Delay analysis of resequencing buffer in Markov environment with HOQ-FIFO-LIFO policy*

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Abstract. Resequencing of customers during the service process results in hard to analyze delay distributions. A set of models with various service and resequencing policies have been analyzed already for memoryless arrival, service and resequencing processes with an intensive use of transform domain descriptions. In case of Markov modulated arrival, service and resequencing processes those methods are not applicable any more. In a previous work we analyzed the Markov modulated case with HOQ-FIFO-FIFO policy (head of queue customer of the higher priority FIFO queue is moved to resequencing FIFO queue). In this work we investigate if the approach remains applicable for different service discipline for the HOQ-FIFO-LIFO policy.
It turns out that the analysis of the new service policy requires the solution of a coupled quadratic matrix equations which were separated in the HOQ-FIFO-FIFO case.

Keywords: Resequencing buffer, Delay analysis, Markov modulated arrival and service process.

1 Introduction

In models with resequencing delay distributions are of primary interest. Usually resequencing is due to some disruptive events but it also may be one of the features, which are inherent to the system (for models in the context of queueing theory see, for example, the reviews [3, 2]). With the evolution and the widespread use of matrix analytic methods [6, 7, 5, 4], there is a belief that the more and more Markov chain based analysis of stochastic models with memoryless components can be extended for the same problem with modulating Markov environment. The transform domain delay analysis of the resequencing

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buffer models in [9] was an example of notoriously hard extension with modulating Markov environment. For the HOQ-FIFO-FIFO policy, which is one of the policies studied in [9], the analysis with modulating Markov environment is presented in [10].

This work is essentially a methodological study to understand if the methodology developed in [10] is general enough for applying in other queueing models, particularly for the same resequencing buffer model **but** with HOQ-FIFO-LIFO policy.

The rest of the paper is structured as follows. In Section 2 the system description is provided. In the next section we summarize the results concerning the joint stationary distribution, which, in fact, coincides with the one for the system from [10]. Section 4 provides the new contribution of the paper, which is the waiting time distribution **for the** HOQ-FIFO-LIFO policy. Some numerical experiments are provided in Section 5 and the paper is concluded **with** Section 6.

2 Model description

The system under consideration is a single server queueing system with two infinite buffers: the regular buffer (or, simply, buffer) and the resequencing buffer. **Regular customers (or, simply, customers)** arrive at the system and occupy one place in the regular buffer. Resequencing signals arrive at the system according to a resequencing process. If the buffer is not empty, then, upon arrival, each resequencing signal moves one customer from the regular buffer to the resequencing buffer and itself leaves the system, otherwise it leaves the system without having any effect on it. A single server serves customers from both queues. Upon service completion one customer from the regular buffer goes to the server and only if there are no regular customers in the buffer, one customer from resequencing buffer enters the server. No service interruption is allowed. The HOQ-FIFO-LIFO policy means that the resequencing signal moves the oldest waiting regular customer to the resequencing buffer (Head Of Queue, HOQ), the service policy of the regular buffer is FIFO and of the resequencing buffer is LIFO.

Since the customers from the resequencing buffer are served if and only if the regular buffer is empty, the considered system is a variant of a priority queue with regular buffer customers as high priority customers and resequencing buffer customers as low priority customers.

We assume that regular customers arrive according to a MAP process with generator matrices $(\mathbf{A_0}, \mathbf{A_1})$ and resequencing signals arrive according to a MAP with $(\mathbf{H_0}, \mathbf{H_1})$. The service process is a MAP with $(\mathbf{S_0}, \mathbf{S_1})$. Let $\mathbf{A_J} = \mathbf{A_0} + \mathbf{A_1}$, $\mathbf{S_J} = \mathbf{S_0} + \mathbf{S_1}$, and $\mathbf{H_J} = \mathbf{H_0} + \mathbf{H_1}$ denote the phase processes of the associated MAPs (see e.g. [5] for details). The block structure of the Markov chain representing the number of high and low priority customers in the system is depicted in Figure 1. The block represents the set of states with the same number of high and low priority customers and with different phases of the MAPs. The letters on the figures describe

- arrival of a customer: $\mathcal{A} = \mathbf{A_1} \otimes I \otimes I$,
- service of a customer: $\mathcal{S} = I \otimes \mathbf{S_1} \otimes I$,
- resequencing of a customer: $\mathcal{H} = I \otimes I \otimes \mathbf{H_1}$,
- phase change when resequencing is possible: $\mathcal{L} = \mathbf{A_0} \oplus \mathbf{S_0} \oplus \mathbf{H_0}$,
- phase change when resequencing is not possible: $\mathcal{L}' = \mathbf{A_0} \oplus \mathbf{S_0} \oplus \mathbf{H_J}$,
- phase change when resequencing is not possible and the service process is stopped: $\mathcal{L}_0 = \mathbf{A_0} \otimes I \oplus \mathbf{H_J} = \mathbf{A_0} \otimes I \otimes I + I \otimes I \otimes \mathbf{H_J}$,

where $\otimes$ ($\oplus$) denotes the Kronecker product (sum) and I the identity matrix of appropriate size. The phase of the service process is frozen (does not change) when the system is empty.

The main goal of the analysis is to evaluate the stationary waiting time distribution of a regular customer arriving at the system.

3 Joint stationary distribution of the number of customers

Before deriving the expressions for the stationary waiting time distribution one has to obtain expressions for joint stationary distribution of number of customers in regular buffer, resequencing buffer and phases of regular and resequencing arrivals and service process. Since the service order does not affect the number of customers in the system, the joint stationary distribution in the HOQ-FIFO-LIFO system is identical with the one of the HOQ-FIFO-FIFO system studied in [10]. In this section we introduce the notation and repeat results from [10], which will be used later on.

