# Analysis of $BMAP/G/1$ vacation model of non-$M/G/1$-type*

Zsolt Saffer and Miklós Telek

Department of Telecommunications,
Technical University of Budapest,
1521 Budapest Hungary
{safferzs,telek}@hit.bme.hu

**Abstract.** In this paper we present the analysis of $BMAP/G/1$ vacation models of non-$M/G/1$-type in a general framework. We provide new service discipline independent formulas for the vector generating function (GF) of the stationary number of customers and for its mean, both in terms of quantities at the start of vacation.
We present new results for vacation models with gated and G-limited disciplines. For both models discipline specific systems of equations are setup. Their numerical solution are used to compute the required quantities at the start of vacation.
Keywords: queueing theory, vacation model, BMAP, M/G/1-type process.

## 1 Introduction

Queueing models with server vacation have been studied in the last decades due to their general modeling capability. In these models the server occasionally takes a vacation period, in which no customer is served. For details on vacation models and for their analysis with Poisson arrival process we refer to the comprehensive survey of Doshi [1] and to the excellent book of Takagi [2].

The batch Markovian arrival process ($BMAP$) introduced by Lucantoni [3] enables more realistic and more accurate traffic modeling than the (batch) Poisson process. Consequently analysis of queueing models with $BMAP$ attracted a great attention. The vast majority of the analyzed $BMAP/G/1$ queueing models exploit the underlying $M/G/1$-type structure of the model, i.e., that the embedded Markov chain at the customer departure epochs is of $M/G/1$-type [4] in which the block size in the transition probability matrix equals to the number of phases of the $BMAP$. Hence most of the analysis of $BMAP/G/1$ vacation models are based on the standard matrix analytic-method pioneered by Neuts [5] and further extended by many others (see e.g., [6]).

Chang and Takine [7] applied the factorization property (presented by Chang et al. [8]) to get analytical results for queueing models of $M/G/1$-type with or

* This work is supported by the NAPA-WINE FP7-ICT (http://www.napa-wine.eu) and the OTKA K61709 projects.

without vacations using exhaustive discipline. The factorization property states that the vector probability-generating function (vector PGF or vector GF) of the stationary queue length is factored into two PGFs of proper random variables. One of them is the vector PGF of the conditional stationary queue length given that the server is on vacation.

The class of $BMAP/G/1$ vacation models, for example with gated discipline can not be described by an $M/G/1$-type Markov chain embedded at the customer departure epochs, because at least one supplementary variable is required to describe the discipline. In case of gated discipline this variable is the number of customers not yet served from those present at the beginning of the vacation.

We define the $BMAP/G/1$ *vacation model of non-$M/G/1$-type* as the vacation model, which can not be described by an $M/G/1$-type Markov chain embedded at the customer departure epochs. Numerous disciplines fall into this category, like e.g. the gated, the E-limited or the G-limited ones, etc.

Very few literature is available on $BMAP/G/1$ vacation models of non-$M/G/1$-type. Ferrandiz [9] used Palm-martingale calculus to analyze a flexible vacation scheme. Shin and Pearce [10] studied queue-length dependent vacation schedules by using the semi-Markov process technique. Recently Banik et al. [11] studied the BMAP/G/1/N queue with vacations and E-limited service discipline. They applied supplementary variable technique to get the queue length distributions and several system performance measures.

The principal goal of this paper is to analyze $BMAP/G/1$ vacation models of non-$M/G/1$-type in a unified framework, which utilizes the advantages of separating the service discipline independent and dependent parts of the analysis.

The contributions of this paper are twofold. The main contribution is the new service discipline independent formulas for the vector GF of the stationary number of customers and for its mean, both in terms of the vector GF of the stationary number of customers at start of vacation, and its factorial moments, respectively.

The second contribution is the new results for the $BMAP/G/1$ vacation models with gated and G-limited disciplines. To the best knowledge of the authors, no results are available for these vacation models of non-$M/G/1$-type. For both models system equations are setup, which can be numerically solved by methods for systems of linear equations. Afterwards the required quantities at the start of the vacation can be computed.

