Mission Time Analysis of Large Dependable Systems

Sándor Rácz, Miklós Telek*
Technical University of Budapest, 1521 Budapest, Hungary,
raczs@ttt-atm.ttt.bme.hu, *telek@hit.bme.hu*

Abstract

*The mission time analysis of large dependable systems that can be described by Markov Reward Models (*MRM*) with phase type distributed impulse and constant rate rewards is considered in the paper. A single Laplace transform domain description of the distribution of completion time is provided through a new analysis approach. Based on this description an effective numerical method is introduced which allows the evaluation of models with large state space (*$\sim 10^6$ *states). The applied analysis approach makes the use of an expanded Markov chain, but the state space expansion is much less than for common "phase type expansion", because the expanded state space is composed by the union (instead of product) of the original state space and the state space of the phase type structure of non-zero impulse rewards. (Roughly speaking, the applied state space expansion is additive in contrast with the multiplicative state space expansion used for phase type modeling.)*

The proposed method, which is a counterpart of the analysis method of accumulated reward of MRMs with rate and impulse rewards, provides the moments of reward measures approximately on the same computational cost and memory requirement as the transient analysis of the expanded Continuous Time Markov Chain. Numerical example demonstrates the abilities of the proposed method.

Key words: *Markov Reward Models, Impulse and Rate Reward, Accumulated Reward, Completion Time, Phase Type Distribution.*

1 Introduction

The performance of real life computer and communication systems can be analyzed through Markov Reward Models that describes the behaviour and the performance of the considered system [6, 2, 4, 8]. Two main different points of view have been assumed in the literature when dealing with MRM [7]. In the *system oriented* point of view the most significant measure is the total amount of work done by the system in a finite interval. This measure is commonly referred to as *performability*. In the *user oriented* (or *task oriented*) point of view the system is regarded as a server, and the emphasis of the analysis is on the time the system needs to accomplish an assigned task. Consequently, the most characterizing measure becomes the mission time that is also referred to as completion time. The numerical analysis of this second measure is considered in the paper. A significant bottle-neck of the application of this models for analysis of real systems is the lack of numerical methods that can be applied for models with large state space and complex (rate and impulse) reward structure. The numerical method presented below relaxes this bottle-neck.

The analytical description of MRMs with only rate rewards are known for more than a decade. Several numerical techniques were proposed for the numerical evaluation of reward measures based on the known analytical description. Some of them provide the distribution of reward measures directly in time domain [2, 3, 4, 8], while some other provide the moments of reward measures [6, 5, 11]. The numerical methods that provide the distribution of reward measures are very elegant, but, in general, the numerical analysis of the moments of reward measures is much less expensive than the direct computation of the distribution, hence much larger models can be analyzed if only the moments of the reward measures are of interest.

In this paper we present a numerical method for the analysis of moments of the completion time that can be used when the underlying Markov chain has a large state space ($\sim 10^6$ states) and there is rate rewards associated with the states and phase type distributed (random) impulse rewards associated with the state transitions of the considered system. This method is an extension of the method presented in [12] for the completion time analysis of MRMs with impulse and rate reward. The presented method has several nice properties. It uses the randomization technique that makes it numerically stable, and allows to provide a general error bound in advance of the computation. It is hardly sensitive to the work requirement of the MRM (i.e., the memory requirement is insensitive, and the computational time

* S. Rácz acknowledges the support HSN Laboratory. M. Telek was supported by OTKA F-23971 and T-30685.

linearly increases).

The duality of the completion time and the accumulated reward measures of MRMs with only rate rewards was discussed in [12]. In this paper we extend that duality concept for MRMs with rate and impulse rewards. In [10] a similar numerical approach is introduced for the computation of moments of the accumulated reward in MRMs with rate and impulse rewards, and in this paper we provide a counterpart of that method using the duality of the completion time and the accumulated reward measures.

There are two important properties of the reward measures that indicate the difficulty of the problem considered in this paper:

- Even the distribution of the accumulated reward and the completion time are closely related to each others there is no way to establish a relation among the first n moments of the accumulated reward and the completion time.

- In MRMs with rate and impulse rewards the n-th moment of the accumulated reward do not depend on the $m > n$ moment of the (random) impulse rewards [10], instead the n-th moment of the completion time depends on all the moments of the (random) impulse rewards.

This second property indicates the complexity of the analysis of the moments of completion time. Indeed the approach introduced in this paper is not applicable with any generally distributed impulse reward, but it is applicable only with (possibly defective) phase type distributed impulse rewards.

The rest of the paper is organized as follows. Section 2 introduces the considered class of MRMs and the applied notation. Section 3 summarizes the main results of [10] for better understanding the duality of completion time and accumulated reward measures and the subsequent results. The main new result of the paper, a single Laplace domain description of the distribution of completion time that allows the effective numerical analysis of these models is given in Section 4. An iterative numerical procedure is provided in Section 5 for the analysis of the moments of completion time. The proposed numerical method is applied to evaluate the numerical examples in Section 6 and the paper is concluded in Section 7.

2 Notations and modeling assumptions

Let $\{\mathcal{Z}(t),\ t \geq 0\}$ be a (right continuous) Continuous Time Markov Chain (CTMC) over the finite state space $\mathcal{S} = \{1, 2, \ldots, M\}$ with generator $\mathbf{Q} = [Q_{ij}]$ and initial distribution $\underline{P} = [p_i]$. A non-negative real constant $(r_i,\ i \in \mathcal{S})$ is associated with each state of the process representing the reward rate (in state i). Let $\mathbf{R}$ be the diagonal matrix of the reward rates (i.e., $\mathbf{R} = \text{diag}(r_1, r_2, \ldots, r_M)$). A non-negative real random variable $(\mathcal{D}_{ij},\ i, j \in \mathcal{S})$ is associated with each possible state transitions of the process representing the amount of reward gained at a state transition (from i to j) (Figure 1). Let $D_{ij}(w)$ be the distribution (i.e., $D_{ij}(w) = Pr\{\mathcal{D}_{ij} \leq w\}$) and $D^{\sim}_{ij}(v)$ be the Laplace-Stieltjes transform (i.e., $D^{\sim}_{ij}(v) = \int_0^\infty e^{-vw} dD_{ij}(w)$) of $\mathcal{D}_{ij}$. The associated matrix is $\mathbf{D}^{\sim}(v) = \left[D^{\sim}_{ij}(v)\right]$ and the matrices of the moments of impulse reward are

$$\mathbf{D}^{(n)} = \left[E(\mathcal{D}^n_{ij})\right] = (-1)^n \frac{\partial^n \mathbf{D}^{\sim}(v)}{\partial v^n}\bigg|_{v=0} .$$

If there is no impulse reward associated with the state transition from i to j then $D_{ij}(w) = \text{UnitStep}(w)$ and $D^{\sim}_{ij}(v) = 1$. The diagonal elements are defined similarly $D_{ii}(w) = \text{UnitStep}(w)$ and $D^{\sim}_{ii}(v) = 1$.