3.1 Censored process

To simplify the analysis and obtain a Markov chain with a regular structure we censor the Markov chain in Figure 1 for the cases when the server is busy. The structure of the censored Markov chain is depicted in Figure 2. The transitions of upper left block of the censored chain is obtained as

$$\mathcal{L}'' = \mathcal{L}' - \mathcal{S}\mathcal{L}_0^{-1}\mathcal{A} = (\mathbf{A_0} \oplus \mathbf{S_0} \oplus \mathbf{H_J}) - (I \otimes \mathbf{S_1} \otimes I)(\mathbf{A_0} \otimes I \oplus \mathbf{H_J})^{-1}(\mathbf{A_1} \otimes I \otimes I).$$

3.2 QBD representation of the censored process

Following, for example, the discussion of Section 13.1 in [5] we can represent the censored Markov chain as QBD process where the levels are composed by the set of states where the number of regular customers is the same (these states form the columns of blocks in Figure 2). The generator $\mathbb{Q}$ of the censored process can be represented in hyper-block tridiagonal form, where the hyper-block refers to

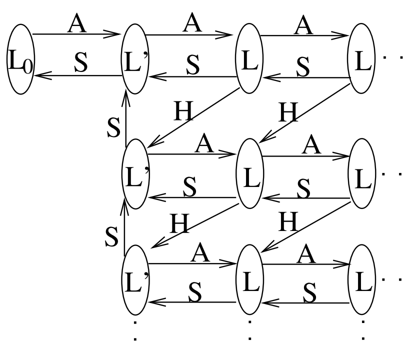

Fig. 1. Block structure of the Markov chain representing the number of regular (high priority) and resequenced (low priority) customers

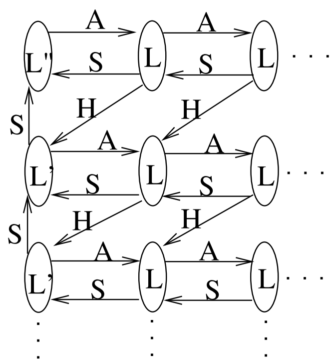

Fig. 2. Block structure of the censored Markov chain representing the number of regular (high priority) and resequenced (low priority) customers

the set of (infinitely many) states on the same level.

$$\mathbb{Q} = \begin{pmatrix} \mathbb{L}' & \mathbb{F} & 0 & 0 & 0 & \cdots \\ \mathbb{B} & \mathbb{L} & \mathbb{F} & 0 & 0 & \cdots \\ 0 & \mathbb{B} & \mathbb{L} & \mathbb{F} & 0 & \cdots \\ 0 & 0 & \mathbb{B} & \mathbb{L} & \mathbb{F} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and, due to the fact that the number of states within each level is infinite, matrices $\mathbb{L}'$, $\mathbb{L}$, $\mathbb{B}$, $\mathbb{F}$ have infinite rows and columns which are associated with the blocks in Figure 2).

$$\mathbb{L}' = \begin{pmatrix} \mathcal{L}'' & 0 & 0 & \cdots \\ \mathcal{S} & \mathcal{L}' & 0 & \cdots \\ 0 & \mathcal{S} & \mathcal{L}' & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathbb{L} = \begin{pmatrix} \mathcal{L} & 0 & 0 & \cdots \\ 0 & \mathcal{L} & 0 & \cdots \\ 0 & 0 & \mathcal{L} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathbb{F} = \begin{pmatrix} \mathcal{A} & 0 & 0 & \cdots \\ 0 & \mathcal{A} & 0 & \cdots \\ 0 & 0 & \mathcal{A} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} \mathcal{S} & \mathcal{H} & 0 & 0 & \cdots \\ 0 & \mathcal{S} & \mathcal{H} & 0 & \cdots \\ 0 & 0 & \mathcal{S} & \mathcal{H} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In the censored Markov chain we denote the stationary probability vector of the set of states with i regular and j delayed customers by π_{ij} $(i, j \geq 0)$ and compose the following row vectors

$$\mathbf{p}_i = (\pi_{i,0}, \pi_{i,1}, \pi_{i,2}, \pi_{i,3}, \dots), \ i \geq 0,$$
$$\mathbf{p} = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \dots).$$

Henceforth we consider the distribution $\mathbf{p}$, which is the solution of the linear infinite system of equations $\mathbf{p}\,\mathbb{Q} = \mathbf{0}$, $\mathbf{p}\,\mathbf{1} = 1$, to be known.

3.3 Distribution right after customer arrival

Notice that as MAP arrivals do not see time averages (that is PASTA property does not hold) one has to calculate stationary probabilities $\tilde{\pi}_{ij}$ that after a customer arrival there are i ($i \geq 1$) customer in the regular buffer and j ($j \geq 0$) in the resequencing buffer. Following the same argument as in [8], we can write

$$\tilde{\pi}_{ij} = \frac{1}{\lambda}\pi_{i-1,j}\mathcal{A}, \ i \geq 1, \ j \geq 0, \quad \text{and} \quad \tilde{\pi}_{00} = \frac{1}{\lambda}\pi_{idle}\mathcal{A}.$$

Here π_{idle} is the stationary distribution of the block of states representing idle server (the left most block in Figure 1). It is found (see the details in [10, Section 3.6]) from the system of linear equations $\pi_{idle}(\mathcal{L}_0 - \mathcal{A}\mathbf{T}_0^{-1}\mathcal{S}) = 0$, $\pi_{idle}\mathbf{1} = 1 - \lambda/\mu$. As usual, λ denotes the average arrival rate and μ denotes the average service rate.

4 Stationary waiting time distribution

The waiting time (W) is understood here, as usual, as the time lapse, starting from the instant when regular customer arrives at the system up to the instant when it enters server. Its stationary distribution will be evaluated in terms of Laplace–Stieltjes transform $\omega(s) = E(e^{-sW})$. Regular customer may enter the server either from the regular buffer or from the resequencing buffer and thus its stationary waiting time distribution can be computed as

$$\begin{aligned}
&\omega(s) = E(e^{-sW}) = \omega_{\mathrm{H}}(s) + \omega_{\mathrm{L}}(s) \\
&= E(e^{-sW} I_{\{\text{served from regular buffer}\}}) + E(e^{-sW} I_{\{\text{served from resequencing buffer}\}})
\end{aligned}$$

where $I_{\{a\}}$ is the indicator of event a.

It is clear that under HOQ-FIFO-LIFO policy the stationary waiting time distribution of the regular customer that receives service from regular buffer coincides with that under the HOQ-FIFO-FIFO policy. Thus we will not repeat these derivations here and refer the reader for the details to the [10, Section 4.1]. Henceforth we consider $\omega_{\mathrm{H}}(s)$ to be known.

