Performability Analysis of Markov Reward Models with Rate and Impulse Reward

Sándor Rácz[1] and Miklós Telek[2] *

[1] Department of Telecommunications and Telematics
Technical University of Budapest, 1521 Budapest, Hungary
raczs@ttt-atm.ttt.bme.hu
[2] Department of Telecommunications
Technical University of Budapest, 1521 Budapest, Hungary
telek@hit.bme.hu

Abstract. The numerical analysis of Markov Reward Models (*MRM*) with random impulse and constant rate reward is considered in this paper. A single Laplace transform domain description of the distribution of accumulated reward is provided through a new argument. Based on this description an effective numerical method is introduced for the analysis of the moments of reward measures which allows the evaluation of models with large state space ($\sim 10^6$ states) and its computational complexity is independen of the number of different reward rates and impulse rewards. The proposed method, which is an extension of a method for the analysis MRMs with only rate reward, provides the moments of reward measures with a computational cost and memory requirement similar to the transient analysis of the underlying Continuous Time Markov Chain. A numerical example with a large model demonstrates the abilities of the proposed method.
Key words: Markov Reward Models with Impulse and Rate Reward, Performability, Accumulated Reward, Completion Time, Uniformization Computational Complexity.

1 Introduction

The increasing dependence on the performance of computer and communication systems arose the need for adequate analysis techniques to evaluate these systems. Stochastic reward processes [10, 7] became commonly accepted as a promising candidate for system performance analysis. There are two main subclass of measures associated with reward models. The random amount of reward accumulated during a given time interval is referred to as *accumulated reward* or in the context of dependable computer systems as *performability*[1] [11], and

* The authors wish to thank Gergely Mátéfi's help in the implementation of the numerical method. S. Rácz thanks the support of HSNLab. M. Telek was supported by OTKA F-23971.

[1] These two terms are interchangeable used in this paper.

the random time to accumulate a given amount of reward is referred to as *completion time*. Closed form expressions are known for the distribution of accumulated reward [8] and completion time [9] in double Laplace transform domain in case when the underlying stochastic process is a Continuous Time Markov Chain (*CTMC*) and only rate reward is accumulated. This case is referred to as Markov Reward Model (*MRM*).

Various numerical techniques were proposed for the evaluation reward measures of *MRM*s. Some of them provides only the mean of the accumulated reward, which is the simplest reward measure, because it can be evaluated based on the transient behaviour of the underlying CTMC.

Among the general methods (with no restriction on the structure of the underlying CTMC) that computes the distribution of reward measures the most promising ones are somehow based on the randomization technique that provides nice numerical properties and an overall error bound. The numerical methods based on this approach [4, 5, 3, 12] differ a lot in complexity and memory requirement.

The numerical analysis of the moments of reward measures is, in general, easier than the direct computation of the distribution of those measures, and the distribution can be estimated based on the evaluated moments. The mean of performability is known from the transient behaviour of the underlying CTMC. A numerical convolution approach is proposed in [8] to evaluated the $n+1$th moment of performability based on its nth moment. A similar approach is followed in [16] to calculate the moments of completion time, but the high computational complexity of the numerical convolution does not allow to apply this approach for the analysis of *MRM* with large (> 100) state spaces. The effective numerical method proposed in [17] avoids the numerical convolution and can evaluate the moments of reward measures of large *MRM*s with more than 10^6 states.

Compare to the literature of MRMs with only rate reward there are very few results available for the analysis of MRMs with impulse and rate reward. Some of these results are based on the evaluation of instantaneous state transition rates (i.e., the probability that a state transition occurs in an infinitesimal time interval) [6], hence they provide only the mean accumulated reward. Neither the higher moments of accumulated reward nor any moment of the completion time can be evaluated based on the instantaneous state transition rates.

Till now the distribution of reward measures of MRMs with impulse and rate reward have been evaluated based on aggregation or the path analysis of the underlying CTMC [4, 14]. The applicability of the path based approaches is restricted to the analysis of accumulated reward over a "short" time interval, or the analysis of completion time of a "small" work requirement, because the number of paths that has to be considered to meet a given error bound increases exponentially with the length of the time interval or the level of work requirement. The method introduced in this paper is an extension from [17] for the analysis of MRMs with impulse and rate reward. In contrast with the path based methods it is hardly sensitive to the time interval of the analysis (i.e., the memory requirement is insensitive, and the computational time linearly increases).

The rest of the paper is organized as follows. Section 2 introduces the considered class of MRMs. A Laplace domain description of the distribution of accumulated reward is given in Section 3 and an iterative procedure for the numerical analysis of the moments of accumulated reward in Section 4. Numerical examples are provided in Section 5 and the paper is concluded in Section 6.

2 MRMs with impulse and rate reward

Let $\{\mathcal{Z}(t),\ t \geq 0\}$ be a (right continuous) Continuous Time Markov Chain (CTMC) over the finite state space $\mathcal{S} = \{1, 2, \ldots, M\}$ with generator $\mathbf{Q} = [q_{ij}]$ and initial distribution $\underline{P} = [p_i]$. A non-negative real constant $(r_i,\ i \in \mathcal{S})$ is associated with each state of the process representing the reward rate (in state i). Let $\mathbf{R}$ be the diagonal matrix of the reward rates (i.e., $\mathbf{R} = \text{diag}(r_1, r_2, \ldots, r_M)$). A non-negative real random variable $(\mathcal{D}_{ij},\ i, j \in \mathcal{S})$ is associated with each possible state transitions of the process representing the amount of reward gained at a state transition (from i to j) (Figure 1). Let $D_{ij}(w)$ be the distribution (i.e., $D_{ij}(w) = Pr\{\mathcal{D}_{ij} \leq w\}$) and $D^{\sim}_{ij}(v)$ be the Laplace-Stieltjes transform (i.e., $D^{\sim}_{ij}(v) = \int_0^\infty e^{-vw} dD_{ij}(w)$) of $\mathcal{D}_{ij}$. The associated matrix is $\mathbf{D}^{\sim}(v) = \left[D^{\sim}_{ij}(v)\right]$ and the matrices of the moments of impulse reward are

$$\mathbf{D}^{(n)} = \left[E(\mathcal{D}^n_{ij})\right] = (-1)^n \frac{\partial^n \mathbf{D}^{\sim}(v)}{\partial v^n}\bigg|_{v=0} .$$

If there is no impulse reward associated with the state transition from i to j then $D_{ij}(w) = \text{UnitStep}(w)$ and $\mathbf{D}^{\sim}_{ij}(v) = 1$. The diagonal elements are defined similarly $D_{ii}(w) = \text{UnitStep}(w)$ and $\mathbf{D}^{\sim}_{ii}(v) = 1$.

