:

$$\begin{aligned} B(t,w) = Pr\,\{B(t) \leq w\} &= Pr\,\{C(w) \geq t\} \\ &= 1 - C(t,w) \end{aligned} \tag{4}$$

For the purposes of the subsequent analysis below we define the following matrix functions $\mathbf{P}(t,w) = \{P_{ij}(t,w)\}$ and $\mathbf{F}(t,w) = \{F_{ij}(t,w)\}$ as:

$$P_{ij}(t,w) = Pr\{Z(t) = j\,,\, B(t) \leq w \,|\, Z(0) = i\} \tag{5}$$

$$F_{ij}(t,w) = Pr\{Z(C(w)) = j\,,\, C(w) \leq t \,|\, Z(0) = i\} \tag{6}$$

- $P_{ij}(t,w)$ is the joint distribution of the accumulated reward and the structure state at time t supposed that the initial state of the structure state process is i.
- $F_{ij}(t,w)$ is the joint distribution of the completion time and the structure state at completion supposed that the initial state of the structure state process is i.

From (5) and (6), it follows for any t and i that $\sum_{j \in \Omega} [\,P_{ij}(t,w) + F_{ij}(t,w)\,] = 1$.

Given that $G(w)$ is the Cdf of the random work requirement W, the distribution of the completion time is:

$$\begin{aligned} \hat{C}(t) &= \int_{w=0}^\infty \left[\sum_{i \in \Omega} \sum_{j \in \Omega} P_i(0) F_{ij}(t,w) \right] dG(w) = \\ &\int_{w=0}^\infty \underline{P}(0)\, \mathbf{F}(t,w)\, \underline{h}^T\, dG(w) \end{aligned} \tag{7}$$

where $\underline{P}(0)$ is the row vector of the initial probabilities, and $\underline{h}^T$ is the column vector with all the entries equal to 1.

Markov Reward Models

The introduced matrix functions can be described in double transform domain based on the infinitesimal generator $\mathbf{A}$ of the subordinated *CTMC*. Detailed derivations are presented in (Kulkarni et al., 1986; Telek, 1994; Bobbio and Telek, 1995). The final expressions take the following matrix forms:

$$\mathbf{F}^{\sim *}(s,v) = (s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1}\mathbf{R} \tag{8}$$

$$\mathbf{P}^{\sim *}(s,v) = \frac{s}{v}\,(s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1} \tag{9}$$

where $^\sim$ denotes the Laplace-Stieltjes transform with respect to $t(\rightarrow s)$, * denotes the Laplace transform with respect to $w(\rightarrow v)$, $\mathbf{I}$ is the identity matrix and $\mathbf{R}$ is the diagonal matrix of the reward rates (r_i); the dimensions of $\mathbf{I}$, $\mathbf{R}$, $\mathbf{A}$, $\mathbf{F}$ and $\mathbf{P}$ are $(n \times n)$.

Starting from Equations (8-9), the evaluation of the reward measures of a *MRM* requires the following steps:

1. derivation of the entries of the $\mathbf{P}^{*\sim}(s,v)$ and $\mathbf{F}^{*\sim}(s,v)$ matrices symbolically in the double transform domain according to (8) and (9);
2. symbolic inverse Laplace-Stieltjes transformation of $\mathbf{P}^{*\sim}(s,v)$ and/or $\mathbf{F}^{*\sim}(s,v)$ with respect to s;
3. numerical inverse Laplace transformation with respect to v;
4. unconditioning the result according to the Cdf of the work requirement defined by (7).

However, this way of the analysis contains some computationally intensive steps, and the whole procedure can be applied to very small scale problems (less than 6-8 states) only.

3 ANALYSIS OF ON-OFF *MRM*s

Definition 3 *The subclass of* MRMs *in which the reward rates can only be* 0 *or a positive value c is called* **on-off MRMs**.

There are several practical examples that result in an on-off *MRM*, moreover most of the classical markovian reliability theory can be described with on-off *MRM*s.

The completion time analysis of an on-off *MRM* can always be transformed into the analysis of an on-off *MRM* with binary reward rates (i.e. the positive reward rates equal to 1). If c is the constant reward rate of the analyzed on-off *MRM*, and the random work requirement is W, then the system is equivalent to the same on-off *MRM* with binary reward rates, where the work requirement is W/c, with distribution $G(w/c)$. In the rest of this paper we consider only binary reward rates.