4.1 Stationary waiting time distribution of the customer that receives service from resequencing buffer

For $i \geq j \geq 0$ and $k > 0$ let $\mathbb{F}(t, i, j, k)$ be the matrix (according to the initial and final phases of the MAPs $(\mathbf{A_0}, \mathbf{A_1})$, $(\mathbf{S_0}, \mathbf{S_1})$ and $(\mathbf{H_0}, \mathbf{H_1})$) of the probabilities that k customers arrive, $i - j$ customers are served and j are moved to the resequencing buffer in time t, when the initial number of customers in the buffer is larger than i. For the Laplace transform $\tilde{\mathbb{F}}(s, i, j, k) = \int_t e^{-st}\mathbb{F}(t, i, j, k)dt$ we have

$$\tilde{\mathbb{F}}(s, 0, 0, 0) = (sI - \mathcal{L})^{-1} = \mathcal{L}(s), \tag{1}$$

and otherwise

$$\tilde{\mathbb{F}}(s,i,j,k) = I_{\{i>j\}}\mathcal{L}(s)\mathcal{S}\tilde{\mathbb{F}}(s,i-1,j,k) + I_{\{j>0\}}\mathcal{L}(s)\mathcal{H}\tilde{\mathbb{F}}(s,i-1,j-1,k) \quad (2)$$
$$+ I_{\{k>0\}}\mathcal{L}(s)\mathcal{A}\tilde{\mathbb{F}}(s,i,j,k-1),$$

where $\mathcal{L}(s)$ is defined in (1). An intuitive explanation of the first term of (2) is as follows. There is no arrival, service and resequencing up to time τ ($\mathcal{L}(s)$) than an service occurs ($\mathcal{S}$) and than $i-1$ services, j resequencing and k arrival occur in (τ, t) ($\tilde{\mathbb{F}}(s,i-1,j,k)$). The other terms follow the same pattern. The cases that the tagged customer moves to the resequencing buffer is described by $\tilde{\mathbb{F}}(s,i,j,k)\mathcal{H}$.

Similarly, let $\tilde{\mathbb{W}}(s,i,j)$ be the matrix (according to the initial and final phases of the MAPs $(\mathbf{A_0}, \mathbf{A_1})$, $(\mathbf{S_0}, \mathbf{S_1})$ and $(\mathbf{H_0}, \mathbf{H_1})$) Laplace–Stieltjes transform of the waiting time of a customer which starts its life in the resequencing buffer in LIFO position j, when the number of customers in the regular buffer is i. The LIFO position is $j = 1$ for the customer which arrived most recently to the resequencing buffer and all existing LIFO positions are increased by one when a new customer arrives to the resequencing buffer. For $i \geq 0, j \geq 1$, we have

$$\tilde{\mathbb{W}}(s,i,j) = I_{\{i>0\}}\mathcal{L}(s)\mathcal{S}\tilde{\mathbb{W}}(s,i-1,j) + I_{\{i=0\}}\mathcal{L}(s)\mathcal{S}\tilde{\mathbb{W}}(s,0,j-1) + \quad (3)$$
$$I_{\{i>0\}}\mathcal{L}(s)\mathcal{H}\tilde{\mathbb{W}}(s,i-1,j+1) + I_{\{i=0\}}\mathcal{L}(s)\mathcal{H}\tilde{\mathbb{W}}(s,0,j) + \mathcal{L}(s)\mathcal{A}\tilde{\mathbb{W}}(s,i+1,j),$$

where $\tilde{\mathbb{W}}(s,0,0) = I$. The solution of $\tilde{\mathbb{W}}(s,i,j)$ is not trivial. We search for the solution in product form $\tilde{\mathbb{W}}(s,i,j) = \widetilde{\mathbf{G}}(s)^i\widehat{\mathbf{G}}(s)^j$. The product from solution satisfies (3) for $i \geq 0, j \geq 1$ if

$$s\widehat{\mathbf{G}}(s) - \mathcal{L}\widehat{\mathbf{G}}(s) = \mathcal{S} + \mathcal{H}\widehat{\mathbf{G}}(s) + \mathcal{A}\widetilde{\mathbf{G}}(s)\widehat{\mathbf{G}}(s), \quad (4)$$
$$s\widetilde{\mathbf{G}}(s) - \mathcal{L}\widetilde{\mathbf{G}}(s) = \mathcal{S} + \mathcal{H}\widehat{\mathbf{G}}(s) + \mathcal{A}\widetilde{\mathbf{G}}^2(s), \quad (5)$$

which are obtained from (3) by substituting the product form at $i+1 = j = 1$ and $i = j+1 = 1$. The equations (4) and (5) form a pair of coupled matrix quadratic equations whose minimal non-negative solution can be computed by efficient iterative numerical methods, but do not exhibit closed form result. A simple linearly convergent iterative method is as follows.

4.2 Iterative solution of the coupled matrix equations

The system of equations (4)-(5) can be re-written as

$$\widehat{\mathbf{G}}(s) = \left(sI - \mathcal{L} - \mathcal{H} - \mathcal{A}\widetilde{\mathbf{G}}(s)\right)^{-1}\mathcal{S}, \quad (6)$$
$$\widetilde{\mathbf{G}}(s) = \left(sI - \mathcal{L} - \mathcal{A}\widetilde{\mathbf{G}}(s)\right)^{-1}\left(\mathcal{S} + \mathcal{H}\widehat{\mathbf{G}}(s)\right). \quad (7)$$

In order to find $\widehat{\mathbf{G}}(s)$ and $\widetilde{\mathbf{G}}(s)$ for the given value of s, we start with $\widetilde{\mathbf{G}}_0(s) = 0$. Then for $i = 1, 2, \ldots$ the next two iterative steps are performed until the convergence is reached

$$\widehat{\mathbf{G}}_i(s) = \Big(sI - \mathcal{L} - \mathcal{H} - \mathcal{A}\widetilde{\mathbf{G}}_{i-1}(s)\Big)^{-1} \mathcal{S}, \tag{8}$$

$$\widetilde{\mathbf{G}}_i(s) = \Big(sI - \mathcal{L} - \mathcal{A}\widetilde{\mathbf{G}}_{i-1}(s)\Big)^{-1} \Big(\mathcal{S} + \mathcal{H}\widehat{\mathbf{G}}_i(s)\Big). \tag{9}$$

4.3 Delay analysis of customer served from the resequencing buffer

Based on the previously computed matrix Laplace–Stieltjes transforms, the waiting time of the customer which enters server from the resequencing buffer can be computed as

$$\begin{aligned}\omega_{\mathrm{L}}(s) &= E(e^{-sW} I_{\{\text{served from resequencing buffer}\}}) \\ &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \tilde{\pi}_{ij} \sum_{\ell=0}^{i-1}\sum_{k=0}^{\infty} \tilde{\mathbb{F}}(s,i-1,\ell,k)\mathcal{H}\widetilde{\mathbf{G}}(s)^k\widehat{\mathbf{G}}(s)\mathbf{1} \\ &= \frac{1}{\lambda}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \pi_{i,j}\mathcal{A} \sum_{\ell=0}^{i}\sum_{k=0}^{\infty} \tilde{\mathbb{F}}(s,i,\ell,k)\mathcal{H}\widetilde{\mathbf{G}}(s)^k\widehat{\mathbf{G}}(s)\mathbf{1}\ .\end{aligned} \tag{10}$$