Definition 1 *The* **accumulated reward**, $\mathcal{B}(t)$*, is a random variable which represents the accumulation of reward in time:*

$$\mathcal{B}(t) = \int_0^t \left(r_{\mathcal{Z}(\tau)} + \delta_\tau \mathcal{D}_{\mathcal{Z}(\tau^-),\mathcal{Z}(\tau)}\right) d\tau \tag{1}$$

and

$$\mathcal{B}_i(t) = \int_0^t \left(r_{\mathcal{Z}(\tau)} + \delta_\tau \mathcal{D}_{\mathcal{Z}(\tau^-),\mathcal{Z}(\tau)}\right) d\tau \ , \quad \text{if } \mathcal{Z}(0) = i \tag{2}$$

where δ_τ *is the unit impulse (also referred Dirac delta) at time* τ*.*

In Definition 1, the first term stands for the rate reward accumulation (the accumulated reward increases at rate r_i while the process stays in state $\mathcal{Z}(t) = i$) and the second term stands for the impulse reward accumulation (as many random impulses are summed up as many state transitions happen). By this definition, $\mathcal{B}(t)$ is a stochastic process that depends on $\mathcal{Z}(u)$ for $0 \leq u \leq t$ and $\mathcal{B}(0) = 0$. According to Definition 1 this paper restricts the attention to the class of models in which no state transition can entail to a loss of the accumulated reward. This kind of accumulation is also referred to as preemptive resume. The distribution of the accumulated reward is defined by

$$B(t, w) = Pr\{\mathcal{B}(t) \leq w\} \ , \quad B_i(t, w) = Pr\{\mathcal{B}_i(t) \leq w\} \ .$$

Note that

$$B(t, w) = \sum_{i \in S} p_i \, B_i(t, w) \ , \tag{3}$$

hence, in the rest of this paper, we use the initial state dependent measures and the global measures can always be evaluated by the mean of this relation.

Definition 2 *The* **completion time**, $\mathcal{C}_i$, *is the random variable representing the time to accumulate the random amount of reward* $\mathcal{W}$

$$\mathcal{C}_i = \min[t \geq 0 \, : \, \mathcal{B}_i(t) \geq \mathcal{W}] \, . \tag{4}$$

The distribution of $\mathcal{C}_i$ is

$$C_i(t) = Pr\{\mathcal{C}_i \leq t\} \, . \tag{5}$$

Let $\mathcal{C}_i(w)$ be the random variable representing the time to accumulate w (fix) amount of reward and $C_i(t, w)$ its distribution, i.e.,

$$\mathcal{C}_i(w) = \min[t \geq 0 \, : \, \mathcal{B}_i(t) \geq w] \tag{6}$$
$$C_i(t, w) = Pr\{\mathcal{C}_i(w) \leq t\} \, . \tag{7}$$

Let $G(w)$ be the distribution of $\mathcal{W}$ with support on $(0, \infty)$. By Definition 2,

$$C_i(t) = \int_0^\infty C_i(t, w) \, dG(w) \, . \tag{8}$$

The distribution of the completion time is closely related to the distribution of the accumulated reward by means of the following relation

$$\begin{aligned} B_i(t, w) = Pr\{\mathcal{B}_i(t) \leq w\} = \\ Pr\{\mathcal{C}_i(w) \geq t\} = 1 - C_i(t, w) \, . \end{aligned} \tag{9}$$

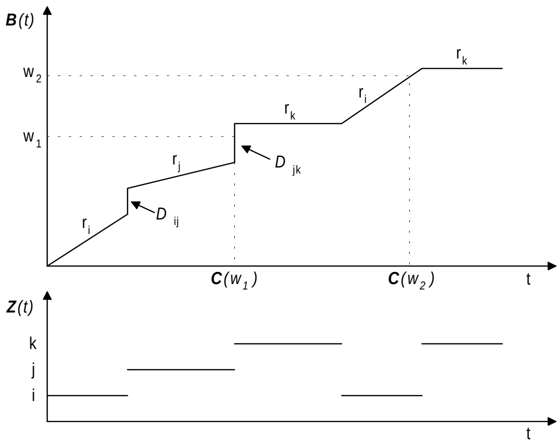

Figure 1. A sample path of $\mathcal{Z}(t)$ and $\mathcal{B}(t)$.

The non-zero impulse rewards are restricted to be phase type distributed in this paper (apart of Section 3). The phase type distributed impulse reward associated with the state transition from i to j (i.e., $\mathcal{D}_{ij}$) is defined by (see [9]):

- the row vector $\underline{\alpha}_{ij}$ that defines the initial probability distribution of $\mathcal{D}_{ij}$ and
- the matrix β_{ij} that defines the state transitions in the phase type structure of $\mathcal{D}_{ij}$. (We refer to β_{ij} as the generator of the phase type distribution.)

Further more the column vector $\underline{\beta}'_{ij}$ denotes the vector of transition rates to the absorbing state of the phase type structure (the row sum of β_{ij} plus $\underline{\beta}'_{ij}$ gives a vector of zeros.) Note that the subscripts ($_{ij}$) with these symbols refer to the state transition of $\mathcal{Z}(t)$.

3 Transform domain description of the accumulated reward

This section summarize the analytical result of [10]. The distribution of accumulated reward is given in single Laplace transform domain. Applying a new analysis approach that allows to utilize the duality of the accumulated reward and the completion time measures.

The results of this section are more general than the class of MRMs considered in this paper. (In this section the underlying CTMC can have any general structure cyclic/acyclic, reducible/irreducible, the reward rate can be any finite non-negative real number and the impulse reward can be any non-negative *generally distributed* random variable including discrete and deterministic distributions, defective distributions, distributions with infinite moments). The description of accumulated reward is given by the following theorem:

Theorem 1 *The column vector of the distribution of the accumulated reward* $\underline{B}(t, w) = [B_i(t, w)]$ *is as follows:*

$$\underline{B}^\sim(t, v) = e^{[\mathbf{Q} \odot \mathbf{D}^\sim(v) - v\mathbf{R}]t} \cdot \underline{h} \, , \tag{10}$$

where $\odot$ *denotes the piecewise matrix multiplication (*$[\mathbf{A} \odot \mathbf{B}]_{ij} = a_{ij} \cdot b_{ij}$*),* $^\sim$ *denotes the Laplace-Stieltjes transform with respect to* $w (\to v)$*, and* $\underline{h}$ *is the column vector with all the entries equal to 1.*

Note that $\mathbf{Q} \odot \mathbf{D}^\sim(v)|_{v=m}$ is a generator matrix of a modified (possible transient) Markov chain for any real $m > 0$, hence $\mathbf{Q} \odot \mathbf{D}^\sim(v)|_{v=m}$ characterize a phase type distribution. Further more, $\mathbf{R}$ is a diagonal matrix of the reward rates associated with this Markov chain.
The proof of the theorem is readable from the following two lemmas.