Definition 1. *The* **accumulated reward**, $\mathcal{B}(t)$, *is a random variable which represents the accumulation of reward in time:*

$$\mathcal{B}(t) = \int_0^t \left(r_{\mathcal{Z}(\tau)} + \delta_\tau \mathcal{D}_{\mathcal{Z}(\tau^-),\mathcal{Z}(\tau)}\right) d\tau \tag{1}$$

and

$$\mathcal{B}_i(t) = \int_0^t \left(r_{\mathcal{Z}(\tau)} + \delta_\tau \mathcal{D}_{\mathcal{Z}(\tau^-),\mathcal{Z}(\tau)}\right) d\tau \ , \quad \text{if } \mathcal{Z}(0) = i \tag{2}$$

where δ_τ is the unit impulse (also referred Dirac delta) at time τ.

By this definition, $\mathcal{B}(t)$ is a stochastic process that depends on $\mathcal{Z}(u)$ for $0 \leq u \leq t$ and $\mathcal{B}(0) = 0$. According to Definition 1 this paper restricts the attention to the class of models in which no state transition can entail to a loss of the accumulated reward. This kind of accumulation is also referred to as preemptive resume. The distribution of the accumulated reward is defined by

$$B(t, w) = Pr\{\mathcal{B}(t) \leq w\} \qquad \text{and} \qquad B_i(t, w) = Pr\{\mathcal{B}_i(t) \leq w\} \ . \tag{3}$$

Note that

$$B(t,w) = \sum_{i \in S} p_i \ B_i(t,w) \ , \tag{4}$$

hence, in the rest of this paper, we use the initial state dependent measures and the global measures can always be evaluated by the mean of this relation.

Definition 2. *The* **completion time***, $\mathcal{C}_i$, is the random variable representing the time to accumulate the random amount of reward $\mathcal{W}$*

$$\mathcal{C}_i = \min[t \geq 0 \ : \ \mathcal{B}_i(t) \geq \mathcal{W}] \ . \tag{5}$$

The distribution of $\mathcal{C}_i$ is

$$C_i(t) = Pr\{\mathcal{C}_i \leq t\} \ . \tag{6}$$

Let $\mathcal{C}_i(w)$ be the random variable representing the time to accumulate w (fix) amount of reward and $C_i(t,w)$ its distribution, i.e.,

$$\mathcal{C}_i(w) = \min[t \geq 0 \ : \ \mathcal{B}_i(t) \geq w] \ \text{ and } C_i(t,w) = Pr\{\mathcal{C}_i(w) \leq t\} \ . \tag{7}$$

Let $G(w)$ be the distribution of $\mathcal{W}$ with support on $(0, \infty)$. By Definition 2,

$$C_i(t) = \int_0^\infty C_i(t,w) \ dG(w) \ . \tag{8}$$

The distribution of the completion time is closely related to the distribution of the accumulated reward by means of the following relation (see Figure 1.)

$$B_i(t,w) = Pr\{\mathcal{B}_i(t) \leq w\} = Pr\{\mathcal{C}_i(w) \geq t\} = 1 - C_i(t,w) \ . \tag{9}$$

3 Transform domain description of Performability

This section presents, as a basic result of the paper, the analytical description of the distribution of accumulated reward in single Laplace transform domain. The result is derived by a new approach that can handle impulse reward in addition to rate reward. The case when only rate reward is in the model can be obtained as a special case.

The provided results are valid for a quite general class of MRMs. The underlying CTMC can have any general structure (cyclic/acyclic, reducible/irreducible), the reward rate can be any finite non-negative real number and the impulse reward can be any non-negative *generally distributed* random variable (including discrete and deterministic distributions, defective distributions, distributions with infinite moments).

The description of accumulated reward is provided by the basic theorem of the paper:

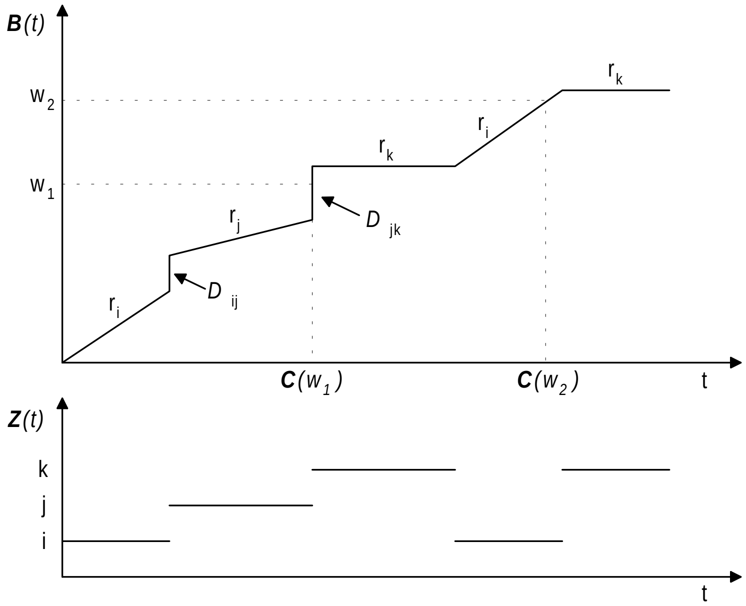

Fig. 1. A sample path of $\mathcal{Z}(t)$ and $\mathcal{B}(t)$.

Theorem 1. *The column vector of the distribution of the accumulated reward* $\underline{B}(t,w) = [B_i(t,w)]$ *is as follows:*

$$\underline{B}^{\sim}(t,v) = e^{[\mathbf{Q} \odot \mathbf{D}^{\sim}(v) - v\mathbf{R}]t} \cdot \underline{h} \ , \tag{10}$$

where $\odot$ *denotes the piecewise matrix multiplication* ($[\mathbf{A} \odot \mathbf{B}]_{ij} = a_{ij} \cdot b_{ij}$), $^{\sim}$ *denotes the Laplace-Stieltjes transform with respect to* $w (\to v)$, *and* $\underline{h}$ *is the column vector with all the entries equal to 1.*

The proof of the theorem is readable from the following two lemmas.