The main part of the analysis of $\omega_{\mathrm{L}}(s)$ is deferred to the next section. But in the course of the subsequent derivations we will make use of several quantities which are better introduced by considering terms of $\omega_{\mathrm{L}}(s)$ with $i = 0$. We represent $\omega_{\mathrm{L}}(s)$ as

$$\begin{aligned}\omega_{\mathrm{L}}(s) =& \omega_{\mathrm{L}}^{i>0}(s) + \omega_{\mathrm{L}}^{i=0}(s) \\ =& \frac{1}{\lambda}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \pi_{i,j}\mathcal{A} \sum_{\ell=0}^{i}\sum_{k=0}^{\infty} \tilde{\mathbb{F}}(s,i,\ell,k)\mathcal{H}\widetilde{\mathbf{G}}(s)^k\widehat{\mathbf{G}}(s)\mathbf{1} \\ &+ \frac{1}{\lambda}\sum_{j=0}^{\infty} \pi_{0,j}\mathcal{A} \sum_{k=0}^{\infty} \underbrace{\tilde{\mathbb{F}}(s,0,0,k)}_{(\mathcal{L}(s)\mathcal{A})^k\mathcal{L}(s)}\mathcal{H}\widetilde{\mathbf{G}}(s)^k\widehat{\mathbf{G}}(s)\mathbf{1}.\end{aligned} \tag{11}$$

In what follows we will need the expressions for probability generating functions $\hat{\pi}_0(z) = \sum_{m=0}^{\infty} \pi_{0,m} z^m$ and $\hat{\pi}_i(z) = \sum_{j=0}^{\infty} \pi_{ij} z^j$, $i \geq 1$, which were obtained in [10]:

$$\hat{\pi}_0(z) = \pi_{0,0}(\mathcal{L}' - \mathcal{L}'' + \frac{1}{z}\mathcal{S})(\mathcal{A}\overline{\mathbf{G}}(z) + \mathcal{L}' + \frac{1}{z}\mathcal{S})^{-1}, \tag{12}$$

$$\hat{\pi}_i(z) = \hat{\pi}_{i-1}(z)\overline{\mathbf{R}}(z),\ i \geq 1, \tag{13}$$

where $\overline{\mathbf{R}}(z)$ is the minimal non-negative solution of the quadratic matrix equation

$$\mathcal{A} + \overline{\mathbf{R}}(z)\mathcal{L} + \overline{\mathbf{R}}^2(z)\,(z\mathcal{H} + \mathcal{S}) = \mathbf{0}. \tag{14}$$

Derivation of $\omega_{\mathrm{L}}^{i=0}(s)$

The methodology from [10], which we apply here in order to obtain the stationary waiting time distribution, is based on the technique which can be referred to as the Kronecker expansion (see [1, 11]). It is based on the identity $vec(ABC) = (C^T \otimes A)vec(B)$. In this identity vec denotes the column stacking vector operator, which transforms a matrix of size $n \times m$ into a vector of size $nm \times 1$. In all further derivations we will make extensive use of the Kronecker expansion, which will appear in seemingly different but, in fact, equal forms (for example, $vec(AB) = (I^T \otimes A)vec(B) = (B^T \otimes A)vec(I) = (B^T \otimes I)vec(A)$).

Coming back to $\omega_{\mathrm{L}}^{i=0}(s)$ and using the identity $vec(ABC) = (C^T \otimes A)vec(B)$, one obtains

$$\begin{aligned}\omega_{\mathrm{L}}^{i=0}(s) &= \frac{1}{\lambda}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\pi_{0,j}\mathcal{A}(\mathcal{L}(s)\mathcal{A})^k\mathcal{L}(s)\mathcal{H}\widetilde{\mathbf{G}}(s)^k\widehat{\mathbf{G}}(s)\mathbf{1}\\ &= \frac{1}{\lambda}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left(\mathbf{1}^T\widehat{\mathbf{G}}(s)^T\widetilde{\mathbf{G}}(s)^{k^T}\otimes\pi_{0,j}\mathcal{A}(\mathcal{L}(s)\mathcal{A})^k\right)vec(\mathcal{L}(s)\mathcal{H})\end{aligned}$$

and

$$\begin{aligned}\omega_{\mathrm{L}}^{i=0}(s) &= \frac{1}{\lambda}\left(\mathbf{1}^T\widehat{\mathbf{G}}(s)^T\otimes 1\right)\\ &\qquad\cdot\underbrace{\sum_{j=0}^{\infty}(I\otimes\pi_{0,j})}_{I\otimes\hat{\pi}_0(1)}(I\otimes\mathcal{A})\underbrace{\sum_{k=0}^{\infty}\left(\widetilde{\mathbf{G}}(s)^{k^T}\otimes(\mathcal{L}(s)\mathcal{A})^k\right)}_{(I-\widetilde{\mathbf{G}}(s)^T\otimes\mathcal{L}(s)\mathcal{A})^{-1}}vec(\mathcal{L}(s)\mathcal{H})\\ &= \frac{1}{\lambda}\left(\mathbf{1}^T\widehat{\mathbf{G}}(s)^T\otimes 1\right)(I\otimes\hat{\pi}_0(1))(I\otimes\mathcal{A})\left(I-\widetilde{\mathbf{G}}(s)^T\otimes\mathcal{L}(s)\mathcal{A}\right)^{-1}\\ &\qquad\qquad\cdot vec(\mathcal{L}(s)\mathcal{H})\\ &= \frac{1}{\lambda}\left(\mathbf{1}^T\widehat{\mathbf{G}}(s)^T\otimes\hat{\pi}_0(1)\mathcal{A}\right)\left(I-\widetilde{\mathbf{G}}(s)^T\otimes\mathcal{L}(s)\mathcal{A}\right)^{-1}vec(\mathcal{L}(s)\mathcal{H}).\end{aligned}$$

Derivation of $\omega_{\mathrm{L}}^{i>0}(s)$

Having found the expression for $\omega_{\mathrm{L}}^{i=0}(s)$ the last unknown quantity in $\omega_{\mathrm{L}}(s)$ is $\omega_{\mathrm{L}}^{i>0}(s)$. In the following we split expression (10) for $\omega_{\mathrm{L}}^{i>0}(s)$ into the following two terms:

$$\omega_{\mathrm{L}}^{i>0}(s) = \omega_{\mathrm{L}}^{k=0}(s) + \omega_{\mathrm{L}}^{k>0}(s),$$

where $\omega_{\mathrm{L}}^{k=0}(s)$ includes only terms of $\omega_{\mathrm{L}}^{i>0}(s)$ with $k = 0$ and $\omega_{\mathrm{L}}^{k>0}(s)$ all other terms. Further we obtain the expressions for each of them individually.