The rest of this paper is organized as follows. In section II we introduce the model and the notations. The derivation of the stationary number of customers in vacation follows in Section III. The new formulas of the vector GF of the stationary number of customers at an arbitrary moment and its mean are derived in section IV. In section V we present the analysis of vacation models of non-M/G/1-type. Numerical example follows in section VI. We give final remarks in section VII.

## 2 Model and Notation

### 2.1 *BMAP* process

We give a brief summary on the *BMAP* related definitions and notations. For more details we refer to [3].

$\Lambda(t)$ denotes the number of arrivals in $(0,t]$. $J(t)$ is the state of a background continuous-time Markov chain (CTMC) at time $t$, which is referred to as phase and phase process, respectively. The *BMAP* batch arrival process is characterized by $\{(\Lambda(t), J(t))\,; t \geq 0\}$ bivariate CTMC on the state space $(\Lambda(t), J(t))$; where $(\Lambda(t) \in \{0,1,\ldots\}, J(t) \in \{1,2,\ldots,L\})$. Its infinitesimal generator is:

$$\begin{pmatrix} \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \ldots \\ \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \ldots \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \ldots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\mathbf{0}$ is an $L \times L$ matrix and $\{\mathbf{D}_k; k \geq 0\}$ is a set of $L \times L$ matrices.

$\mathbf{D}_0$ and $\{\mathbf{D}_k; k \geq 1\}$ govern the transitions corresponding to no arrivals and to batch arrivals with size k, respectively. The irreducible infinitesimal generator of the phase process is $\mathbf{D} = \sum_{k=0}^{\infty} \mathbf{D}_k$. Let $\boldsymbol{\pi}$ be the stationary probability vector of the phase process. Then $\boldsymbol{\pi}\mathbf{D} = \mathbf{0}$ and $\boldsymbol{\pi}\mathbf{e} = 1$ uniquely determine $\boldsymbol{\pi}$, where $\mathbf{e}$ is the column vector having all elements equal to one. $\widehat{\mathbf{D}}(z)$, the matrix generating function of $\mathbf{D}_k$ is defined as

$$\widehat{\mathbf{D}}(z) = \sum_{k=0}^{\infty} \mathbf{D}_k z^k, \quad |z| \leq 1. \tag{1}$$

The stationary arrival rate of the BMAP,

$$\lambda = \boldsymbol{\pi} \, \frac{d}{dz} \widehat{\mathbf{D}}(z) \bigg|_{z=1} \mathbf{e} = \boldsymbol{\pi} \sum_{k=0}^{\infty} k \mathbf{D}_k \mathbf{e}, \tag{2}$$

is supposed to be positive and finite.

### 2.2 The *BMAP/G/1* queue with server vacation

Batch of customers arrive to the infinite buffer queue according to a *BMAP* process defined by $\widehat{\mathbf{D}}(z)$. The service times are independent and identically distributed. $B$, $B(t)$, $b$, $b^{(2)}$ denote the service time r.v., its cumulated distribution function and its first two moments, respectively. The mean service time is positive and finite, $0 < b < \infty$.

The server occasionally takes vacations, in which no customer is served. After finishing the vacation the server continues to serve the queue. The model with this strategy is called *queue with single vacation.* If no customer is present in the queue after finishing the vacation, the server immediately takes the next vacation. We define the *cycle time* as a service period and a vacation period together. The server utilization is $\rho = \lambda b$. On the vacation model we impose the following assumptions:

**A.1** Independence property: The arrival process and the customer service times are mutually independent. In addition the customer service time is independent of the sequence of vacation periods that precede it.

**A.2** Customer loss-free property: All customers arriving to the system will be served. Thus the system has infinite queue and $\rho < 1$.

**A.3** Nonpreemtive service property: The service is nonpreemtive. Hence the service of the actual customer is finished before the server goes to vacation.

**A.4** Phase independent vacation property: The length of the vacation period is independent of the arrival process and from the customer service times.

In the following $[Y]_{i,j}$ stands for the $i,j$-th element of matrix $\mathbf{Y}$. Similarly $[y]_j$ denotes the $j$-th element of vector $\mathbf{y}$.