Lemma 1. *Let* $\underline{C}_m(t)$ *be the column vector of the completion time when the work requirement is exponentially distributed with parameter* m *(i.e.,* $\underline{C}_m(t) = [C_i(t) \mid G(w) = 1 - e^{-mw}]$*), and* $\hat{\underline{C}}_m(t)$ *is the analytical continuation of* $\underline{C}_m(t)$*.* $\underline{B}^{\sim}(t,v)$ *satisfies*

$$\underline{B}^{\sim}(t,v) = \underline{h} - \hat{\underline{C}}_m(t)\Big|_{m=v} \ . \tag{11}$$

Proof of Lemma 1 From (8) and (9) we have

$$C_i(t) = \int_0^{\infty} \Big(1 - B_i(t,w)\Big) \, dG(w) = m \int_0^{\infty} \Big(1 - B_i(t,x)\Big) \cdot e^{-mx} \, dx \ . \tag{12}$$

(12) can be rewritten using the Laplace-Stieltjes transform of the accumulated reward

$$C_i(t) = 1 - vB_i^*(t,v)\Big|_{v=m} = 1 - B_i^{\sim}(t,v)\Big|_{v=m} \tag{13}$$

which, in vector form, is

$$\underline{C}_m(t) = \underline{h} - \underline{B}^{\sim}(t,v)\Big|_{v=m} \ . \tag{14}$$

Since (11) is analytical for $\Re(v) \geq 0$ the lemma is given. □

Note that Lemma 1 is applicable with more general models than the one discussed in this paper, e.g., Lemma 1 holds with time inhomogeneous reward rates or impulse reward distributions.

Lemma 2. *The completion time of an exponentially distributed work requirement is a phase type distributed random variable (even with generally distributed impulse reward), and it can be evaluated as*

$$\underline{C}_m(t) = \underline{h} - e^{[\mathbf{Q} \odot \mathbf{D}^{\sim}(m) - m\mathbf{R}]t} \cdot \underline{h} \tag{15}$$

where $\mathbf{Q} \odot \mathbf{D}^{\sim}(m) - m\mathbf{R}$ *is the generator of the phase type distribution.*

The same result was obtained in [2] for MRMs with only rate reward. That case is captured when $D_{ij}^{\sim}(m) = 1, \forall i,j$.
Proof of Lemma 2 Due to the memoryless property of the exponentially distributed work requirement the remaining work to complete is exponentially distributed with the same parameter at any instance of time before completion. At a state transition from state i to j the completion occurs if the impulse reward $\mathcal{D}_{ij}$ is not less than the remaining work to complete, $\mathcal{W}_r$, i.e., the completion occurs with the following probability:

$$\begin{aligned} &Pr\{\text{completion}\} = Pr\{\mathcal{D}_{ij} > \mathcal{W}_r\} = \textstyle\int_0^\infty Pr\{\mathcal{D}_{ij} > w\}\ dG(w) = \\ &1 - \textstyle\int_0^\infty Pr\{\mathcal{D}_{ij} \leq w\}\ dG(w) = 1 - \int_0^\infty D_{ij}(w)\ dG(w) = \\ &1 - m \textstyle\int_0^\infty D_{ij}(w)\ e^{-mw}\ dw = 1 - m\ D_{ij}^*(m) = 1 - D_{ij}^{\sim}(m) \end{aligned} \tag{16}$$

Assuming the process stays in state i at time t before completion the following cases can occur in the interval $(t, t+dt)$ (see also [1]):

- *no state transition and no completion* occurs with probability $1 + (q_{ii} - r_1 m)dt + \sigma(dt)$,
- *no state transition and completion* occurs with probability $r_1 m dt + \sigma(dt)$,
- *state transition to j and no completion* occurs with probability $D_{ij}^{\sim}(m) q_{ij} dt + \sigma(dt)$,
- *state transition to j and completion* occurs with probability $(1 - D_{ij}^{\sim}(m)) q_{ij} dt + \sigma(dt)$,
- *any other cases* occur with probability $\sigma(dt)$.

Based on this behaviour a new CTMC can be defined by adding an absorbing state, $M+1$, to the state space of the underlying CTMC, defining state transitions from $\forall i \in \mathcal{S}$ to $M+1$ with rate $mr_i + \sum_{j,j\neq i} q_{ij}(1 - D^{\sim}_{ij}(m))$ and setting the transition rates between the states in $\mathcal{S}$ according to the above described behaviour. The absorbing state represents the completion of the exponentially distributed work requirement. The new CTMC defines a phase type distribution of order $\#\mathcal{S}$. Its $\#\mathcal{S} \times \#\mathcal{S}$ generator is $\mathbf{Q} \odot \mathbf{D}^{\sim}(m) - m\mathbf{R}$, and its time to absorption (i.e., completion time) is given by (15) (see, e.g., [13]). □

Note that Theorem 1 and Lemma 2 also hold, when the underlying CTMC have absorbing sets within $\mathcal{S}$. In this case the completion time of the exponentially distributed work requirement is a defective phase type distribution.

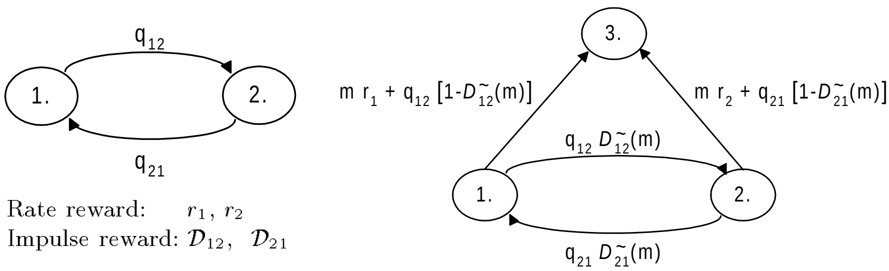

Fig. 2. A simple two-state system.