Lemma 1 *Let* $\underline{C}_m(t)$ *be the column vector of the completion time when the work requirement is exponentially distributed with parameter* m *(i.e.,* $\underline{C}_m(t) = [C_i(t) \mid G(w) = 1 - e^{-mw}]$*), and* $\hat{\underline{C}}_m(t)$ *is the analytical continuation of* $\underline{C}_m(t)$*.* $\underline{B}^\sim(t, v)$ *satisfies*

$$\underline{B}^\sim(t, v) = \underline{h} - \hat{\underline{C}}_m(t) \Big|_{m=v} \, . \tag{11}$$

Proof of Lemma 1 From (8) and (9) we have

$$\begin{aligned} C_i(t) &= \int_0^\infty \Big(1 - B_i(t,w)\Big)\, dG(w) \\ &= m \int_0^\infty \Big(1 - B_i(t,x)\Big) \cdot e^{-mx}\, dx \; . \end{aligned} \quad (12)$$

(12) can be rewritten using the Laplace-Stieltjes transform of the accumulated reward

$$C_i(t) = 1 - B_i^\sim(t,v)\Big|_{v=m} \quad (13)$$

which, in vector form, is

$$\underline{C}_m(t) = \underline{h} - \underline{B}^\sim(t,v)\Big|_{v=m} \; . \quad (14)$$

Since (11) is analytical for $\Re(v) \geq 0$ the lemma is given. □

Lemma 2 *The completion time of an exponentially distributed work requirement is a phase type distributed random variable (even with generally distributed impulse reward), and it can be evaluated as*

$$\underline{C}_m(t) = \underline{h} - e^{[\mathbf{Q}\odot\mathbf{D}^\sim(m) - m\mathbf{R}]t} \cdot \underline{h} \quad (15)$$

where $\mathbf{Q}\odot\mathbf{D}^\sim(m) - m\mathbf{R}$ is the generator of the phase type distribution.

Proof of Lemma 2 Due to the memoryless property of the exponentially distributed work requirement the remaining work to complete is exponentially distributed with the same parameter at any instance of time before completion. At a state transition from state i to j the completion occurs if the impulse reward $\mathcal{D}_{ij}$ is not less than the remaining work to complete, $\mathcal{W}_r$, i.e., the completion occurs with the following probability:

$$\begin{aligned} Pr\{\text{compl.}\} &= Pr\{\mathcal{D}_{ij} > \mathcal{W}_r\} \\ &= \int_0^\infty Pr\{\mathcal{D}_{ij} > w\}\, dG(w) \\ &= 1 - \int_0^\infty D_{ij}(w)\, dG(w) \\ &= 1 - m\int_0^\infty D_{ij}(w)\, e^{-mw}\, dw \\ &= 1 - D_{ij}^\sim(m) \end{aligned} \quad (16)$$

Assuming the process stays in state i at time t before completion the following cases can occur in the interval $(t, t+dt)$:

- *no state transition and no completion* occurs with probability $1 + (Q_{ii} - r_i m)dt + \sigma(dt)$,
- *no state transition and completion* occurs with probability $r_i m dt + \sigma(dt)$,
- *state transition to j and no completion* occurs with probability $D_{ij}^\sim(m) Q_{ij} dt + \sigma(dt)$,
- *state transition to j and completion* occurs with probability $(1 - D_{ij}^\sim(m))Q_{ij}dt + \sigma(dt)$,
- *any other cases* occur with probability $\sigma(dt)$.

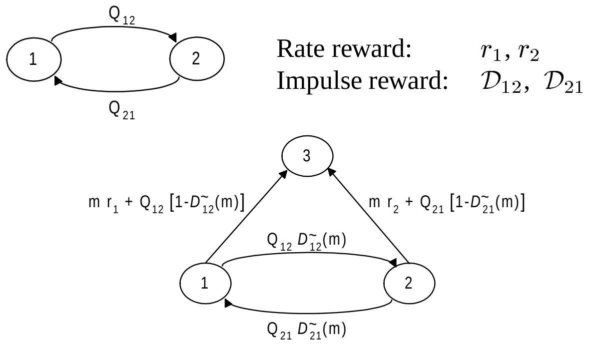

Figure 2. A simple two-state system.

Based on this behaviour a new CTMC can be defined by adding an absorbing state, $M + 1$, to the state space of the underlying CTMC, defining state transitions from $\forall i \in \mathcal{S}$ to $M+1$ with rate $mr_i + \sum_{j, j\neq i} Q_{ij}(1 - D_{ij}^\sim(m))$ and setting the transition rates between the states in $\mathcal{S}$ according to the above described behaviour. The absorbing state represents the completion of the exponentially distributed work requirement. The new CTMC defines a phase type distribution of order $\#\mathcal{S}$. Its $\#\mathcal{S} \times \#\mathcal{S}$ generator is $\mathbf{Q}\odot\mathbf{D}^\sim(m) - m\mathbf{R}$, and its time to absorption (i.e., completion time) is given by (15) (see, e.g., [9]). □

Lemma 2 is demonstrated through a simple example of a two-state system shown in Figure 2. The MRM of the example is defined by the generator of the underlying CTMC, $\mathbf{Q}$, the diagonal matrix of the reward rates, $\mathbf{R}$, and the impulse reward matrix, $\mathbf{D}^\sim(v)$:

$$\mathbf{Q} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \;\; \mathbf{R} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix},$$

$$\mathbf{D}^\sim(v) = \begin{pmatrix} 1 & D_{12}^\sim(v) \\ D_{21}^\sim(v) & 1 \end{pmatrix} \; .$$

In this example

$$C_1(t) = P_{13}(t) = 1 - (P_{11}(t) + P_{12}(t)) \; , \quad (17)$$

$$C_2(t) = P_{23}(t) = 1 - (P_{21}(t) + P_{22}(t)) \; . \quad (18)$$

The generator of the CTMC with the additional absorbing state, which describes the phase type distribution, is

$$\begin{pmatrix} Q_{11} - mr_1 & Q_{12}D_{12}^\sim(m) & mr_1 + Q_{12}(1 - D_{12}^\sim(m)) \\ Q_{21}D_{21}^\sim(m) & Q_{22} - mr_2 & mr_2 + Q_{12}(1 - D_{21}^\sim(m)) \\ 0 & 0 & 0 \end{pmatrix}$$

whose $\#\mathcal{S} \times \#\mathcal{S}$ (2×2) upper left submatrix has the form $\mathbf{Q}\odot\mathbf{D}^\sim(m) - m\mathbf{R}$.