According to the associated reward rates the states of on-off *MRM*s can be divided into two parts, namely R and $R^c = \Omega - R$, where R contains the states with positive reward rates. Suppose that R contains m states out of n. Thus we can renumber the states in Ω in a way that the states numbered $1, 2, \ldots, m$ belong to R and the states numbered $m+1, m+2, \ldots, n$ belong to R^c. By this ordering of the states, $\mathbf{A}$ can be partitioned into the following form $\mathbf{A} = \left[\begin{array}{c|c} \mathbf{A}_1 & \mathbf{A}_2 \\ \hline \mathbf{A}_3 & \mathbf{A}_4 \end{array}\right]$, where $\mathbf{A}_1$ describes the transitions inside R, $\mathbf{A}_2$ contains the intensity of the transitions from R to R^c, $\mathbf{A}_3$ the transitions from R^c to R, and $\mathbf{A}_4$ the transitions inside R^c. If there is no absorbing state group in R^c, i.e. the completion time of a finite work requirement w is finite with probability 1, then $\mathbf{A}_4^{-1}$ exists. By the renumbering of states the diagonal matrix of the reward rates has the form $\mathbf{R} = \left[\begin{array}{c|c} \mathbf{I}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right]$, where $\mathbf{I}_1$ is a unity matrix with cardinality $m \times m$.

The Moments Of The Completion Time Of On-Off *MRM*s

In this section we calculate the moments of the completion time using the Laplace-Stieltjes transform, and we propose a recursive method to calculate the moments in a computationally effective way. We make use of the idea proposed by Iyer et al. for the analysis of the accumulated reward (Iyer et al., 1986). The nth moment of the completion time of w amount of work is defined by

$$M^{(n)}(w) = E\{C(w)^n\} = \int\limits_{t=0}^{\infty} t^n \, d\, C(t,w).$$

Theorem 1 *The nth moment of the completion time of an on-off* MRM *with binary reward rates and work requirement w is:*

$$M^{(n)}(w) = n! \; \underline{P}(0) \; \mathrm{LT}^{-1} \left[(\mathbf{R}v - \mathbf{A})^{-(n+1)} \mathbf{R} \right] \; \underline{h}^T \tag{10}$$

Proof: The moments can be calculated using the Laplace-Stieltjes transform of the completion time and substituting Equation (8):

$$\begin{aligned} M^{(n)}(w) &= (-1)^n \lim_{s\to 0} \frac{\partial^n \, \mathrm{LT}^{-1}\left[C^{\sim *}(s,v)\right]}{\partial s^n} = \\ & (-1)^n \lim_{s\to 0} \frac{\partial^n \, \mathrm{LT}^{-1}\left[\underline{P}(0)\; \mathbf{F}^{\sim *}(s,v)\; \underline{h}^T\right]}{\partial s^n} = \\ & (-1)^n \underline{P}(0) \lim_{s\to 0} \frac{\partial^n \, \mathrm{LT}^{-1}\left[\mathbf{F}^{\sim *}(s,v)\right]}{\partial s^n} \underline{h}^T = \\ & (-1)^n \underline{P}(0) \lim_{s\to 0} \frac{\partial^n \, \mathrm{LT}^{-1}\left[(s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1}\mathbf{R}\right]}{\partial s^n} \underline{h}^T . \end{aligned} \tag{11}$$

In the above formula the order of the inversion and the derivation can be changed:

$$M^{(n)}(w) = (-1)^n \underline{P}(0)\, \mathrm{LT}^{-1} \left[\lim_{s\to 0} \frac{\partial^n \, (s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1}\mathbf{R}}{\partial s^n} \right] \underline{h}^T .$$

The derivation can be accomplished using Leibniz's rule, and setting the value of s to 0:

$$M^{(n)}(w) = n! \; \underline{P}(0) \; \mathrm{LT}^{-1} \left[(v\mathbf{R} - \mathbf{A})^{-(n+1)}\mathbf{R} \right] \; \underline{h}^T . \quad \square$$

Because of the inverse Laplace transformation contained in Equation (10) the calculation of the moments is a computationally intensive task. Begain et al. (Begain et al., 1995) proposed an effective method to calculate the first moment, i.e. the mean value of the completion time:

Theorem 2 *The expected time while an on-off* MRM *with binary reward rates completes w amount of work is: (Begain et al., 1995)*

$$E\{C(w)\} = \underline{P}(0) \left[\begin{array}{cc} \mathbf{L}(w) & -\mathbf{L}(w)\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3\mathbf{L}(w) & \phi \end{array} \right] \underline{h}^T \tag{12}$$

where

$$\beta = \mathbf{A}_1 - \mathbf{A}_2\mathbf{A}_4^{-1}\mathbf{A}_3 \; , \quad \mathbf{L}(w) = \int_0^w e^{\beta u} du.$$

and

$$\phi = -\mathbf{A}_4^{-1} + \mathbf{A}_4^{-1}\mathbf{A}_3\mathbf{L}(w)\mathbf{A}_2\mathbf{A}_4^{-1}$$

Proof: See (Begain et al., 1995). $\square$

Here we propose a recursive method to calculate the higher moments. First we introduce some notation. Let $M_{ij}^{(n)}(w)$ be the nth moment of the completion time assuming that the process was started in state i, the work requirement was completed in state j and the work requirement was w. Let $\mathbf{M}^{(n)}(w)$ be a matrix with entries $M_{ij}^{(n)}(w)$, and $\mathbf{M}^{*\,(n)}(v)$ be the Laplace transform of $\mathbf{M}^{(n)}(w)$. Let $\mathbf{F}^{\sim *\,(n)}(0,v) = \left.\frac{\partial^n \mathbf{F}^{\sim *}(s,v)}{\partial s^n}\right|_{s=0}$.

Theorem 3 *The nth moment ($n \geq 2$) of the completion time of an on-off* MRM *with binary reward rates and work requirement w can be obtained as*

$$\begin{aligned} M^{(n)}(w) \;&=\; \underline{P}(0)\; \mathbf{M}^{(n)}(w)\; \underline{h}^T = \\ & n\; \underline{P}(0) \int\limits_{y=0}^{w} \mathbf{\Theta}(w-y)\, \mathbf{M}^{(n-1)}(y)\; \underline{h}^T\, d\,y \\ & + n\; \underline{P}(0)\; \hat{\mathbf{A}}\, \mathbf{M}^{(n-1)}(w)\; \underline{h}^T \end{aligned} \tag{13}$$

where

$$\mathbf{\Theta}(w) = \left[\begin{array}{cc} e^{\beta\, w} & -e^{\beta\, w}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3 e^{\beta\, w} & \mathbf{A}_4^{-1}\mathbf{A}_3 e^{\beta\, w}\mathbf{A}_2\mathbf{A}_4^{-1} \end{array} \right]$$

and

$$\hat{\mathbf{A}} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{A}_4^{-1} \end{array}\right].$$

Proof: From Equation (8)

$$(s\mathbf{I} + v\mathbf{R} - \mathbf{A})\mathbf{F}^{\sim *}(s, v) = \mathbf{R} \tag{14}$$

Using Leibniz's rule, the differentiation of Equation (14) $n+1$ times with respect to s and setting $s = 0$ yields

$$\mathbf{F}^{\sim *\,(n+1)}(0, v) = -(n+1)(\mathbf{R}v - \mathbf{A})^{-1}\mathbf{F}^{\sim *\,(n)}(0, v) \tag{15}$$

Because $\mathbf{M}^{*\,(n)}(v) = (-1)^n \mathbf{F}^{\sim *\,(n)}(0, v)$ according to Equation (11), Equation (15) can be rewritten as

$$\mathbf{M}^{*\,(n+1)}(v) = (n+1)\,(\mathbf{R}v - \mathbf{A})^{-1}\mathbf{M}^{*\,(n)}(v). \tag{16}$$

Since $\mathrm{LT}^{-1}\left[\,(v\mathbf{R} - \mathbf{A})^{-1}\right] = \mathbf{\Theta}(w) + \hat{\mathbf{A}}\delta(w)$, where $\delta(w)$ denotes the Dirac delta function, the inversion and the integration yields the theorem. □

To apply the result of Theorem 3 for the evaluation of the first moment we shall define in accordance with Equation (11)

$$\begin{aligned} M^{(0)}(w) &= \mathrm{LT}^{-1}\left[C^{\sim *}(0, v)\right] = \\ & \quad \mathrm{LT}^{-1}\left[\underline{P}(0)\ \mathbf{F}^{\sim *}(0, v)\ \underline{h}^T\right] \end{aligned}$$

and $\mathbf{M}^{*\,(0)}(v) = \mathbf{F}^{\sim *}(0, v)$. To express the first moment first we use Equation (13) then Equation (16) to obtain