Lemma 2 is demonstrated through a simple example of a two-state system shown in Figure 2. The MRM of the example is defined by the the generator of the underlying CTMC, $\mathbf{Q}$, the rate reward matrix, $\mathbf{R}$, and the impulse reward matrix, $\mathbf{D}^{\sim}(v)$:

$$\mathbf{Q} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \quad \mathbf{D}^{\sim}(v) = \begin{pmatrix} 1 & D^{\sim}_{12}(v) \\ D^{\sim}_{21}(v) & 1 \end{pmatrix}. \tag{17}$$

In this example

$$C_1(t) = P_{13}(t) = 1 - (P_{11}(t) + P_{12}(t)) \text{ and } C_2(t) = P_{23}(t) = 1 - (P_{21}(t) + P_{22}(t)) .$$

The generator of the CTMC with the additional absorbing state, which describes the phase type distribution, is

$$\begin{pmatrix} q_{11} - mr_1 & q_{12}D^{\sim}_{12}(m) & mr_1 + q_{12}(1 - D^{\sim}_{12}(m)) \\ q_{21}D^{\sim}_{21}(m) & q_{22} - mr_2 & mr_2 + q_{12}(1 - D^{\sim}_{21}(m)) \\ 0 & 0 & 0 \end{pmatrix}, \tag{18}$$

whose $\#\mathcal{S} \times \#\mathcal{S}$ (2×2) upper left submatrix has the form $\mathbf{Q} \odot \mathbf{D}^{\sim}(m) - m\mathbf{R}$.

In the special case when only identical deterministic impulse reward is associated with a subset of transitions, the accumulated reward counts the occurrences of marked transitions.

4 Numerical analysis of MRMs with impulse and rate reward

In this section a numerical method is introduced which provides the moments of accumulated reward based on its transform domain description.

4.1 Moments of the Accumulated reward

Let $m_i^{(n)}(t) = E\{\mathcal{B}_i(t)^n\}$ be the nth moment of the reward accumulated in $(0,t)$. The column vector $\underline{m}^{(n)}(t) = [m_i^{(n)}(t)]$ can be evaluated based on $\underline{B}^{\sim}(t,v)$ as

$$\underline{m}^{(n)}(t) = (-1)^n \left. \frac{\partial^n \underline{B}^{\sim}(t,v)}{\partial v^n} \right|_{v=0} . \tag{19}$$

The following theorem provides a recursive method for the numerical analysis of the moments of accumulated reward.

Theorem 2. *The nth moment ($n \geq 1$) of the accumulated reward is*

$$\underline{m}^{(n)}(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \cdot \mathbf{N}^{(n)}(i) \cdot \underline{h} \tag{20}$$

where $\mathbf{N}^{(n)}(i)$ *is defined as*

$$\mathbf{N}^{(n)}(i) = \begin{cases} \mathbf{Q}^i & \text{if } i \geq 0, n = 0 , \\ \mathbf{0} & \text{if } i = 0, n \geq 1 , \\ \mathbf{Q} \odot \mathbf{D}^{(1)} + \mathbf{R} & \text{if } i = 1, n = 1 , \\ \mathbf{Q} \odot \mathbf{D}^{(n)} & \text{if } i = 1, n \geq 2 , \\ \displaystyle\sum_{l=0}^{n} \binom{n}{l} \mathbf{N}^{(l)}(1) \cdot \mathbf{N}^{(n-l)}(i-1) & \text{if } i \geq 2, n \geq 1 . \end{cases} \tag{21}$$

Proof of Theorem 2 From (19) and (10)

$$\underline{m}^{(n)}(t) = (-1)^n \left. \frac{\partial^n}{\partial v^n} e^{[\mathbf{Q}\odot\mathbf{D}^{\sim}(v) - v\mathbf{R}]t} \right|_{v=0} \cdot \underline{h} \tag{22}$$

$$= (-1)^n \left. \frac{\partial^n}{\partial v^n} \sum_{i=0}^{\infty} \frac{t^i}{i!} \left[\mathbf{Q} \odot \mathbf{D}^{\sim}(v) - v\mathbf{R}\right]^i \right|_{v=0} \cdot \underline{h} \tag{23}$$

$$= (-1)^n \sum_{i=0}^{\infty} \frac{t^i}{i!} \left. \frac{\partial^n}{\partial v^n} \left[\mathbf{Q} \odot \mathbf{D}^{\sim}(v) - v\mathbf{R}\right]^i \right|_{v=0} \cdot \underline{h} \tag{24}$$

Let

$$\mathbf{N}^{(n)}(i) = \left. \frac{\partial^n}{\partial v^n} \left[\mathbf{Q} \odot \mathbf{D}^{\sim}(v) - v\mathbf{R}\right]^i \right|_{v=0} \qquad \text{for } \forall n, i \geq 0 . \tag{25}$$

From the Leibniz rule[2] it follows

$$\mathbf{N}^{(n)}(i) = [\mathbf{N}^{(0)}(1) \cdot \mathbf{N}^{(0)}(i-1)]^{(n)} \tag{26}$$

$$= \sum_{l=0}^{n} \binom{n}{l} \mathbf{N}^{(l)}(1) \cdot \mathbf{N}^{(n-l)}(i-1) \quad \text{if } \; i \geq 2, n \geq 1 \tag{27}$$

with the initial conditions

$$\begin{aligned}
&\mathbf{N}^{(0)}(0) = \mathbf{I}, \\
&\mathbf{N}^{(0)}(i) = \mathbf{Q}^{i} \quad i > 0, \\
&\mathbf{N}^{(n)}(0) = \mathbf{0} \quad n \geq 1, \\
&\mathbf{N}^{(1)}(1) = \mathbf{Q} \odot \mathbf{D}^{(1)} + \mathbf{R} \\
&\mathbf{N}^{(n)}(1) = \mathbf{Q} \odot \mathbf{D}^{(n)} \quad n \geq 2 \ .
\end{aligned}$$

This completes the proof. □

Based on (21) the nth moment of the accumulated reward is finite if all the moments of the impulse reward from the 1st to the nth one are finite (independent of the higher moments); and the nth moment of the accumulated reward can become infinite if at least one moment of the impulse reward is infinite.