4 Transform domain description of the completion time

This section provides the main result of the paper, a single Laplace transform domain expression of the completion time that allows an effective numerical analysis. Unfortunately, it is much harder to evaluate the moments of completion time of MRM with impulse and rate reward than the moments of accumulated reward. The n-th moment of the accumulated reward depends only on the $m \leq n$ moments of the impulse reward [10], while the n-th moment of the completion time is affected by all the moments of the impulse rewards. That is why we need to restrict the impulse rewards to be phase type distributed to provide an effective numerical analysis method. The analysis approach applied in this section is based on the duality of accumulated reward and completion time. The duality is obtained by interchanging the role of the time and the reward variables in the analytical description of these measures [12]. Unfortunately, the duality of these reward measures is not complete when both impulse and rate reward is accumulated, as the above mentioned differences indicate. Some of the results in this section are obtained as the dual counterpart of the results of [10] (that is summarized in Section 3) and the rest of the results are based on the special features of the completion time measure.

4.1 MRM with positive rate and phase type distributed impulse reward

In this subsection, the main theorem is provided for MRMs with strictly positive rate and phase type distributed impulse reward, but in the next subsection we also consider the case of zero reward rate.

Theorem 2 *The single Laplace transform of the completion time of a MRM with pozitive rate and phase type distributed impulse reward can be computed by the following expression:*

$$\underline{C}^{\sim}(s,w) = \mathbf{\Gamma} \cdot \left(e^{(\mathbf{T}-s\mathbf{F})w} \cdot \underline{h} \right) \tag{19}$$

where

$\mathbf{T}$ *is a generator matrix of a modified Markov chain over an enlarged state space $\mathcal{G}$ ($\mathcal{S} \subset \mathcal{G}$), that is composed by the state space of the original CTMC and the phase type structures of the impulse rewards. The matrix elements representing state transitions between states in $\mathcal{S}$ are set to $T_{ij} = Q_{ij}/r_i$ if there is no impulse reward associated with the state transition from i to j. If there is an impulse reward associated with the state transition from $i \in \mathcal{S}$ to $j \in \mathcal{S}$ the T_{ij} matrix element is set to 0 and there is a state transition with rate Q_{ij}/r_i to the phase type structure of the associated impulse reward. The exit of this phase type structure is directed to state j. (This way the enlarged CTMC is such that the rate of the state transitions in $\mathcal{S}$ are rescaled and the state transition from state i to j goes through the phase type structure of the associated impulse reward, if any, i.e., a phase type distributed time, equal to the associated impulse reward, is added to the rescaled time of state transition.) The structure of $\mathbf{T}$ is shown in Figure 3.*

Matrix $\mathbf{F}$ is a diagonal matrix of cardinality $\#\mathcal{G}$ whose diagonal element associated with $i \in \mathcal{S}$ equals to $1/r_i$ and with $i \in \mathcal{G} \setminus \mathcal{S}$ equals to 0. ($\mathbf{F}$ can be viewed as the reward matrix of the enlarged Markov chain.)

$\mathbf{\Gamma} = [\,\mathbf{I} \mid \mathbf{0}\,]$ *is a filter matrix of size $\#\mathcal{S} \times \#\mathcal{G}$ composed by a unity matrix of cardinality $\#\mathcal{S}$ and an $\#\mathcal{S} \times (\#\mathcal{G} - \#\mathcal{S})$ matrix of zeros. (The only role of the $\mathbf{\Gamma}$ matrix is that the multiplication with $\mathbf{\Gamma}$ eliminates the $\#\mathcal{G} - \#\mathcal{S}$ extra vector elements.)*

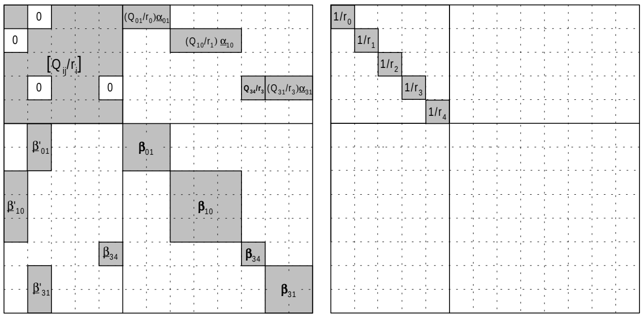

Figure 3. The structure of the generator and reward matrix of the enlarged Markov chain.

The proof of the theorem is provided through the following lemmas. First, the counterpart of Lemma 1 is provided.

Lemma 3 *The column vector of the distribution of completion time $\underline{C}(t,w) = [C_i(t,w)]$ satisfies:*

$$\underline{C}^{\sim}(s,w) = \underline{h} - \underline{\hat{B}}_u(w)\Big|_{u=s} \tag{20}$$

where $^{\sim}$ denotes the Laplace-Stieltjes transform with respect to $t(\rightarrow s)$, $\underline{B}_u(w)$ is the column vector of the distribution of reward accumulated during an interval that is exponentially distributed with parameter u, i.e., $\underline{B}_u(w) = [B_i(\mathcal{T},w) \mid \mathcal{T}$ is exp. with parameter $u]$, and $\underline{\hat{B}}_u(w)$ is the analytical continuation of $\underline{B}_u(w)$.

Proof of Lemma 3 Using the properties of the Laplace-Stieltjes transforms and Equation (9) $\underline{C}^{\sim}(s,w)$ can be written as

$$\begin{aligned} \underline{C}^{\sim}(s,w) &= s \int_0^{\infty} \underline{C}(t,w) e^{-st} dt \\ &= \underline{h} - s \int_0^{\infty} \underline{B}(t,w) e^{-st} dt \,. \end{aligned} \tag{21}$$

When s takes the positive real value u we have

$$\underline{C}^{\sim}(u,w) = \underline{h} - \int_0^\infty \underline{B}(t,w) u e^{-ut} dt = \underline{B}_u(w) \ , \quad (22)$$

and since $\underline{C}^{\sim}(s,w)$ is analytical for $\Re(s) \geq 0$ the lemma is given. $\square$

Note that the role of the accumulated reward and the completion time are interchanged in Lemma 1 and Lemma 3.

Based on Lemma 3 the remaining task to obtain Theorem 2 is to determine the amount of reward accumulated by a MRM with positive rate and phase type distributed impulse reward during an exponentially distributed period of time.

Lemma 4 *The amount of reward accumulated by a MRM with positive rate and phase type distributed impulse reward during an exponentially distributed period of time is phase type distributed, and the generator matrix of this phase type distribution is* $\mathbf{T} - u\mathbf{F}$*, i.e.,*

$$\underline{B}_u(w) = \mathbf{\Gamma} \cdot \left(\underline{h} - e^{(\mathbf{T}-u\mathbf{F})w} \cdot \underline{h} \right) \ . \quad (23)$$

where u *is the parameter of the exponential distribution of the accumulation period.*

Proof of Lemma 4 The proof is based on the fact that the accumulation period is exponentially distributed, hence at any instant of time of the accumulation the remaining time till the end of the accumulation period is exponentially distributed with the same parameter.