Derivation of $\omega_{\mathrm{L}}^{k=0}(s)$

In order to compute $\omega_{\mathrm{L}}^{k=0}(s)$ we perform the Kronecker expansion and apply the relation $vec(ABC) = (C^T \otimes A)vec(B)$ two times. We have

$$\omega_{\mathrm{L}}^{k=0}(s) = \frac{1}{\lambda}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\pi_{i,j}\mathcal{A}\underbrace{\sum_{\ell=0}^{i}\tilde{\mathbb{F}}(s,i,\ell,0)\mathcal{H}}_{\widehat{\mathcal{F}}_{k=0}(s,i)}\widehat{\mathbf{G}}(s)\mathbf{1}$$

$$=\frac{1}{\lambda}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\pi_{i,j}\mathcal{A}\widehat{\mathcal{F}}_{k=0}(s,i)\widehat{\mathbf{G}}(s)\mathbf{1}=\frac{1}{\lambda}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\Big(\mathbf{1}^T\widehat{\mathbf{G}}(s)^T\otimes\pi_{i,j}\mathcal{A}\Big)vec(\widehat{\mathcal{F}}_{k=0}(s,i))$$

$$= \frac{1}{\lambda}\Big(\mathbf{1}^T\widehat{\mathbf{G}}(s)^T\otimes 1\Big)\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\Big(I\otimes\pi_{i,j}\Big)\Big(I\otimes\mathcal{A}\Big)vec(\widehat{\mathcal{F}}_{k=0}(s,i))$$

$$= \frac{1}{\lambda}\Big(\mathbf{1}^T\widehat{\mathbf{G}}(s)^T\otimes 1\Big)\underbrace{\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\Big[vec(\widehat{\mathcal{F}}_{k=0}(s,i))^T\otimes\Big(I\otimes\pi_{i,j}\Big)\Big]}_{\mathbf{M}(s)}vec\Big(I\otimes\mathcal{A}\Big).$$

Here the only unknown quantity is $\mathbf{M}(s)$. We will show now that the matrix $\mathbf{M}(s)$ can be expressed in the form $\mathbf{M}(s) = M_1(s) + \mathbf{M}(s)M_2(s)$, where $M_1(s)$ and $M_2(s)$ are known matrices. Thus for any given s it can be computed as $\mathbf{M}(s) = (I - M_2(s))^{-1}M_1(s)$. Summing over $j \geq 0$ (remembering (13)) and extracting the term with $i = 1$, one can write

$$\begin{aligned}\mathbf{M}(s) &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\Big[vec(\widehat{\mathcal{F}}_{k=0}(s,i))^T\otimes\Big(I\otimes\pi_{i,j}\Big)\Big]\\ &= \sum_{i=1}^{\infty}\Big[vec(\widehat{\mathcal{F}}_{k=0}(s,i))^T\otimes\Big(I\otimes\hat{\pi}_i(1)\Big)\Big]\\ &= vec(\widehat{\mathcal{F}}_{k=0}(s,1))^T\otimes\Big(I\otimes\hat{\pi}_1(1)\Big)\\ &\quad+\sum_{i=2}^{\infty}vec(\widehat{\mathcal{F}}_{k=0}(s,i))^T\otimes\Big(I\otimes\hat{\pi}_i(1)\Big).\end{aligned}\tag{15}$$

In order to obtain the expression for the only unknown quantity $vec(\widehat{\mathcal{F}}_{k=0}(s,i))^T$ we revisit the definition of $\widehat{\mathcal{F}}_{k=0}(s,i)$. By applying (2) when $i>0$, we obtain

$$\begin{aligned}
&\widehat{\mathcal{F}}_{k=0}(s,i) = \sum_{\ell=0}^{i} \tilde{\mathbb{F}}(s,i,\ell,0)\mathcal{H} \\
&= \sum_{\ell=1}^{i-1} \tilde{\mathbb{F}}(s,i,\ell,0)\mathcal{H} + \tilde{\mathbb{F}}(s,i,0,0)\mathcal{H} + \tilde{\mathbb{F}}(s,i,i,0)\mathcal{H} \\
&= \sum_{\ell=1}^{i-1} \mathcal{L}(s)\mathcal{S}\tilde{\mathbb{F}}(s,i-1,\ell,0)\mathcal{H} + \sum_{\ell=1}^{i-1} \mathcal{L}(s)\mathcal{H}\tilde{\mathbb{F}}(s,i-1,\ell-1,0)\mathcal{H} \\
&+ \mathcal{L}(s)\mathcal{S}\tilde{\mathbb{F}}(s,i-1,0,0)\mathcal{H} + \mathcal{L}(s)\mathcal{H}\tilde{\mathbb{F}}(s,i-1,i-1,0)\mathcal{H} \\
&= \mathcal{L}(s)\mathcal{S} \sum_{\ell=0}^{i-1} \tilde{\mathbb{F}}(s,i-1,\ell,0)\mathcal{H} + \mathcal{L}(s)\mathcal{H} \sum_{\ell=0}^{i-1} \tilde{\mathbb{F}}(s,i-1,\ell,0)\mathcal{H} \\
&= \mathcal{L}(s)\left(\mathcal{S}+\mathcal{H}\right) \sum_{\ell=0}^{i-1} \tilde{\mathbb{F}}(s,i-1,\ell,0)\mathcal{H} \ ,
\end{aligned}$$

or, equivalently, in terms of $\widehat{\mathcal{F}}_{k=0}(s,i)$:

$$\widehat{\mathcal{F}}_{k=0}(s,i) = \mathcal{L}(s)\left(\mathcal{S}+\mathcal{H}\right)\widehat{\mathcal{F}}_{k=0}(s,i-1) \ , \ i \geq 1. \tag{16}$$

By applying vec operator to (16) one finds the following expression for $vec(\widehat{\mathcal{F}}_{k=0}(s,i))^T$, $i \geq 1$:

$$vec(\widehat{\mathcal{F}}_{k=0}(s,i))^T = vec(\widehat{\mathcal{F}}_{k=0}(s,i-1))^T \left[I \otimes \mathcal{L}(s)\left(\mathcal{S}+\mathcal{H}\right) \right]^T, i \geq 1. \tag{17}$$