The iterative procedure to evaluate $\mathbf{N}^{(n)}(i)$ has the following properties:

- it is not possible to evaluate the nth moment itself, but to obtain the nth moment all the previous moments (or at least the associated $\mathbf{N}^{(n)}(i)$ terms) must be computed,
- matrix-matrix multiplications are computed in each iteration steps,
- numerical problems are possible due to the repeated multiplication with $\mathbf{Q}$, which contains both positive and negative elements.

Hence the method in Theorem 2 is not directly applicable for numerical analysis.

4.2 A Numerical Method Based on Randomization

The previous iterative procedure provides an effective iterative method to evaluate the terms needed to compute the moments of accumulated reward, but due to the properties of digital computers with limited available memory and floating point aritmetics a direct application of Theorem 2 would result in numerical problems such as instabilities, “ringing” (negative probabilities), etc. To avoid these problems a modified procedure is proposed based on the concept of randomization. Let

$$\mathbf{A} = \frac{\mathbf{Q}}{q} + \mathbf{I} \;\; \text{and} \;\; \mathbf{S} = \frac{\mathbf{R}}{q} \ , \tag{28}$$

where $q = \max_{i,j \in \mathcal{S}} (|q_{ij}|)$. By this definition $\mathbf{A}$ is a stochastic matrix ($0 \leq a_{i,j} \leq 1, \forall i, j \in \mathcal{S}$ and $\sum_{j \in \mathcal{S}} a_{i,j} = 1, \forall i \in \mathcal{S}$). Using these matrices we have

[2] $(f_1(z_0) \cdot f_2(z_0))^{(n)} = \sum_{l=0}^{n} \binom{n}{l} f_1^{(k)}(z_0) \cdot f_2^{(n-k)}(z_0)$

$$\begin{aligned}\underline{B}^{\sim}(t,v) &= e^{[\mathbf{Q}\odot\mathbf{D}^{\sim}(v)-v\mathbf{R}]t}\cdot\underline{h}\\ &= e^{[(\mathbf{A}-\mathbf{I})\odot\mathbf{D}^{\sim}(v)-v\mathbf{S}]qt}\cdot\underline{h}\\ &= e^{[\mathbf{A}\odot\mathbf{D}^{\sim}(v)-v\mathbf{S}]qt}\cdot\underline{h}\cdot e^{-qt}\end{aligned}\tag{29}$$

Lemma 3. *The moments of accumulated reward can be computed using only matrix-vector multiplications and saving only vectors of size* $\#\mathcal{S}$ *in each step of the iteration as*

$$\underline{m}^{(n)}(t) = \sum_{i=0}^{\infty}\underline{U}^{(n)}(i)\cdot\frac{(qt)^i}{i!}e^{-qt}\tag{30}$$

where $\underline{U}^{(n)}(i)$ *is defined as*

$$\underline{U}^{(n)}(i) = \begin{cases}\underline{h} & \text{if } \ i\geq 0, n=0\ ,\\ \underline{0} & \text{if } \ i=0, n\geq 1\ ,\\ \displaystyle\sum_{l=0}^{n}\binom{n}{l}\mathbf{V}^{(l)}\cdot\underline{U}^{(n-l)}(i-1) & \text{if } \ i\geq 1, n\geq 1\ ,\end{cases}\tag{31}$$

and $\mathbf{V}^{(n)}$ *is defined as*

$$\mathbf{V}^{(n)} = \begin{cases}\mathbf{A} & \text{if } \ n=0\ ,\\ \mathbf{A}\odot\mathbf{D}^{(1)}+\mathbf{S} & \text{if } \ n=1\ ,\\ \mathbf{A}\odot\mathbf{D}^{(n)} & \text{if } \ n\geq 2\ ,\end{cases}\tag{32}$$

where $\mathbf{D}^{(n)}$ *is the matrix of the nth moment of impulse reward (as it is defined in Section 2).*

Proof of Lemma 3 Starting from (29) the proof of Lemma 3 follows the same pattern as the proof of Theorem 2. □

To demonstrate the iterative procedure of computing $\underline{U}^{(n)}(i)$ the first elements of the recursion are provided in Table 1 and the dependence of the consecutive term is depicted in Figure 3.

	$\mathbf{V}^{(n)}$	$\underline{U}^{(n)}(0)$	$\underline{U}^{(n)}(1)$	$\underline{U}^{(n)}(2)$	$\underline{U}^{(n)}(3)$
$n=0$	$\mathbf{A}$	$\underline{h}$	$\underline{h}$	$\underline{h}$	$\underline{h}$
$n=1$	$\mathbf{A}\odot\mathbf{D}^{(1)}+\mathbf{S}$	$\underline{0}$	$\mathbf{V}^{(1)}\cdot\underline{h}$	$\mathbf{V}^{(0)}\cdot\underline{U}^{(1)}(1)+$ $\mathbf{V}^{(1)}\cdot\underline{U}^{(0)}(1)$	$\mathbf{V}^{(0)}\cdot\underline{U}^{(1)}(2)+$ $\mathbf{V}^{(1)}\cdot\underline{U}^{(0)}(2)$
$n=2$	$\mathbf{A}\odot\mathbf{D}^{(2)}$	$\underline{0}$	$\mathbf{V}^{(2)}\cdot\underline{h}$	$\mathbf{V}^{(0)}\cdot\underline{U}^{(2)}(1)+$ $2\cdot\mathbf{V}^{(1)}\cdot\underline{U}^{(1)}(1)+$ $\mathbf{V}^{(2)}\cdot\underline{U}^{(0)}(1)$	$\mathbf{V}^{(0)}\cdot\underline{U}^{(2)}(2)+$ $2\cdot\mathbf{V}^{(1)}\cdot\underline{U}^{(1)}(2)+$ $\mathbf{V}^{(2)}\cdot\underline{U}^{(0)}(2)$
$n=3$	$\mathbf{A}\odot\mathbf{D}^{(3)}$	$\underline{0}$	$\mathbf{V}^{(3)}\cdot\underline{h}$	$\mathbf{V}^{(0)}\cdot\underline{U}^{(3)}(1)+$ $3\cdot\mathbf{V}^{(1)}\cdot\underline{U}^{(2)}(1)+$ $3\cdot\mathbf{V}^{(2)}\cdot\underline{U}^{(1)}(1)+$ $\mathbf{V}^{(3)}\cdot\underline{U}^{(0)}(1)$	$\mathbf{V}^{(0)}\cdot\underline{U}^{(3)}(2)+$ $3\cdot\mathbf{V}^{(1)}\cdot\underline{U}^{(2)}(2)+$ $3\cdot\mathbf{V}^{(2)}\cdot\underline{U}^{(1)}(2)+$ $\mathbf{V}^{(3)}\cdot\underline{U}^{(0)}(2)$