According to (2) $\mathcal{B}_i(\mathcal{T})$ denotes the (random) amount of reward accumulated during the exponentially distributed period $\mathcal{T}$ when $\mathcal{Z}(0) = i$. Let τ_i denote the first sojourn time in state i. τ_i is exponentially distributed with parameter $q_i = -Q_{ii}$. To evaluate the reward accumulated during a $\mathcal{T}$ long period starting from state i the following cases has to be considered:

- $\mathcal{T} \leq \tau_i$: the probability of this case is

  $$Pr\{\mathcal{T} \leq \tau_i\} = \frac{u}{u+q_i} \ .$$

  Under this condition the accumulation period is exponentially distributed with parameter $(u+q_i)$, because:

  $$Pr\{\mathcal{T} \leq t | \mathcal{T} \leq \tau_i\} = 1 - e^{-(u+q_i)t} \ .$$

  The amount of reward accumulated in this case is also exponentially distributed with parameter $(u+q_i)/r_i$.

- $\mathcal{T} > \tau_i$: the probability of this case is

  $$Pr\{\mathcal{T} > \tau_i\} = \frac{q_i}{u+q_i} \ .$$

  Under this condition the sojourn time is exponentially distributed with parameter $(u+q_i)$, since:

  $$Pr\{\tau_i \leq t | \mathcal{T} > \tau_i\} = 1 - e^{-(u+q_i)t} \ .$$

  The amount of reward accumulated when $\mathcal{T} > \tau_i$ is composed by three parts: the reward accumulated in state i, the phase type distributed impulse reward associated with the state transition from i to j, and the reward accumulated starting from state j, $\mathcal{B}_j(\mathcal{T})$, since the remaining time $\mathcal{T} - \tau_i$ is also exponentially distributed with the same parameter. Assuming $\mathcal{T} > \tau_i$ the reward accumulated in state i is exponentially distributed with parameter $(u+q_i)/r_i$, as before, since τ_i is exponentially distributed with parameter $(u+q_i)$.

Considering these cases the Laplace-Stieltjes transform of $\mathcal{B}_i(\mathcal{T})$, $B_i^{\sim}(\mathcal{T},v) = E(e^{-v\mathcal{B}_i(\mathcal{T})})$ satisfies:

$$\begin{aligned} B_i^{\sim}(\mathcal{T},v) &= \frac{u}{u+q_i} \, \frac{\frac{u+q_i}{r_i}}{v + \frac{u+q_i}{r_i}} \quad (24) \\ &+ \frac{q_i}{u+q_i} \left(\frac{\frac{u+q_i}{r_i}}{v + \frac{u+q_i}{r_i}} \sum_{j \in \mathcal{S}, j \neq i} \frac{Q_{ij}}{q_i} \, D_{ij}^{\sim}(v) \, B_j^{\sim}(\mathcal{T},v) \right) \\ &= \frac{u}{r_i v + u + q_i} + \sum_{j \in \mathcal{S}, j \neq i} \frac{Q_{ij}}{r_i v + u + q_i} \, D_{ij}^{\sim}(v) \, B_j^{\sim}(\mathcal{T},v) \end{aligned}$$

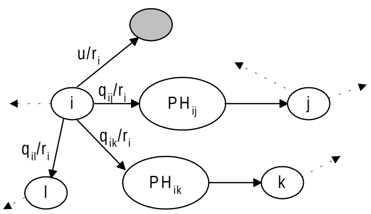

Figure 4. The structure of the phase type distribution characterized by $\mathbf{T} - u\mathbf{F}$.

Now, we evaluate the initial state dependent time to absorption of the phase type distribution characterized by $\mathbf{T} - u\mathbf{F}$. $\mathcal{K}_i$ denotes the time to absorption starting from i. We utilize that due to the given structure of $\mathbf{T}$ and $\mathbf{F}$ a phase type distributed time $\mathcal{D}_{ij}$ is spent in $\mathcal{G} \backslash \mathcal{S}$ if j is the next state in $\mathcal{S}$ that is visited after a sojourn in $i \in \mathcal{S}$. Further more, the row sum of $\mathbf{T} - u\mathbf{F}$ equals to 0 for the states in $\mathcal{G} \setminus \mathcal{S}$, i.e., there is no direct transition to the absorbing state from $\mathcal{G} \setminus \mathcal{S}$. Figure 4 shows the schematic structure of the phase type distribution characterized by $\mathbf{T} - u\mathbf{F}$. Based on this structure the Laplace Stieltjes transform of $\mathcal{K}_i$, $K_i^{\sim}(v)$, can be provided as a function of the Laplace Stieltjes transform of $\mathcal{K}_j; j \in \mathcal{S}$ as follows:

$$\begin{aligned} K_i^{\sim}(v) &= \frac{u/r_i}{v + (u+q_i)/r_i} \quad (25) \\ &+ \sum_{j \in \mathcal{S}, j \neq i} \frac{Q_{ij}/r_i}{v + (u+q_i)/r_i} D_{ij}^{\sim}(v) K_j^{\sim}(v) \end{aligned}$$

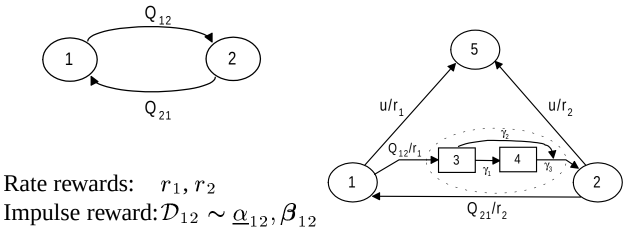

Figure 5. A simple two-state system with Phase Type distributed impulse reward.