$$\begin{aligned} M^{(1)}(w) &= \mathrm{LT}^{-1}\left[\underline{P}(0)\ \mathbf{M}^{*\,(1)}(v)\ \underline{h}^T\right] = \\ & \quad \mathrm{LT}^{-1}\left[\underline{P}(0)\ (\mathbf{R}v - \mathbf{A})^{-1}\,\mathbf{M}^{*\,(0)}(v)\ \underline{h}^T\right], \end{aligned}$$

which is by definition

$$\begin{aligned} &\mathrm{LT}^{-1}\left[\underline{P}(0)\ (\mathbf{R}v - \mathbf{A})^{-1}\,\mathbf{F}^{\sim *}(0, v)\ \underline{h}^T\right] = \\ &\mathrm{LT}^{-1}\left[\underline{P}(0)\ (\mathbf{R}v - \mathbf{A})^{-2}\,\mathbf{R}\ \underline{h}^T\right] = \\ &\mathrm{LT}^{-1}\left[\underline{P}(0)\ \tfrac{1}{v}\,(\mathbf{R}v - \mathbf{A})^{-1}\ \underline{h}^T\right], \end{aligned}$$

since $(\mathbf{R}v - \mathbf{A})^{-2}\,\mathbf{R}\ \underline{h}^T = 1/v\ (\mathbf{R}v - \mathbf{A})^{-1}\ \underline{h}^T$, because p $\mathbf{A}\,\underline{h}^T = \underline{0}^T$. The inverse transform gives the result of Theorem 2.

If the system is started from operational states, which is a rather realistic assumption, (i.e. $P_i(0) = 0$ if $i \in R^c$), then one can neglect the second term of the rhs of Equation (13). This term stands for the time needed to start the reward accumulation (i.e. to enter R) when the system starts from R^c.

Another interesting analysis problem of on-off *MRMs* is the probability distribution of the structure state process at completion, i.e. $P^c_{ij} = Pr\{Z(C) = j|Z(0) = i\}$. A closed form solution of this problem is presented in (Begain et al., 1995).

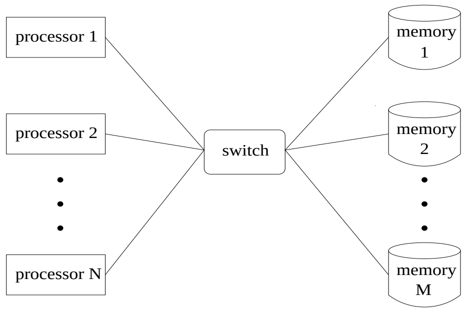

Figure 1: Example system structure

4 NUMERICAL EXAMPLE

The results of this paper are demonstrated by the analysis of a simple multiprocessor system. The system is similar to the Carnegie-Mellon multiprocessor system, presented in (Smith et al., 1988). The system consists of N processors, M memories, and an interconnection network (i.e., a crossbar switch) that allows any processor to access any memory (Figure 1). The failure rates per hour for the system are set to be 0.2, 0.1 and 0.05 for the processors, memories and the switch respectively.

Viewing the interconnecting network as one switch and modeling the system at the processor-memory-switch level, the switch becomes essential for the system operation. It is also clear that a minimum number of processors and memories are necessary for the system to be operational. Each state is thus specified by a triple (i, j, k) indicating the number of operating processors, memories, and networks, respectively. We augment the states with the nonoperational state F. Events that decrease the number of operational units are the failures and events that increase the number of operational elements are the repairs. We assume that failures do not occur when the system is not operational. When a component fails, a recovery action must be taken (e.g., shutting down the a failed processor, etc.), or the whole system will fail and enter state F. The probability that the recovery action is successfully completed is known as *coverage*.

Two kinds of repair actions are considered, global repair which restores the system to state $(N, M, 1)$ with rate $\mu = 0.2$ per hour from state F, and local repair, which can be thought of as a repair person beginning to fix a component of the system as soon as a component failure occurs. We assume that there is only one repair person for each component type. Let the local repair rates be 2.0, 1.0 and 0.5 for the processors, memories and the switch, respectively.

The studied system has two processors, two memories, and one connections network, thus the state space consists of 13 states. For this case, the minimal configuration is supposed to have one processor, one memory and one interconnection switch. The value of the coverage was set to 0.90. This is a simple system, however a system of this size would be untractable using the double transformation method. We emphasize that it is just a demonstrative example, the performance of larger systems can also be calculated using the proposed method. More work has to be done to learn the limitations of the proposed method.