Table 1: First terms of the iterative procedure

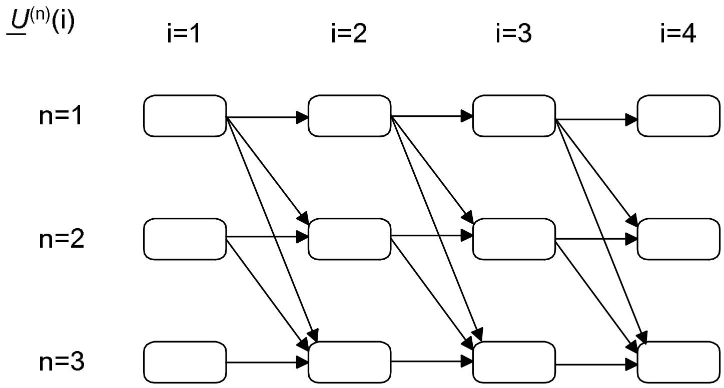

Fig. 3. The dependency structure of the iteration steps

To evaluate the error incurred by applying a finite summation instead of (30) the following vector norms are introduced.

$$d_1 = \max_j \; [(\mathbf{A} \odot \mathbf{D}^{(1)} + \mathbf{S}) \cdot \underline{h}]_j \tag{33}$$

$$d_n = \max_j \; [(\mathbf{A} \odot \mathbf{D}^{(n)}) \cdot \underline{h}]_j \quad , \quad n \geq 2 \tag{34}$$

The norm of $\underline{U}^{(n)}(i)$ is upper bounded by $u^{(n)}(i) \geq \max_j \; [\underline{U}^{(n)}(i)]_j$ which can be calculated recursively in a similar manner like $\underline{U}^{(n)}(i)$:

$$u^{(n)}(i) = \begin{cases} 1 & \text{if } \; i \geq 0, n = 0 \; , \\ 0 & \text{if } \; i = 0, n \geq 1 \; , \\ \sum_{l=0}^{n} \binom{n}{l} d_l \cdot u^{(n-l)}(i-1) & \text{if } \; i \geq 1, n \geq 1 \; . \end{cases} \tag{35}$$

Let $a_n = \max_{\ell \in \{1,\ldots,n\}} d_\ell$ be the largest of the norms $d_1, \ldots, d_n$. From (35), it can be seen that

$$\max_j \; [\underline{U}^{(n)}(i)]_j \leq u^{(n)}(i) \leq a_n (a_n 2^n)^{i-1} \tag{36}$$

Lemma 4 gives a procedure of accuracy control based on (30) and (36).

Lemma 4. *The nth moment of accumulated reward can be calculated as a finite sum and an error part, where the maximum allowed error is* ε

$$\underline{m}^{(n)}(t) = \sum_{i=0}^{G} \underline{U}^{(n)}(i) \cdot \frac{(qt)^i}{i!} e^{-qt} + \underline{\xi}(G) \tag{37}$$

where

$$G = \min \left(g \in \mathbb{N} \;\middle|\; 2^{-n} e^{-qt(1-a_n 2^n)} \sum_{i=g+1}^{\infty} \frac{(qta_n 2^n)^i}{i!} \cdot e^{-qta_n 2^n} \leq \varepsilon \right) \tag{38}$$

and the $\underline{0} \leq \underline{\xi}(G) \leq \varepsilon \cdot \underline{h}$ *inequality holds for all the elements of the vectors.*

Proof of Lemma 4 The error vector $\underline{\xi}(G)$ piecewise satisfies the following inequality:

$$\begin{aligned} \underline{\xi}(G) &= \sum_{i=G+1}^{\infty} \underline{U}^{(n)}(i) \cdot \frac{(qt)^i}{i!} e^{-qt} \leq \sum_{i=G+1}^{\infty} a_n (a_n 2^n)^{i-1} \frac{(qt)^i}{i!} e^{-qt} \cdot \underline{h} \\ &= 2^{-n} e^{-qt(1-a_n 2^n)} \sum_{i=G+1}^{\infty} \frac{(qta_n 2^n)^i}{i!} e^{-qta_n 2^n} \cdot \underline{h} \end{aligned} \tag{39}$$

From which the lemma comes. □

In (38) the tail of a Poisson distribution with parameter $qta_n 2^n$ is multiplied by a positive number $2^{-n} e^{-qt(1-a_n 2^n)}$. The tail of a Poisson distribution is quickly decreasing from its mean $(qta_n 2^n)$ hence G is mainly determined by the parameter of the Poisson distribution $(qta_n 2^n)$ and almost independent of the coefficient $(2^{-n} e^{-qt(1-a_n 2^n)})$. If $qta_n 2^n < 100$ the error can be calculated as

$$\underline{\xi}(G) = 2^{-n} e^{-qt(1-a_n 2^n)} \left(1 - \sum_{i=0}^{G} \frac{(qta_n 2^n)^i}{i!} e^{-qta_n 2^n} \right) \cdot \underline{h} \, , \tag{40}$$

and if $qta_n 2^n > 100$ then G and $qta_n 2^n$ are of the same order of magnitude $(G > qta_n 2^n)$ and G can be approximated using the normal distribution with the same mean and variance.

4.3 Computational complexity and memory requirement

The numerical procedure provided by Lemma 4 is very effective compare to the previously known methods both in terms of computational complexity and memory requirement. Broadly speaking, the computational complexity of the analysis of nth moment is n times more than the complexity of transient analysis of the underlying CTMC using randomization [15], and the memory requirement to evaluate one moment is the same (excluding the representation of the input data $\mathbf{Q}, \mathbf{R}, \mathbf{D}^{(i)}$).