By the equivalence of (24) and (25) the lemma is given. □

Lemma 4 is demonstrated through the following example of a two state CTMC (see Figure 5.) where a phase type distributed impulse reward (characterized by $\underline{\alpha}_{12}$ and $\boldsymbol{\beta}_{12}$) is associated with the state transition from 1 to 2.

$$\mathbf{Q} = \begin{pmatrix} -Q_{12} & Q_{12} \\ Q_{21} & -Q_{21} \end{pmatrix}, \ \underline{\alpha}_{12} = (\,1 \quad 0\,),$$
$$\boldsymbol{\beta}_{12} = \begin{pmatrix} -(\gamma_1+\gamma_2) & \gamma_1 \\ 0 & -\gamma_3 \end{pmatrix}, \ \underline{\beta}'_{12} = \begin{pmatrix} \gamma_2 \\ \gamma_3 \end{pmatrix}.$$

By Theorem 2 the $\mathbf{T}$ and $\mathbf{F}$ matrices are as follows:

$$\mathbf{T} = \begin{pmatrix} -Q_{12}/r_1 & 0 & Q_{12}/r_1 & 0 \\ Q_{21}/r_2 & -Q_{21}/r_2 & 0 & 0 \\ 0 & \gamma_2 & -(\gamma_1+\gamma_2) & \gamma_1 \\ 0 & \gamma_3 & 0 & -\gamma_3 \end{pmatrix},$$
$$\mathbf{F} = \begin{pmatrix} 1/r_1 & 0 & 0 & 0 \\ 0 & 1/r_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For easier understanding of the phase type characteristic of $\mathbf{T} - u\mathbf{F}$ an absorbing state (state 5) is also depicted in Figure 5. Having the 5-state CTMC in Figure 5 and denoting the state transition probability of this CTMC by $\hat{P}_{ij}(t)$

$$\underline{B}_u(w) = \begin{pmatrix} \hat{P}_{15}(t)|_{t=w} \\ \hat{P}_{25}(t)|_{t=w} \end{pmatrix}. \qquad (26)$$

Note, that the case of exponentially distributed impulse reward can be captured as a special case, as a phase type distribution of order 1.

4.2 MRM with non-negative rate and phase type distributed impulse reward

The case of zero reward rate can be viewed as the limiting case when the reward rate tend to 0. Assuming that the reward rate of state i ($i \in \mathcal{S}$) decreases to zero the associated transitions rates of matrix $\mathbf{T}$ increases to infinity. The limiting case when $r_i = 0$ can be handled based on the concept of Generalized Stochastic Petri Nets [1] that has two kinds of "states" (referred to as marking):

- regular state – that is visited for an exponentially distributed period of time (referred to as tangible marking),
- immediate state – that is visited instantaneously (referred to as vanishing marking).

If $r_i = 0$ ($i \in \mathcal{S}$) state i becomes an immediate state, that performs only a random switching, i.e., the process stays in state i for zero time and after it visits

- the absorbing state – with probability $\dfrac{u}{u+q_i}$,
- state j, if there is no impulse reward associated with the state transitions from i to j – with probability $\dfrac{Q_{ij}}{u+q_i}$,
- the phase type structure of the i to k state transition (if there is an impulse reward associated with the i to k state transition) – with probability $\dfrac{Q_{ik}}{u+q_i}$.

As it is shown in [1] the obtained stochastic process is a CTMC, hence the amount of reward accumulated by a MRM with non-negative rate and phase type distributed impulse reward during an exponentially distributed period of time is phase type distributed as well. It is also possible to provide a numerical method to evaluate the completion time of MRMs with non-negative rate and phase type distributed impulse reward using a similar approach as in [1], but it is not considered in this paper.

5 Numerical analysis of the moments of completion time

In this section we introduce a numerical method which provides the moments of the completion time using the single transform domain description obtained in the previous Section. We also discuss the numerical properties of the proposed method.

Let $s_i^{(n)}(w) = E\{\mathcal{C}_i(w)^n\}$ be the n-th moment of the time to accumulate w amount of reward. The column vector $\underline{s}^{(n)}(w) = [s_i^{(n)}(w)]$ can be evaluated based on $\underline{C}^{\sim}(s,w)$ as

$$\underline{s}^{(n)}(w) = (-1)^n \frac{\partial^n \underline{C}^{\sim}(s,w)}{\partial s^n}\bigg|_{s=0} \qquad (27)$$

Theorem 3 *The n-th moment ($n \geq 1$) of the completion time is*

$$\underline{s}^{(n)}(w) = (-1)^n\, \mathbf{\Gamma} \cdot \left(\sum_{i=0}^{\infty} \frac{t^i}{i!} \cdot \mathbf{M}^{(n)}(i) \cdot \underline{h} \right) \qquad (28)$$

where $\mathbf{M}^{(n)}(i)$ is defined as

$$\mathbf{M}^{(n)}(i) = \begin{cases} \mathbf{I} & \text{if } i = n = 0 , \\ \mathbf{0} & \text{if } i = 0, n \geq 1 , \\ \mathbf{T}^i & \text{if } i \geq 1, n = 0 , \\ \mathbf{T} \cdot \mathbf{M}^{(n)}(i-1) - \\ -n\, \mathbf{F} \cdot \mathbf{M}^{(n-1)}(i-1) & \text{if } i \geq 1, n \geq 1 . \end{cases}$$

Proof of Theorem 3 From (27) and (19)

$$\begin{aligned} \underline{s}^{(n)}(w) &= (-1)^n\, \mathbf{\Gamma} \cdot \left(\left. \frac{\partial^n e^{(\mathbf{T}-s\mathbf{F})w}}{\partial s^n} \right|_{s=0} \cdot \underline{h} \right) \\ &= (-1)^n\, \mathbf{\Gamma} \cdot \left(\left. \frac{\partial^n}{\partial s^n} \sum_{i=0}^{\infty} \frac{w^i}{i!} \, (\mathbf{T} - s\mathbf{F})^i \right|_{s=0} \cdot \underline{h} \right) \\ &= (-1)^n\, \mathbf{\Gamma} \cdot \left(\sum_{i=0}^{\infty} \frac{w^i}{i!} \left. \frac{\partial^n}{\partial s^n} \, (\mathbf{T} - s\mathbf{F})^i \right|_{s=0} \cdot \underline{h} \right) \end{aligned}$$

Let

$$\mathbf{M}^{(n)}(i) = \left. \frac{\partial^n}{\partial s^n} \, (\mathbf{T} - s\mathbf{F})^i \right|_{s=0} \quad \text{for } \forall n, i \geq 0 . \qquad (29)$$

From Leibnitz rule[1] it follows

$$\begin{aligned} &\mathbf{M}^{(n)}(i) = [\mathbf{M}^{(0)}(1) \cdot \mathbf{M}^{(0)}(i-1)]^{(n)} \\ &= \sum_{l=0}^{n} \binom{n}{l} \mathbf{M}^{(l)}(1) \cdot \mathbf{M}^{(n-l)}(i-1) \quad \text{if } i \geq 2, n \geq 1 \end{aligned}$$

with the initial conditions $\mathbf{M}^{(0)}(0) = \mathbf{I}$, $\mathbf{M}^{(0)}(i) = \mathbf{T}^i$ and $\mathbf{M}^{(n)}(0) = \mathbf{0}$ $n \geq 1$. This completes the proof. □