We define matrix $\mathbf{A}_k$, whose $(i,j)$-th element denotes the conditional probability that during a customer service time the number of arrivals is $k$ and the initial and final phases of the $BMAP$ are $i$ and $j$, respectively. That is, for $k \geq 0$, $1 \leq i,j \leq L$,

$$[\mathbf{A}_k]_{i,j} = P\left\{\mathit{\Lambda}(B) = k, J(B) = j | J(0) = i\right\}.$$

The matrix GF $\widehat{\mathbf{A}}(z)$ is defined as $\widehat{\mathbf{A}}(z) = \sum_{k=1}^{\infty} \mathbf{A}_k z^k$. $\widehat{\mathbf{A}}(z)$ can be expressed explicitly as [3]

$$\widehat{\mathbf{A}}(z) = \int_{t=0}^{\infty} e^{\widehat{\mathbf{D}}(z)t} dB(t).$$

For later use we also express $\boldsymbol{\pi}\left(\mathbf{I} - \left.\frac{d\widehat{\mathbf{A}}(z)}{dz}\right|_{z=1}\right)\mathbf{e}$, where $\mathbf{I}$ denotes the unity matrix. To this end we rewrite the term $\boldsymbol{\pi} \left.\frac{d\widehat{\mathbf{A}}(z)}{dz}\right|_{z=1} \mathbf{e}$ as

$$\boldsymbol{\pi} \left.\frac{d\widehat{\mathbf{A}}(z)}{dz}\right|_{z=1} \mathbf{e} = \boldsymbol{\pi} \left.\frac{dE\left(e^{\widehat{\mathbf{D}}(z)B}\right)}{dz}\right|_{z=1} \mathbf{e} = \boldsymbol{\pi} E\left(\sum_{k=0}^{\infty} \left.\frac{d\left(\widehat{\mathbf{D}}(z)^k\right)}{dz}\right|_{z=1} \mathbf{e} \frac{B^k}{k!}\right) =$$
$$E\left(\sum_{k=1}^{\infty} \boldsymbol{\pi}\mathbf{D}^{k-1} \left.\frac{d\widehat{\mathbf{D}}(z)}{dz}\right|_{z=1} \mathbf{e} \frac{B^k}{k!}\right) = \boldsymbol{\pi} \left.\frac{d\widehat{\mathbf{D}}(z)}{dz}\right|_{z=1} \mathbf{e}\, E(B) = \lambda b = \rho,$$

where we used that $\boldsymbol{\pi}\mathbf{D} = 0$.

Now the term $\boldsymbol{\pi}\left(\mathbf{I} - \left.\frac{d\widehat{\mathbf{A}}(z)}{dz}\right|_{z=1}\right)\mathbf{e}$ can be given explicitly as

$$\boldsymbol{\pi}\left(\mathbf{I} - \left.\frac{d\widehat{\mathbf{A}}(z)}{dz}\right|_{z=1}\right)\mathbf{e} = 1 - \rho. \tag{3}$$

$V$, $V(t)$, $v$ denote the vacation time r.v., its cumulated distribution function and its mean, respectively. The mean vacation time is positive and finite, $0 <$

$v < \infty$. Similar to the quantities associated with the service period, we define matrix $\mathbf{U}_k$, whose elements, for $k \geq 0$, $1 \leq i, j \leq L$, are

$$[\mathbf{U}_k]_{i,j} = P\left\{\Lambda(V) = k, J(V) = j | J(0) = i\right\},$$

and the matrix GF $\widehat{\mathbf{U}}(z) = \sum_{k=1}^{\infty} \mathbf{U}_k z^k = \int_{t=0}^{\infty} e^{\widehat{\mathbf{D}}(z)t} dV(t)$.

Our vacation model is similar to the *generalized vacation model* for the $M/G/1$ queue defined in Fuhrmann and Cooper [12]. The phase independent vacation property **A.4** corresponds to the independence assumption 6. of [12].