An iteration step for the evaluation of the moments from the first to the nth one consists of $n(n+1)/2$ matrix vector multiplications. The complexity of matrix vector multiplications is determined by the number of non-zero element in $\mathbf{Q}$ (not by the size of the state space). The matrices multiplied during the iteration $(\mathbf{V}^{(0)}, \ldots, \mathbf{V}^{(n)})$ are as sparse as $\mathbf{Q}$ is. The problem of "fill up" does not occur, i.e., if $\mathbf{Q}$ is sparse the same sparse matrices are multiplied during the whole procedure.

The analysis of the first n moments requires to store $2n$ vectors of size $\#\mathcal{S}$, i.e., two vectors per moments are needed, one for the actual value of $\underline{m}^{(n)}(t)$ and one for $\underline{U}^{(n)}(i)$. E.g., the analysis of the first 4th moments consists of 10 times more matrix vector multiplications than the transient analysis of the underlying CTMC, and requires 4 times more memory for the computation (disregarding the input data).

A Pascal like description of the proposed numerical method is provided in Figure 4.

1. Compute G;
2. **For** $j := 0$ **To** n **Do**
   Begin
   $\underline{U}^{(j)} := \mathbf{V}^{(j)} \cdot \underline{h}$;
   $\underline{m}^{(j)}(t) := \underline{U}^{(j)} \cdot \text{Poisson}(1; qt)$;
   End;
3. **For** $i := 2$ **To** G **Do**
   Begin
   For $j := n$ **DownTo** 0 **Do**
   Begin
   For $k := 0$ **To** j **Do**
   $\underline{U}^{(j)} := \binom{j}{k} \cdot \mathbf{V}^{(k)} \cdot \underline{U}^{(j-k)}$;
   $\underline{m}^{(j)}(t) := \underline{m}^{(j)}(t) + \underline{U}^{(j)} \cdot \text{Poisson}(i; qt)$;
   End;
   End;

Fig. 4. A Pascal-like description of the proposed numerical method

5 Numerical Examples

Two numerical examples are introduced in this section. The first one indicates the ability of the proposed method to evaluate models with a large state space. The second one emphasizes the applicability of MRMs with impulse and rate reward for the analysis of real systems.

5.1 A synthetic example with large state space

The performability of an MRM with 10^5 states, numbered from 0 to $99,999$, is evaluated in this example. The underlying CTMC is a birth-death process with birth rate equals to 100 and death rate equals to 1. The state space of the example is shown in Figure 5. The reward rate associated with states $85,000 - 99,999$ equals to 10 and each downward transitions above state $80,000$ incur an impulse reward equals to 1. The rest of the states and state transitions does not incur any reward accumulation.

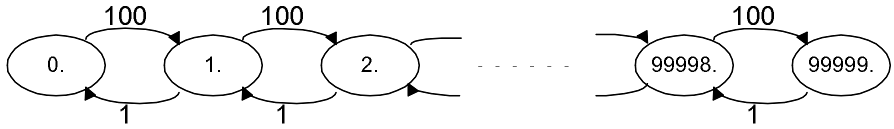

Fig. 5. The state space of the synthetic example

The diagram in Figure 6 (Figure 7) shows the mean and the coefficient of variation of the accumulated reward versus time, when the initial state is state $80,000$ $(79,950)$, while an uniform initial distribution is assumed in Figure 8 (i.e., $p_i = 10^{-5}, \forall i \in \mathcal{S}$). In the figures the thick lines shows the mean and the thin lines the coefficient of variation of performability and the associated scales are shown on the left and right vertical accesses, respectively. In Figure 7, there is an interval of time $(0, 0.5)$ while the mean accumulated reward is close to zero and the associated curve is horizontal. This is the mean time to arrive to the first state (state $80,000$) where the reward accumulation starts.

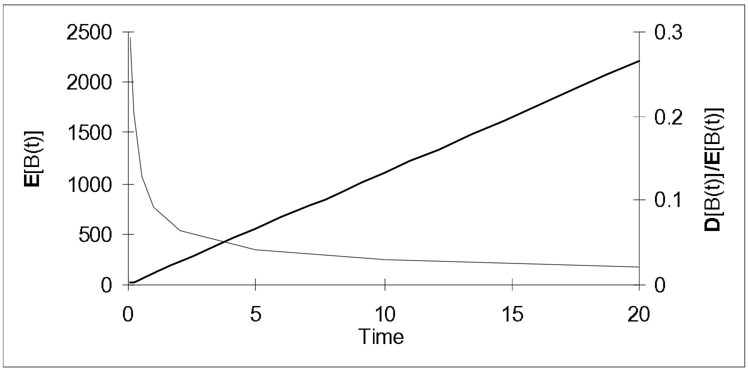

Fig. 6. The mean and coef. of variation of accumulated reward when $p_{80,000} = 1$

The example was evaluated using a personal computer with 32Mb memory and 200MHz Pentium processor. The size of the example was chosen to fit to the operative memory of the computer to avoid swapping. Applying more powerful computers or allowing swapping makes possible to evaluate larger examples, and to calculate higher moments as well. Our test implementation required 15 minutes to compute each of Figures 6 - 8.

5.2 Analysis of a complex system behaviour

A dependable equipment (a computer or a machine producing goods) is operated according to the following rules. Three operational conditions (states) are dis-

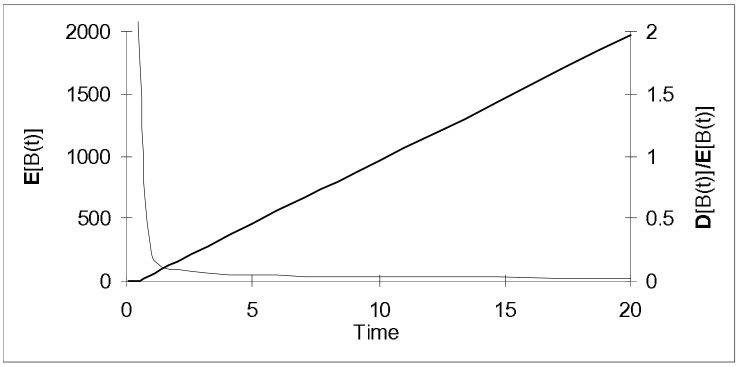

Fig. 7. The mean and coef. of variation of accumulated reward when $p_{79,950} = 1$