5.1 Numerical method using randomization

The iterative procedure presented above is not tuned to have nice numerical properties. To avoid numerical problems like instability, "ringing" (negative probabilities), etc., a modified procedure is proposed. Let

$$\mathbf{H} = \frac{\mathbf{T}}{z} + \mathbf{I} , \quad \mathbf{L} = \frac{\mathbf{F}}{z\, f} \qquad (30)$$

where $z = \max_{i,j \in \mathcal{G}} (|t_{ij}|)$ and $f = \max_{i \in \mathcal{G}} (r_i)/z$. By this definition $\mathbf{H}$ is a stochastic matrix ($0 \leq h_{i,j} \leq 1, \forall i, j \in \mathcal{G}$ and $\sum_{j \in \mathcal{G}} h_{i,j} = 1, \forall i \in \mathcal{G}$) and $\mathbf{S}$ is a diagonal matrix such that $0 \leq s_{i,i} \leq 1, \forall i \in \mathcal{S}$. Using these matrices

$$\underline{C}^{\sim}(s, w) = \mathbf{\Gamma} \cdot e^{(\mathbf{T}-s\mathbf{F})w} \cdot \underline{h} = \mathbf{\Gamma} \cdot e^{(\mathbf{H}-sf\mathbf{S})zw} \cdot \underline{h}\, e^{-zw} . \qquad (31)$$

Theorem 4 *The moments of completion time can be computed using only matrix-vector multiplications and saving only vectors of size $\#\mathcal{G}$ in each step of the iteration as*

$$\underline{s}^{(n)}(w) = n!\, f^n \cdot \mathbf{\Gamma} \cdot \sum_{i=0}^{\infty} \underline{U}^{(n)}(i) \frac{(zw)^i}{i!} e^{-zw} \qquad (32)$$

where

$$\underline{U}^{(n)}(i) = \begin{cases} \underline{0}, & \textit{if } i = 0, n \geq 1 , \\ \underline{h}, & \textit{if } i \geq 0, n = 0 , \\ \mathbf{H} \cdot \underline{U}^{(n)}(i-1) & \\ +\mathbf{L} \cdot \underline{U}^{(n-1)}(i-1), & \textit{if } i \geq 1, n \geq 1 . \end{cases} \qquad (33)$$

Proof of Theorem 4 Starting from (31) the proof of Theorem 4 follows the same pattern as the proof of Theorem 3. □

To demonstrate the iterative procedure of computing $\underline{U}^{(n)}(i)$ the first elements of $\underline{U}^{(n)}(i)$ provided by (33) are shown in Table 1.

	i=0	i=1	i=2	i=3
n=0	$\underline{h}$	$\underline{h}$	$\underline{h}$	$\underline{h}$
n=1	$\underline{0}$	$\mathbf{L}\underline{h}$	$\mathbf{HL}\underline{h} + \mathbf{L}\underline{h}$	$\mathbf{HHL}\underline{h} + \mathbf{HL}\underline{h} + \mathbf{L}\underline{h}$
n=2	$\underline{0}$	$\underline{0}$	$\mathbf{LL}\underline{h}$	$\mathbf{HLL}\underline{h} + \mathbf{LHL}\underline{h} + \mathbf{LL}\underline{h}$
n=3	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\mathbf{LLL}\underline{h}$

Table 1. The first element of $\underline{U}^{(n)}(i)$

Suppose one is interested in the first 3 moments of the completion time. To perform the computation 3 vectors of size $\#\mathcal{G}$ is needed to store $\underline{U}^{(n)}(i), n = 1, 2, 3$. In each iteration step $i = 1, 2, 3, \ldots$ matrix-vector multiplications and vector summations has to be performed according to (33) using the vectors of the previous iteration step and the constant matrices $\mathbf{H}$ and $\mathbf{L}$. Figure 6 shows the dependency structure of the computation. To calculate the i-th iteration of $\underline{U}$ only the result of the $(i-1)$-th iteration of $\underline{U}$ is used. Note that $\mathbf{L}$ is a diagonal matrix and $\mathbf{H}$ is as sparse as $\mathbf{Q}$ is. Further 3 vectors of size $\#\mathcal{S}$ is needed to store the "actual value" of $\underline{s}^{(n)}(w), n = 1, 2, 3$ according to (32).
The following theorem provides a global error bound of the procedure.

Theorem 5 *The n-th moment of the completion time can be calculated as a finite sum and an error part, where the maximum allowed error is ε*

[1] $(f_1 \cdot f_2)^{(n)}(z_0) = \sum_{l=0}^{n} \binom{n}{l} f_1^{(k)}(z_0) \cdot f_2^{(n-k)}(z_0)$

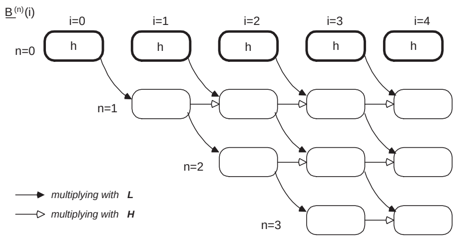

Figure 6. The dependency structure of the iteration steps

$$\underline{s}^{(n)}(w) = n!\, f^n \cdot \mathbf{\Gamma} \cdot \sum_{i=0}^{G-1} \underline{U}^{(n)}(i) \frac{(zw)^i}{i!} e^{-zw} + \underline{\xi}(G) \quad (34)$$

where

$$G = \min_{g \in \mathbb{N}} \left((zw)\, n!\, f^n \sum_{i=g-1}^{\infty} \frac{(zw)^i}{i!} e^{-zw} \leq \varepsilon \right) \quad (35)$$

and the $\underline{0} \leq \underline{\xi}(G) \leq \underline{h}\, \varepsilon$ *inequality holds for all elements of the vectors.*

Proof of Theorem 5 By the definition of $\mathbf{H}$ and $\mathbf{L}$

$$\underline{0} \leq \mathbf{L} \cdot \underline{h} \leq \underline{h} \quad \text{and} \quad \underline{0} \leq \mathbf{H} \cdot \mathbf{L} \cdot \underline{h} \leq \underline{h} \quad (36)$$

hold piece-wise (as all the subsequent vector inequalities), hence $\underline{U}^{(n)}(i)$ is bounded by

$$\underline{0} \leq \underline{U}^{(n)}(i) \leq i\, \underline{h}. \quad (37)$$

The error, $\underline{\xi}(g)$, incurred when eliminating the tail of the infinite sum is also bounded by

$$\underline{\xi}(g) = n!\, f^n \sum_{i=g}^{\infty} \underline{U}^{(n)}(i) \frac{(zw)^i}{i!} e^{-zw} \quad (38)$$

$$\leq n!\, f^n \sum_{i=g}^{\infty} \underline{h}\, i \frac{(zw)^i}{i!} e^{-zw} \quad (39)$$

$$\leq (zw)\, n!\, f^n \sum_{i=g-1}^{\infty} \underline{h} \frac{(zw)^i}{i!} e^{-zw} \quad (40)$$

which gives the theorem. □

6 Numerical Examples

There are two numerical examples evaluated by the proposed method. The first example indicates the ability of evaluating models with a large state space, i.e., 10^5 states. The second example of a two state system introduces an interesting behaviour of the completion time when impulse and rate reward are considered in the system.