Vacation models are distinguished by their (service) discipline that is the set of rules determining the beginning and the end of the vacation (service). Commonly applied service disciplines are, e.g., the exhaustive, the gated, the limited, etc. In case of exhaustive discipline, the server continues serving the customers until the queue is emptied. Under gated discipline only those customers are served, which are present at the beginning of the service period. In case of E-limited discipline either N customers are served in a service period or the queue becomes empty before and the service period ends. In case of G-limited discipline at most N customers are served among the customers, which are present at the beginning of the service period.

## 3 Stationary number of customers in the vacation period

We define $N(t)$ as the number of customers in the system at time $t$, and $\widehat{\mathbf{q}}(z)$ and $\widehat{\mathbf{q}}^v(z)$ as the vector GFs of the stationary number of customers and of the stationary number of customers during the vacation period, respectively. The elements of $\widehat{\mathbf{q}}(z)$ and $\widehat{\mathbf{q}}^v(z)$ are defined as

$$[\widehat{\mathbf{q}}(z)]_j = \lim_{t\to\infty} \sum_{n=0}^{\infty} P\left\{N(t) = n, J(t) = j\right\} z^n, \quad |z| \leq 1, \text{ and}$$

$$[\widehat{\mathbf{q}}^v(z)]_j = \lim_{t\to\infty} \sum_{n=0}^{\infty} P\left\{N(t) = n, J(t) = j \mid t \in \textit{vacation period}\right\} z^n, \quad |z| \leq 1,$$

respectively.

Furthermore, $t_k^m$ denotes the start of vacation (the instant just after the completion of service) in the $k$-th cycle. The vector GF, $\widehat{\mathbf{m}}(z)$, is defined by its elements as

$$[\widehat{\mathbf{m}}(z)]_j = \lim_{k\to\infty} \sum_{n=0}^{\infty} P\left\{N(t_k^m) = n, J(t_k^m) = j\right\} z^n, \quad |z| \leq 1.$$

**Theorem 1.** *The following relation holds for the vector GF of the stationary number of customers in the vacation period:*

$$\widehat{\mathbf{q}}^v(z)\,\widehat{\mathbf{D}}(z) = \frac{\widehat{\mathbf{m}}(z)\left(\widehat{\mathbf{U}}(z) - \mathbf{I}\right)}{v}. \tag{4}$$

*Proof.* The matrix GF of the number of customers arriving during the vacation period is $E\left(e^{\widehat{\mathbf{D}}(z)V}\right)$, from which

$$E\left(e^{\widehat{\mathbf{D}}(z)V}\right) = \int_{t=0}^{\infty} e^{\widehat{\mathbf{D}}(z)t} dV(t) = \widehat{\mathbf{U}}(z). \tag{5}$$

The vector GF of the stationary number of customers in the system at instant $\tau$ in the vacation period is $\widehat{\mathbf{m}}(z)\ e^{\widehat{\mathbf{D}}(z)\tau}$, where the first term stands for the stationary number of customers in the system at the beginning of the vacation and the second term stands for the number of customers arriving in the $(0,\tau)$ interval of the vacation period. To obtain the stationary number of customers during the vacation period we need to average the number of customers in the system over the duration of the vacation period

$$\widehat{\mathbf{q}}^{v}(z) = \frac{\widehat{\mathbf{m}}(z)E\left(\int_{\tau=0}^{V}\ e^{\widehat{\mathbf{D}}(z)\tau} d\tau\right)}{E\left(V\right)}. \tag{6}$$

Based on the definition of $\widehat{\mathbf{q}}^{v}(z)$, we have $\widehat{\mathbf{q}}^{v}(1)\ \mathbf{e} = 1$. Indeed the numerator of (6) at $z = 1$ multiplied by $\mathbf{e}$ can be written as

$$E\left(\int_{\tau=0}^{V} \widehat{\mathbf{m}}(1)\ \left(e^{\mathbf{D}\tau}\ \mathbf{e}\right)\ d\tau\right) = E\left(\int_{\tau=0}^{V} \widehat{\mathbf{m}}(1)\ \mathbf{e}\ d\tau\right) = E\left(\int_{\tau=0}^{V} 1\ d\tau\right) = E(V), \tag{7}$$