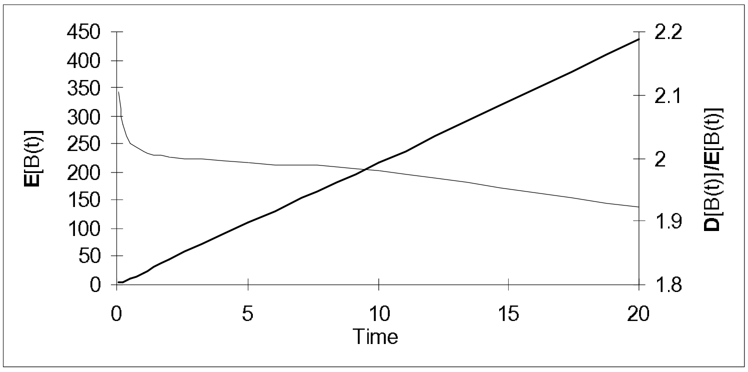

Fig. 8. The mean and coef. of variation of accumulated reward with uniform initial distribution $(p_i = 10^{-5}, \forall i \in \mathcal{S})$

tinguished, namely `perfect, good, adequate`. The system degradation (transition form state `perfect` to state `good`, and from state `good` to state `adequate`) occurs at a constant rate. The equipment is periodically stopped for a preventive maintenance. If the system state is `adequate` a complete repair is initiated instead of a preventive maintenance. System failure can occur in any operational states at the same constant failure rate. A system failure results in a complete repair as well, i.e, at the end of the repair the system is restored to the "as good as new" condition represented by state `perfect`.

The cost of preventive maintenance has a fix and a time dependent component which can depend on the system state as well. The cost of complete repair also has a fix and a time dependent component, but there is a correlation between the repair time and the associated fix cost. An additional fix cost is assigned to the longer repair periods (E.g., in some cases the complete repair requires the renewal of some special parts that is expensive and time consuming). Some cost can be associated with the system performance degradation in state `good` and `adequate`.

Assuming all the mentioned state transitions occur at a constant rate the system behaviour can be described by an MRM with impulse and rate reward as it is shown in Figure 9.

Based on the MRM of the system the following performance parameters can be evaluated:

- interval availability
- operational cost during a time interval
- number of different failures during a time interval

Among these performance parameters the analysis of the operational cost requires the use of impulse and rate reward at the same time. We have evaluated the first four moments of the operational cost assuming the state transition rates shown in Figure 9. The shaded states have an associated rate reward and the transitions drawn as thick arrows have an associated impulse reward. In each cases the impulse reward is assumed to be deterministic. The following impulse and rate reward values were used:

rate reward	impulse reward
$r_{LS1} = 1$	$d_{LS1} = 0.2$
$r_{LS2} = 1.5$	$d_{LS2} = 0.2$
$r_{S1} = 10$	$d_{S1} = 10$
$r_{S2} = 20$	$d_{S2} = 5$
$r_G = 0.1$	
$r_A = 0.2$	

Table 2. contains the moments of the operational cost for different time intervals and initial states. The three data of each boxes are calculated assuming the initial state is the `perfect`, the `good`, and the `adequate` state, respectively.

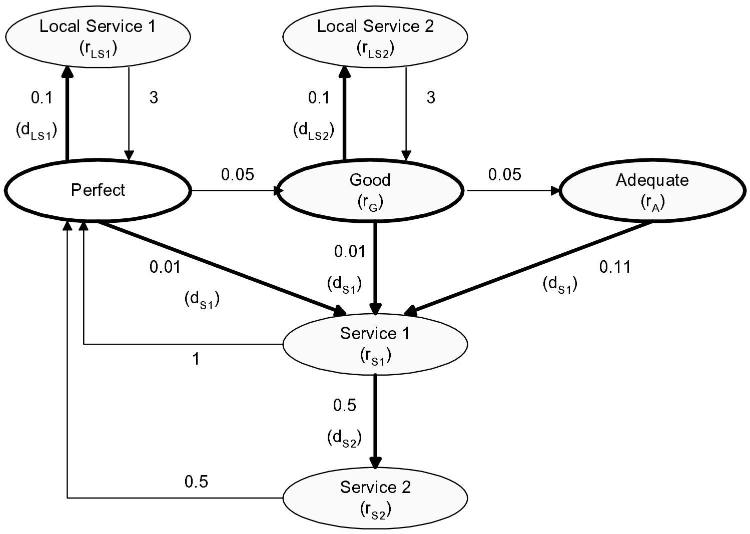

Fig. 9. The state structure of the system

	$E(B(t))$	$E(B(t)^2)$	$E(B(t)^3)$	$E(B(t)^4))$
$t = 1$	0.1918	2.6511	56.22	1404
	0.3298	3.1877	65.65	1608
	1.7694	28.56	613.50	15382
$t = 10$	3.9198	164.29	12890	1341083
	7.9019	325.25	24579	2462323
	21.50	1149	98501	10945246
$t = 100$	83.04	10271	1720854	362927485
	94.95	12627	2213071	482081341
	106.91	15440	2869241	653702868
$t = 1000$	913.24	869967	863458021	$8.91 \cdot 10^{11}$
	925.15	892099	895609345	$9.35 \cdot 10^{11}$
	937.11	914768	929167155	$9.81 \cdot 10^{11}$
$t = 10000$	9215	85283872	$7.92 \cdot 10^{11}$	$7.39 \cdot 10^{15}$
	9227	85503760	$7.95 \cdot 10^{11}$	$7.43 \cdot 10^{15}$
	9239	85724994	$7.98 \cdot 10^{11}$	$7.47 \cdot 10^{15}$

Table 2: The first four moments of accumulated reward with different initial state

6 Conclusion

Markov Reward models with impulse and rate reward are studied in this paper. As a novel contribution, the distribution of performability of these models is derived in closed form in Laplace transform domain. Based on the transform domain description an effective numerical method is obtained that computes the moments of performability. The proposed method has low computational complexity and memory requirement and nice numerical properties due to the use of randomization technique. An MRM with 10^5 states is evaluated on a simple personal computer to illustrate the analysis power of the method, and a complex system behaviour is considered to indicate the modelling power of MRMs with impulse and rate reward.

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