	$w = 0.01$	$w = 0.1$	$w = 1$	$w = 5$
$E\{\mathcal{C}(w)\}$	$3.99 \cdot 10^{-7}$	$3.96 \cdot 10^{-6}$	$3.86 \cdot 10^{-5}$	$1.92 \cdot 10^{-4}$
$E\{\mathcal{C}(w)^2\}$	$1.59 \cdot 10^{-13}$	$1.57 \cdot 10^{-11}$	$1.50 \cdot 10^{-9}$	$3.71 \cdot 10^{-8}$
$E\{\mathcal{C}(w)^3\}$	$6.39 \cdot 10^{-20}$	$6.30 \cdot 10^{-17}$	$5.89 \cdot 10^{-14}$	$7.17 \cdot 10^{-12}$

Table 2. The first three moments of completion time

System with large state space

The examined system has 10^5 states, that are numbered as $1, 2, \ldots, 10^5$. Different positive rate rewards are associated with each system state. The rate reward associated with state n equals to n. Each forward transition (i.e., from n to $n+1$) incurs an Erlang(4) distributed impulse reward with mean 0.2. (Erlang(4) is the phase type distribution of order 4 with the lowest possible squared coefficient of variation, that is 0.25.) The enlarged CTMC, generated according to Theorem 2, has $5 \cdot 10^5 - 4$ states because every state transition which possesses impulse reward turned into 4 extra states (i.e., $100,000 + 99,999 \cdot 4$).

Figure 7 shows the structure of the underlying CTMC, where "thick" arrows indicate that impulse reward is associated with the transition. The underlying CTMC is a finite birth-death process with state dependent death rate and with $100,000$ states. The constant birth and the state dependent death rates are $\lambda = 5000$ and $\mu_n = 0.2(n-1)$, respectively.

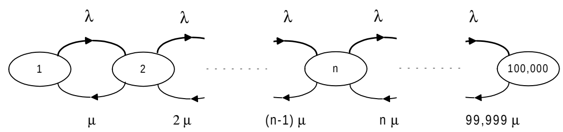

Figure 7. The state structure of the system

The first three moments of completion time is summarized in Table 2 assuming the system starts in steady state.

Two-state model

The following system is considered to verify the results obtained by the proposed numerical method against the intuitive understanding of the model behaviour. Consider a two-state underlying CTMC where the transition rate from state 1 to 2 is $\lambda = 1$ and from state 2 to 1 is $\mu = 0.1$. An impulse reward is associated with the transition from 1 to 2. This impulse reward is Erlang(10) distributed with mean 1 (i.e., it is almost deterministic). The rate reward associated with state 1 and 2 equals to 0.001.

The diagrams on Figure 8-9 show the mean and the coefficient of variation of the completion time versus the work requirement. The "tick" ("slim") curve is obtained when the system starts from state 2 (1). The "tick" curve in Figure 8 is shifted with ~ 10 units above the "slim" one, because the mean sojourn time in state 2 is 10, while the rate reward accumulated in 2 is negligible. The maximum of the "slim" coefficient of variation curve is due to the uncertainty of the state transition from 1 to 2, because when $w < 1$ a single state transition from 1 to 2 result in the completion. The "long" sojourn time in 2 smoothes out this effect on the "tick" curve.

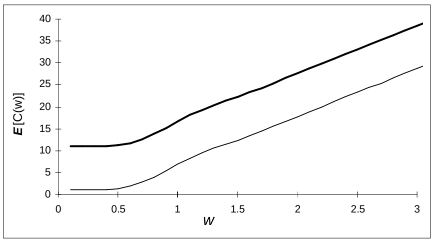

Figure 8. The mean value of completion time

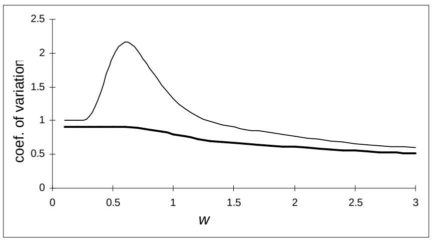

Figure 9. The coefficient of variation of completion time

7 Conclusion

The paper presents an analytical description of the completion time of MRMs that allows its numerical analysis with large state space ($\sim 10^6 states$). The analytical result is obtained by elaborating the duality of the accumulated reward and the completion time measures of Markov reward models with positive rate and phase type distributed impulse rewards.

Numerical examples indicate the abilities of the obtained analysis technique.

References

[1] M. Ajmone Marsan, G. Balbo, and G. Conte. A class of generalized stochastic Petri nets for the performance evaluation of multiprocessor systems. *ACM Transactions on Computer Systems*, 2:93–122, 1984.

[2] E. de Souza e Silva and H.R. Gail. Calculating availability and performability measures of repairable computer systems using randomization. *Journal of the ACM*, 36:171–193, 1989.

[3] E. de Souza e Silva and R. Gail. An algorithm to calculate transient distributions of cummulative rate and impulse based reward. *Commun. in Statist. – Stochastic Models*, 14(3):509–536, 1998.

[4] L. Donatiello and V. Grassi. On evaluating the cumulative performance distribution of fault-tolerant computer systems. *IEEE Transactions on Computers*, 1991.

[5] W.K. Grassmann. Means and variances of time averages in markovian environment. *European Journal of Operational Research*, 31:132–139, 1987.

[6] B.R. Iyer, L. Donatiello, and P. Heidelberger. Analysis of performability for stochastic models of fault-tolerant systems. *IEEE Transactions on Computers*, C-35:902–907, 1986.

[7] V.G. Kulkarni, V.F. Nicola, and K. Trivedi. On modeling the performance and reliability of multi-mode computer systems. *The Journal of Systems and Software*, 6:175–183, 1986.

[8] H. Nabli and B. Sericola. Performability analysis: A new algorithm. *IEEE Transactions on Computers*, C-45(4):491–494, 1996.

[9] M.F. Neuts. *Matrix Geometric Solutions in Stochastic Models*. Johns Hopkins University Press, Baltimore, 1981.

[10] S. Rácz and M. Telek. Performability analysis of Markov reward models with rate and impulse reward. In M. Silva B. Plateau, W. Stewart, editor, *Int. Conf. on Numerical solution of Markov chains*, pages 169–187, Zaragoza, Spain, 1999.

[11] M. Telek, A. Pfening, and G. Fodor. An effective numerical method to compute the moments of the completion time of Markov reward models. *Computers and mathematics with applications*, 36:8:59–65, 1998.

[12] M. Telek and S. Rácz. Numerical analysis of large Markovian reward models. *Performance Evaluation*, 36&37:95–114, Aug 1999.