because $e^{\mathbf{D}\tau}$ is a stochastic matrix and consequently $e^{\mathbf{D}\tau}\ \mathbf{e} = \mathbf{e}$. Multiplying both sides of (6) by $\widehat{\mathbf{D}}(z)$ and using $E(V) = v$ we have

$$\widehat{\mathbf{q}}^{v}(z)\ \widehat{\mathbf{D}}(z) = \frac{1}{v}\ \widehat{\mathbf{m}}(z)E\left(\int_{\tau=0}^{V}\ e^{\widehat{\mathbf{D}}(z)\tau}\ \widehat{\mathbf{D}}(z)\ d\tau\right). \tag{8}$$

The integral term can be rewritten as

$$\begin{aligned} &\int_{\tau=0}^{V}\ e^{\widehat{\mathbf{D}}(z)\tau}\ \widehat{\mathbf{D}}(z)\ d\tau = \int_{\tau=0}^{V}\ \sum_{k=0}^{\infty} \frac{\tau^{k}\widehat{\mathbf{D}}(z)^{k}}{k!}\ \widehat{\mathbf{D}}(z)\ d\tau = \\ &\sum_{k=0}^{\infty} \int_{\tau=0}^{V} \tau^{k}\ d\tau\ \frac{\widehat{\mathbf{D}}(z)^{k+1}}{k!} = \sum_{k=0}^{\infty} \frac{V^{k+1}}{k+1}\ \frac{\widehat{\mathbf{D}}(z)^{k+1}}{k!} = e^{\widehat{\mathbf{D}}(z)V} - \mathbf{I}. \end{aligned} \tag{9}$$

Using (5) and (9) the theorem comes from (8). Q.E.D.

# 4 Service discipline independent stationary relations

## 4.1 Vector GF of the stationary number of customers

**Theorem 2.** *The following service discipline independent relation holds for the vector GF of the stationary number of customers at an arbitrary instant:*

$$\widehat{\mathbf{q}}(z)\,\widehat{\mathbf{D}}(z)\left(z\mathbf{I}-\widehat{\mathbf{A}}(z)\right)=\frac{\widehat{\mathbf{m}}(z)\left(\widehat{\mathbf{U}}(z)-\mathbf{I}\right)}{v}(1-\rho)(z-1)\,\widehat{\mathbf{A}}(z). \tag{10}$$

*Proof.* Chang et al. [8] provided a factorization formula for the $BMAP/G/1$ queue with generalized vacations:

$$\widehat{\mathbf{q}}(z)\left(z\mathbf{I}-\widehat{\mathbf{A}}(z)\right)=\widehat{\mathbf{q}}^{v}(z)(1-\rho)(z-1)\,\widehat{\mathbf{A}}(z). \tag{11}$$

The theorem can be obtained by multiplying both sides of (11) by $\widehat{\mathbf{D}}(z)$ from the right and applying (4), because $\widehat{\mathbf{A}}(z)$ and $\widehat{\mathbf{D}}(z)$ commute, as can be seen from the Taylor expansion of $\widehat{\mathbf{A}}(z)$. Q.E.D.

Note that the contribution of the concrete service discipline to the relation (10) is incorporated by the quantity $\widehat{\mathbf{m}}(z)$.

### 4.2 The mean of the stationary number of customers

This subsection presents the service discipline independent solution for the mean of the stationary number of customers in the system based on its vector GF (10). To this end, we introduce the following notations. When $\widehat{\mathbf{Y}}(z)$ is a GF, $\mathbf{Y}^{(i)}$ denotes it $i$-th ($i \geq 1$) factorial moment, i.e., $\mathbf{Y}^{(i)}=\frac{d^i}{dz^i}\widehat{\mathbf{Y}}(z)|_{z=1}$, and $\mathbf{Y}$ denotes its value at $z=1$, i.e., $\mathbf{Y}=\widehat{\mathbf{Y}}(1)$. We apply these conventions for $\widehat{\mathbf{D}}(z)$, $\widehat{\mathbf{A}}(z)$, $\widehat{\mathbf{U}}(z)$, $\widehat{\mathbf{q}}(z)$, $\widehat{\mathbf{m}}(z)$ and for the later defined $\widehat{\mathbf{r}}(z)$ and $\widehat{\mathbf{t}}(z)$.