By substituting the (17) into (15) and remembering that according to (13) $\hat{\pi}_i(1) = \hat{\pi}_{i-1}(1)\overline{\mathbf{R}}(1)$, we find the sought-for representation for $\mathbf{M}(s)$:

$$\begin{aligned}
\mathbf{M}(s) =& vec(\widehat{\mathcal{F}}_{k=0}(s,1))^T \otimes \Big(I \otimes \hat{\pi}_1(1) \Big) + \sum_{i=1}^{\infty} \left[vec(\widehat{\mathcal{F}}_{k=0}(s,i))^T \otimes \Big(I \otimes \hat{\pi}_i(1) \Big) \right] \\
=& \left(vec(\underbrace{\widehat{\mathcal{F}}_{k=0}(s,0)}_{\mathcal{L}(s)\mathcal{H}})^T \left[I \otimes \mathcal{L}(s)\left(\mathcal{S}+\mathcal{H}\right) \right]^T \right) \otimes \Big(I \otimes \hat{\pi}_0(1)\overline{\mathbf{R}}(1) \Big) \\
&+ \sum_{i=2}^{\infty} \left[vec(\widehat{\mathcal{F}}_{k=0}(s,i-1))^T \left[I \otimes \mathcal{L}(s)\left(\mathcal{S}+\mathcal{H}\right) \right]^T \otimes \Big(I \otimes \hat{\pi}_i(1) \Big) \right]
\end{aligned}$$

$$
\begin{aligned}
&= \left(vec(\mathcal{L}(s)\mathcal{H})^T \left[I \otimes \mathcal{L}(s)\,(\mathcal{S}+\mathcal{H}) \right]^T \right) \otimes \left(I \otimes \hat{\pi}_0(1)\overline{\mathbf{R}}(1) \right) \\
&+ \sum_{i=1}^{\infty} \left[vec(\widehat{\mathcal{F}}_{k=0}(s,i))^T \left[I \otimes \mathcal{L}(s)\,(\mathcal{S}+\mathcal{H}) \right]^T \otimes \left(I \otimes \hat{\pi}_i(1) \right) \left(I \otimes \overline{\mathbf{R}}(1) \right) \right] \\
&= \underbrace{\left(vec(\mathcal{L}(s)\mathcal{H})^T \left[I \otimes \mathcal{L}(s)\,(\mathcal{S}+\mathcal{H}) \right]^T \right)}_{M_1(s)} \otimes \left(I \otimes \hat{\pi}_0(1)\overline{\mathbf{R}}(1) \right) \\
&+ \mathbf{M}(s) \underbrace{\left[\left(I \otimes \mathcal{L}(s)\,(\mathcal{S}+\mathcal{H}) \right)^T \otimes \left(I \otimes \overline{\mathbf{R}}(1) \right) \right]}_{M_2(s)} .
\end{aligned}
$$

Derivation of $\omega_{\mathrm{L}}^{k>0}(s)$

Now we tackle the most complex case – the analysis of $\omega_{\mathrm{L}}^{k>0}(s)$. For $\omega_{\mathrm{L}}^{k>0}(s)$ the Kronecker expansion has to be applied multiple times. At first we recall that the definition of $\omega_{\mathrm{L}}^{k>0}(s)$ is

$$
\omega_{\mathrm{L}}^{k>0}(s) = \frac{1}{\lambda} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \pi_{i,j} \mathcal{A} \underbrace{\sum_{\ell=0}^{i} \sum_{k=1}^{\infty} \tilde{\mathbb{F}}(s,i,\ell,k) \mathcal{H} \widetilde{\mathbf{G}}(s)^k}_{\mathcal{F}(s,i)} \widehat{\mathbf{G}}(s) \mathbf{1}.
$$

Let us now consider term $\mathcal{F}(s,i)$. Applying *vec* operator to $\mathcal{F}(s,i)$ according to the following Kronecker expansion

$$
\begin{aligned}
&vec(ABCD) = (D^T \otimes A) vec(BC) = (vec(BC)^T \otimes (D^T \otimes A)) vec(I) \\
&= (vec(I)^T \otimes I \otimes I)(C \otimes B^T \otimes D^T \otimes A) vec(I),
\end{aligned}
$$

one gets

$$
\begin{aligned}
&vec(\mathcal{F}(s,i)) \\
&= (vec(I)^T \otimes I \otimes I) \underbrace{\sum_{\ell=0}^{i} \sum_{k=1}^{\infty} \left(\widetilde{\mathbf{G}}(s)^k \otimes \mathcal{H}^T \otimes I^T \otimes \tilde{\mathbb{F}}(s,i,\ell,k) \right)}_{\mathcal{F}^{\otimes}(s,i)} vec(I) \\
&= (vec(I)^T \otimes I \otimes I) \mathcal{F}^{\otimes}(s,i) vec(I).
\end{aligned}
$$

By considering the expression for $\mathcal{F}(s,i)$ and using (2), when $i>0$ and $k>0$, we obtain

$$\begin{aligned}\mathcal{F}(s,i) &= \sum_{\ell=0}^{i}\sum_{k=1}^{\infty}\tilde{\mathbb{F}}(s,i,\ell,k)\mathcal{H}\widetilde{\mathbf{G}}(s)^k \\ &= \sum_{\ell=0}^{i-1}\sum_{k=1}^{\infty}\mathcal{L}(s)\mathcal{S}\tilde{\mathbb{F}}(s,i-1,\ell,k)\mathcal{H}\widetilde{\mathbf{G}}(s)^k \\ &+ \sum_{\ell=0}^{i-1}\sum_{k=1}^{\infty}\mathcal{L}(s)\mathcal{H}\tilde{\mathbb{F}}(s,i-1,\ell,k)\mathcal{H}\widetilde{\mathbf{G}}(s)^k \\ &+ \sum_{\ell=0}^{i}\sum_{k=0}^{\infty}\mathcal{L}(s)\mathcal{A}\tilde{\mathbb{F}}(s,i,\ell,k)\mathcal{H}\widetilde{\mathbf{G}}(s)^{k+1}\ . \end{aligned} \tag{18}$$

Having such expression for $\mathcal{F}(s,i)$m one can write out relation for the term $\mathcal{F}^{\otimes}(s,i)$ in the following form:

$$\begin{aligned}&\mathcal{F}^{\otimes}(s,i) \\ &= \underbrace{\Big[\Big(I\otimes I\otimes I\otimes\mathcal{L}(s)\mathcal{S}\Big)+\Big(I\otimes I\otimes I^T\otimes\mathcal{L}(s)\mathcal{H}\Big)\Big]}_{\mathbf{L}(s)}\mathcal{F}^{\otimes}(s,i-1) \\ &+ \underbrace{\Big(\widetilde{\mathbf{G}}(s)\otimes I\otimes I\otimes\mathcal{L}(s)\mathcal{A}\Big)}_{\mathbf{K}(s)}\Big(\mathcal{F}^{\otimes}(s,i)+\widehat{\mathcal{F}}^{\otimes}_{k=0}(s,i)\Big) \\ &= [I-\mathbf{K}(s)]^{-1}[\mathbf{L}(s)\mathcal{F}^{\otimes}(s,i-1)+\mathbf{K}(s)\widehat{\mathcal{F}}^{\otimes}_{k=0}(s,i)]\ ,\end{aligned} \tag{19}$$

where we have introduced the notation

$$\widehat{\mathcal{F}}^{\otimes}_{k=0}(s,i)=\sum_{\ell=0}^{i}\Big(I\otimes\mathcal{H}^T\otimes I^T\otimes\tilde{\mathbb{F}}(s,i,\ell,0)\Big),\ i\ge 0.$$