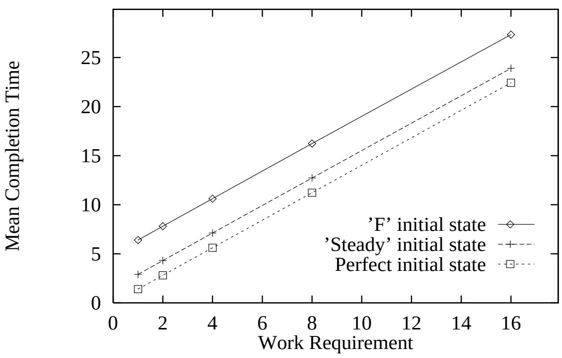

Figure 2: The mean value of the completion time

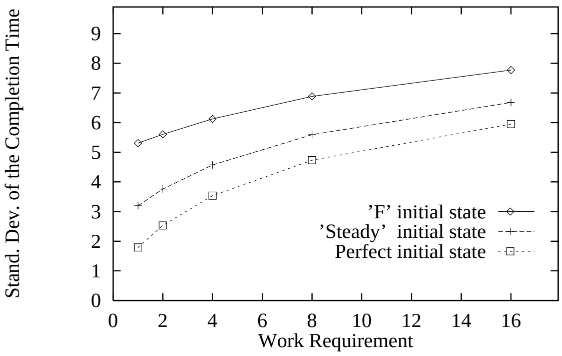

Figure 3: The standard deviation of the completion time

The mean value and the standard deviation of the completion time were calculated, the former using Theorem 2, the latter using Theorem 3 and the well known formula $\sigma(w) = (M^{(2)}(w) - (M^{(1)}(w))^2)^{1/2}$. The work requirement was chosen to take values from the interval $[1, 16]$ (in work hours). In Figures 2, 3 the mean value and the standard deviation of the completion time are shown, assuming that the system was started from the perfect state $(N, M, 1)$, from state F and from the steady state distribution. The integral values were calculated numerically in an iterative way. In each step twice as many sample points were evaluated, and the process was stopped when the maximal relative change of the values was less than 2%.

The mean completion time is higher if the system is started in the steady state instead of the perfect $(N, M, 1)$ state, or if the system is started in the F state instead of the steady state. The difference between the perfect and the F initial state curves refers to the mean time to get from state F to the perfect state. The curves of the standard deviation of the completion time show a similar picture. We have to note that the 2% accuracy limit brings more inaccuracy for higher values (8,16). The curve referring to the F state at time 0 takes the value of the standard deviation of the time to get from state F to the perfect state.

5 CONCLUSION

Markov Reward Models (*MRM*s) have been widely used to model performance and reliability of computer and communication systems. *On-off MRM*s represent a subclass of *MRM*s of practical interest in many real situations. We discussed the analytical description of *MRM*s, focusing on on-off assignment of reward variables. A numerically effective computation method is described for the moments of the completion time of an on-off *MRM*. Performance parameters of a Carnegie-Mellon multiprocessor system are evaluated by the proposed method as an example.

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REFERENCES

Barlow, R. and Proschan, F. 1975. *Statistical Theory of Reliability and Life Testing.* Holt, Rinehart and Winston, New York.

Begain, K., Jereb, L., Puliafito, A., and Telek, M. 1995. "On-off markov reward models." In *Relectronic'95 - Ninth Symposium on Quality and Reliability in Electronics, Budapest*, Budapest.

Bobbio, A. and Telek, M. 1995 "Markov regenerative SPN with non-overlapping activity cycles." In *International Computer Performance and Dependability Symposium - IPDS95.*

Howard, R. 1971. *Dynamic Probabilistic Systems, Volume II: Semi-Markov and Decision Processes.* John Wiley and Sons, New York.

Iyer, B., Donatiello, L., and Heidelberger, P. 1986. "Analysis of performability for stochastic models of fault-tolerant systems." *IEEE Transactions on Computers*, C-35:902–907.

Kulkarni, V., Nicola, V., and Trivedi, K. 1986. "On modeling the performance and reliability of multi-mode computer systems." *The Journal of Systems and Software*, 6:175–183.

McLean, R. and Neuts, M. 1967. "The integral of a step function defined on a Semi-Markov process." *SIAM Journal on Applied Mathematics*, 15:726–737.

Meyer, J. 1982. "Closed form solution of performability." *IEEE Transactions on Computers*, C-31:648–657.

Smith, R., Trivedi, K., and Ramesh, A. 1988. "Performability analysis: Measures, an algorithm and a case study." *IEEE Transactions on Computers*, C-37:406–417.

Telek, M. 1994. *Some advanced reliability modelling techniques.* Phd Thesis, Hungarian Academy of Science.