**Theorem 3.** *The service discipline independent solution for the mean of the stationary number of customers at an arbitrary instant is given by:*

$$\begin{aligned}
\mathbf{q}^{(1)}=&\frac{\mathbf{m}^{(1)}}{\lambda v}\left(\mathbf{U}^{(1)}\mathbf{e}\boldsymbol{\pi}+(\mathbf{U}-\mathbf{I})\left(\mathbf{A}^{(1)}-\mathbf{A}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{D}^{(1)}\right)\mathbf{e}\boldsymbol{\pi}\right)\\
&+\frac{\mathbf{m}}{\lambda v}\left(\frac{1}{2}\mathbf{U}^{(2)}\mathbf{e}\boldsymbol{\pi}+\frac{1}{2}(\mathbf{U}-\mathbf{I})\mathbf{A}^{(2)}\mathbf{e}\boldsymbol{\pi}+\mathbf{U}^{(1)}\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi}\right)\\
&-\frac{\mathbf{m}}{\lambda v}\left(\mathbf{U}^{(1)}\mathbf{A}+(\mathbf{U}-\mathbf{I})\mathbf{A}^{(1)}\right)(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi}\\
&+\frac{\mathbf{m}}{\lambda v}\left(\mathbf{U}^{(1)}\mathbf{A}\mathbf{e}\boldsymbol{\pi}+(\mathbf{U}-\mathbf{I})\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi}\right)\left(\frac{\mathbf{C_2}\mathbf{e}\boldsymbol{\pi}}{\lambda}+(1-\rho)\mathbf{C_1}\right)\\
&+\frac{\mathbf{m}}{\lambda v}(\mathbf{U}-\mathbf{I})\mathbf{A}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\left(\lambda\mathbf{I}-\mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi}\right)\left(\frac{\mathbf{C_2}\mathbf{e}\boldsymbol{\pi}}{\lambda}+(1-\rho)\mathbf{C_1}\right)\\
&+\boldsymbol{\pi}\left(\frac{\mathbf{A}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2(1-\rho)}-\left(\mathbf{I}-\mathbf{A}^{(1)}\right)\mathbf{C_1}\right),
\end{aligned} \tag{12}$$

*where matrices* $\mathbf{C_1}$ *and* $\mathbf{C_2}$ *are defined as*

$$\mathbf{C_1} = (\mathbf{I} - \mathbf{A} + \mathbf{e}\boldsymbol{\pi})^{-1} \left( \frac{\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi}}{(1-\rho)} + \mathbf{I} \right), \quad \mathbf{C_2} = \mathbf{D}^{(1)} (\mathbf{D} + \mathbf{e}\boldsymbol{\pi})^{-1} \mathbf{D}^{(1)} - \frac{1}{2}\mathbf{D}^{(2)}.$$

To prove the theorem we need the following lemmas.

**Lemma 1.** *The term* $\mathbf{q}^{(1)}$ *can be expressed from (14) in terms of* $\mathbf{r}^{(1)}$ *and* $\mathbf{r}^{(2)}\mathbf{e}$ *as follows:*

$$\mathbf{q}^{(1)} = \frac{\mathbf{r}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2(1-\rho)} + \mathbf{r}^{(1)}\mathbf{C_1} + \boldsymbol{\pi} \left( \frac{\mathbf{A}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2(1-\rho)} - \left(\mathbf{I} - \mathbf{A}^{(1)}\right) \mathbf{C_1} \right), \tag{13}$$

*where vector* $\widehat{\mathbf{r}}(z)$ *is defined as*

$$\widehat{\mathbf{r}}(z) = \widehat{\mathbf{q}}(z) \left( z\mathbf{I} - \widehat{\mathbf{A}}(z) \right). \tag{14}$$