From (2) it follows that

$$\begin{aligned}\mathcal{F}^{\otimes}(s,0) &= \sum_{k=1}^{\infty}\Big(\widetilde{\mathbf{G}}(s)^k\otimes\mathcal{H}^T\otimes I\otimes(\mathcal{L}(s)\mathcal{A})^k\mathcal{L}(s)\Big) \\ &= \Big[I-\Big(\widetilde{\mathbf{G}}(s)\otimes I\otimes I\otimes\mathcal{L}(s)\mathcal{A}\Big)\Big]^{-1}\Big(\widetilde{\mathbf{G}}(s)\otimes\mathcal{H}^T\otimes I\otimes\mathcal{L}(s)\mathcal{A}\mathcal{L}(s)\Big)\ ,\end{aligned}$$

and $\widehat{\mathcal{F}}^{\otimes}_{k=0}(s,0)=I\otimes\mathcal{H}^T\otimes I^T\otimes\mathcal{L}(s)$. For $i\ge 1$ from (16) we have

$$\widehat{\mathcal{F}}^{\otimes}_{k=0}(s,i)=\mathbf{L}(s)\ \widehat{\mathcal{F}}^{\otimes}_{k=0}(s,i-1)\ ,i\ge 1.$$

Now we go back to $\omega_{\mathrm{L}}^{k>0}(s)$ and apply *vec* operator multiple times in the following way:

$$
\begin{aligned}
\omega_{\mathrm{L}}^{k>0}(s) &= \frac{1}{\lambda}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\pi_{i,j}\mathcal{A}\mathcal{F}(s,i)\widehat{\mathbf{G}}(s)\mathbf{1} \\
&= \frac{1}{\lambda}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left(\mathbf{1}^T\widehat{\mathbf{G}}(s)^T \otimes \pi_{i,j}\mathcal{A}\right) vec\Big(\mathcal{F}(s,i)\Big) \\
&= \frac{1}{\lambda}\left(\mathbf{1}^T\widehat{\mathbf{G}}(s)^T \otimes 1\right)\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\Big(I\otimes\pi_{i,j}\Big)\Big(I\otimes\mathcal{A}\Big) vec\Big(\mathcal{F}(s,i)\Big) \\
&= \frac{1}{\lambda}\left(\mathbf{1}^T\widehat{\mathbf{G}}(s)^T \otimes 1\right)\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\left[vec\Big(\mathcal{F}(s,i)\Big)^T \otimes \Big(I\otimes\pi_{i,j}\Big)\right] vec\Big(I\otimes\mathcal{A}\Big) \\
&= \frac{1}{\lambda}\left(\mathbf{1}^T\widehat{\mathbf{G}}(s)^T \otimes 1\right)\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\Big[vec(I)^T\mathcal{F}^{\otimes}(s,i)^T(vec(I)^T\otimes I\otimes I)^T \\
&\qquad \otimes\Big(I\otimes\pi_{i,j}\Big)\Big] vec\Big(I\otimes\mathcal{A}\Big) \\
&= \frac{1}{\lambda}\left(\mathbf{1}^T\widehat{\mathbf{G}}(s)^T \otimes 1\right)\Big[vec(I)^T\otimes I\Big]\underbrace{\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\Big[\mathcal{F}^{\otimes}(s,i)^T\otimes\Big(I\otimes\pi_{i,j}\Big)\Big]}_{\mathbf{N}(s)} \\
&\qquad \cdot\Big[(vec(I)^T\otimes I\otimes I)^T\otimes I\Big] vec\Big(I\otimes\mathcal{A}\Big)\ .
\end{aligned}
$$

The only unknown quantity in the expression for $\omega_{\mathrm{L}}^{k>0}(s)$ is $\mathbf{N}(s)$. It can be found from (19) in the manner similar to $\mathbf{M}(s)$. We have

$$
\begin{aligned}
\mathbf{N}(s) &= \Big[\mathcal{F}^{\otimes}(s,1)^T\otimes\underbrace{\sum_{j=0}^{\infty}\Big(I\otimes\pi_{1,j}\Big)}_{I\otimes\hat{\pi}_0(1)\overline{\mathbf{R}}(1)}\Big] + \sum_{i=2}^{\infty}\sum_{j=0}^{\infty}\Big[\mathcal{F}^{\otimes}(s,i)^T\otimes\Big(I\otimes\pi_{i,j}\Big)\Big] \\
&= \Big[\mathcal{F}^{\otimes}(s,1)^T\otimes\big(I\otimes\hat{\pi}_0(1)\overline{\mathbf{R}}(1)\big)\Big] \\
&\qquad + \underbrace{\sum_{i=2}^{\infty}\Big[\widehat{\mathcal{F}}^{\otimes}_{k=0}(s,i)^T\otimes\Big(I\otimes\hat{\pi}_i(1)\Big)\Big]}_{\mathbf{Z}(s)}\Big(\mathbf{K}(s)^T[I-\mathbf{K}(s)]^{-1^T}\otimes I\Big) \\
&\qquad + \underbrace{\sum_{i=2}^{\infty}\Big[\mathcal{F}^{\otimes}(s,i-1)^T\mathbf{L}(s)^T[I-\mathbf{K}(s)]^{-1^T}\otimes\Big(I\otimes\hat{\pi}_{i-1}(1)\Big)\Big(I\otimes\overline{\mathbf{R}}(1)\Big)\Big]}_{\mathbf{N}(s)\left(\mathbf{L}(s)^T[I-\mathbf{K}(s)]^{-1^T}\otimes\left[I\otimes\overline{\mathbf{R}}(1)\right]\right)}\ .
\end{aligned}
$$

For $\mathbf{Z}(s)$, using properties of the Kronecker product, one obtains the following relation:

$$\begin{aligned}
\mathbf{Z}(s) &= \sum_{i=2}^{\infty}\left[\widehat{\mathcal{F}}^{\otimes}_{k=0}(s,i)^T \otimes \Big(I \otimes \hat{\pi}_i(1)\Big)\right] \\
&= \sum_{i=2}^{\infty}\left[\mathcal{F}^{\otimes}_{k=0}(s,i-1)^T \mathbf{L}(s)^T \otimes \Big(I \otimes \hat{\pi}_i(1)\Big)\right] \\
&= \sum_{i=1}^{\infty}\left[\mathcal{F}^{\otimes}_{k=0}(s,i)^T \mathbf{L}(s)^T \otimes \Big(I \otimes \hat{\pi}_i(1)\overline{\mathbf{R}}(1)\Big)\right] \\
&= \sum_{i=1}^{\infty}\left[\widehat{\mathcal{F}}^{\otimes}_{k=0}(s,i)^T \mathbf{L}(s)^T \otimes \Big(I \otimes \hat{\pi}_i(1)\Big)\left(I \otimes \overline{\mathbf{R}}(1)\right)\right] \\
&= \sum_{i=1}^{\infty}\left[\widehat{\mathcal{F}}^{\otimes}_{k=0}(s,i)^T \otimes \Big(I \otimes \hat{\pi}_i(1)\Big)\right]\Big(\mathbf{L}(s)^T \otimes \left(I \otimes \overline{\mathbf{R}}(1)\right)\Big) \\
&= \left[\Big(\widehat{\mathcal{F}}^{\otimes}_{k=0}(s,1)^T \otimes \Big(I \otimes \hat{\pi}_0(1)\overline{\mathbf{R}}(1)\Big)\Big) + \mathbf{Z}(s)\right]\Big(\mathbf{L}(s)^T \otimes \left(I \otimes \overline{\mathbf{R}}(1)\right)\Big).
\end{aligned}$$

The latter relation allows computation of $\mathbf{Z}(s)$ and subsequently $\mathbf{N}(s)$ and $\omega_{\mathrm{L}}^{k>0}(s)$. Thus the expression for $\omega_{\mathrm{L}}(s)$ is obtained.

5 Numerical example

In order to give a more complete picture of how the service and the resequencing policies influence the waiting time of an arbitrary customer, we present a simple numerical example. Due to the Little's law the mean waiting times of arbitrary customer under the HOQ-FIFO-FIFO and HOQ-FIFO-LIFO policies coincide. Thus we dwell on comparison of the standard deviation of the waiting time.

Two use cases are considered. The first one is taken from [10], where the regular customers and resequencing signals arrive according to Poisson processes with rates λ and γ, respectively. The service process has the phase-type distribution with the representation:

$$\boldsymbol{\beta} = (0.5, 0.5)\,, \quad \mathbf{B} = \begin{pmatrix} -4 & 2 \\ 1 & -4 \end{pmatrix}, \text{ from which } \mathbf{S_0} = \begin{pmatrix} -4 & 2 \\ 1 & -4 \end{pmatrix}, \quad \mathbf{S_1} = \begin{pmatrix} 1 & 1 \\ 1.5 & 1.5 \end{pmatrix}.$$

The service rate is $\mu = -1/(\boldsymbol{\beta}\mathbf{B}^{-1}\mathbf{1}) = 2.5$ and consequently $\lambda = 2.5\rho$, where ρ and γ are the parameters of the example. As the second use case we take the same service process $(\mathbf{S_0}, \mathbf{S_1})$, but the arrival process of regular and resequencing customers are characterized by

$$\mathbf{A_0} = \begin{pmatrix} -5 & 1.5 \\ 2 & -3 \end{pmatrix}, \mathbf{A_1} = \begin{pmatrix} 3.5p & 3.5(1-p) \\ p & (1-p) \end{pmatrix}, \mathbf{H_0} = \begin{pmatrix} -7 & 0 \\ 0 & -7q \end{pmatrix}, \mathbf{H_1} = \begin{pmatrix} 7q & 7(1-q) \\ 7q^2 & 7q(1-q) \end{pmatrix}.$$

Indeed they mean order 2 phase-type renewal processes with mean intensity $\lambda = \frac{120}{70-25p}$ $(\rho = \frac{240}{350-125p})$ and $\gamma = \frac{7q}{1-q+q^2}$. By tuning the values of p and q we

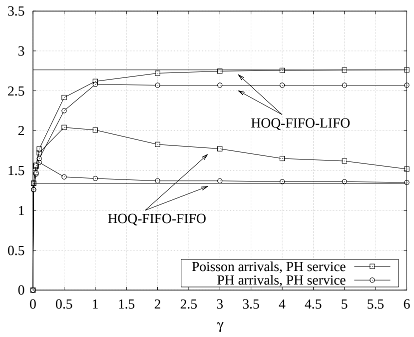

Fig. 3. $\rho = 0.72$

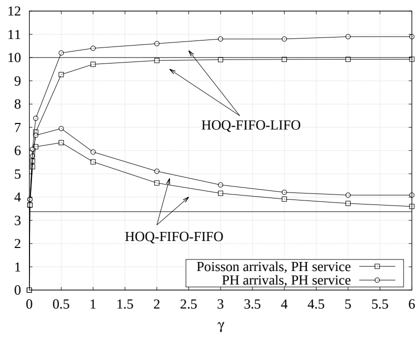

Fig. 4. $\rho = 0.88$

Standard deviation of customer's waiting time as function of resequencing intensity (γ) for two different load (ρ) levels, two different policies and two use cases.

can set the load and the resequencing rate. In Fig. 3 and Fig. 4 one can see the graphs of the standard deviation of the waiting times as function of resequencing rate γ for two arbitrary values of load $\rho = 0.72$ and $\rho = 0.88$ and both use cases.

When γ is low the second order characteristics of the waiting time are almost the same. As the resequencing rate γ grows, the difference in the behaviour of the both curves becomes more significant. This difference comes from the following fact. As the resequencing rate γ grows almost all customers get resequenced. Thus under the HOQ-FIFO-FIFO policy they are served according to FIFO and under the HOQ-FIFO-LIFO policy – according to LIFO. Intuitively in the latter case the variance of the waiting time is bigger because LIFO policy can generate some extremely high response times. Indeed we may have to wait for a very long time in order to take care of the first arrival to the resequencing buffer.

Finally, as γ grows the standard deviations of waiting time under the HOQ-FIFO-FIFO and HOQ-FIFO-LIFO policies tend to the standard deviations of the waiting time (horizontal lines in the figures for the Poisson arrival case) in the standard $M/PH/1$ FIFO and $M/PH/1$ LIFO queues respectively. At $\gamma = 0$ we also have the case of pure FIFO queue.

6 Conclusion

The delay analysis of the HOQ-FIFO-LIFO policy shows that the majority of the analysis steps (recursive evolution equation like description of properly chosen performance measures, Kronecker expansion based treatment of non-commuting matrices, describing the relation of infinite summations from 0 to ∞ with the one from 1 to ∞) remain applicable, but also new analysis elements are required. In particular, the analysis of the HOQ-FIFO-LIFO service policy requires the solution of a coupled quadratic matrix equation, which was separated in the

HOQ-FIFO-FIFO case. In spite, the computational complexity of the HOQ-FIFO-LIFO case is not higher than the one of the HOQ-FIFO-FIFO case, because the solution of the coupled equation is comparable with the solution of two separate ones.

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