*Proof.* Starting from (14) we apply the method used by Lucantoni in [3] and Neuts in [4], which utilizes that $(\mathbf{I} - \mathbf{A} + \mathbf{e}\boldsymbol{\pi})$ is nonsingular. Taking the first two derivatives of (14) at $z = 1$, we get:

$$\mathbf{q}^{(1)} (\mathbf{I} - \mathbf{A}) = \mathbf{r}^{(1)} - \boldsymbol{\pi} \left( \mathbf{I} - \mathbf{A}^{(1)} \right), \tag{15}$$

$$\mathbf{q}^{(2)} (\mathbf{I} - \mathbf{A}) = \mathbf{r}^{(2)} - 2\mathbf{q}^{(1)} \left( \mathbf{I} - \mathbf{A}^{(1)} \right) + \boldsymbol{\pi}\mathbf{A}^{(2)}. \tag{16}$$

Adding $\mathbf{q}^{(1)}\mathbf{e}\boldsymbol{\pi}$ to both sides of (15) and using $\boldsymbol{\pi} (\mathbf{I} - \mathbf{A} + \mathbf{e}\boldsymbol{\pi})^{-1} = \boldsymbol{\pi}$ leads to

$$\mathbf{q}^{(1)} = \quad \mathbf{q}^{(1)}\mathbf{e} \quad \boldsymbol{\pi} + \quad \mathbf{r}^{(1)} - \boldsymbol{\pi} \quad \mathbf{I} - \mathbf{A}^{(1)} \quad (\mathbf{I} - \mathbf{A} + \mathbf{e}\boldsymbol{\pi})^{-1}. \tag{17}$$

The next step is to get the unknown term $\left(\mathbf{q}^{(1)}\mathbf{e}\right)$ in (17). Post-multiplying (16) by $\mathbf{e}$ and post-multiplying (17) by $\left(\mathbf{I} - \mathbf{A}^{(1)}\right)\mathbf{e}$ and rearranging gives

$$\mathbf{q}^{(1)} \left( \mathbf{I} - \mathbf{A}^{(1)} \right) \mathbf{e} = \frac{1}{2}\mathbf{r}^{(2)}\mathbf{e} + \frac{1}{2}\boldsymbol{\pi}\mathbf{A}^{(2)}\mathbf{e}, \tag{18}$$

$$\begin{aligned} \mathbf{q}^{(1)} \left( \mathbf{I} - \mathbf{A}^{(1)} \right) \mathbf{e} = {} & \left( \mathbf{q}^{(1)}\mathbf{e} \right) \boldsymbol{\pi} \left( \mathbf{I} - \mathbf{A}^{(1)} \right) \mathbf{e} \\ & + \left( \mathbf{r}^{(1)} - \boldsymbol{\pi} \left( \mathbf{I} - \mathbf{A}^{(1)} \right) \right) (\mathbf{I} - \mathbf{A} + \mathbf{e}\boldsymbol{\pi})^{-1} \left( \mathbf{I} - \mathbf{A}^{(1)} \right) \mathbf{e}, \end{aligned} \tag{19}$$

respectively. Combining (18) and (19) and applying $\boldsymbol{\pi} \left( \mathbf{I} - \mathbf{A}^{(1)} \right) \mathbf{e} = 1 - \rho$ results in the expression of the required term:

$$\begin{aligned} \mathbf{q}^{(1)}\mathbf{e} = {} & \frac{1}{2(1-\rho)} \left( \mathbf{r}^{(2)}\mathbf{e} + \boldsymbol{\pi}\mathbf{A}^{(2)}\mathbf{e} \right) + \\ & \frac{1}{(1-\rho)} \left( \boldsymbol{\pi} \left( \mathbf{I} - \mathbf{A}^{(1)} \right) - \mathbf{r}^{(1)} \right) (\mathbf{I} - \mathbf{A} + \mathbf{e}\boldsymbol{\pi})^{-1} \left( \mathbf{I} - \mathbf{A}^{(1)} \right) \mathbf{e}. \end{aligned} \tag{20}$$

We can simplify (20) by using $(\mathbf{I}-\mathbf{A}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{e} = \mathbf{e}$ and $\left(\mathbf{r}^{(1)}-\boldsymbol{\pi}\left(\mathbf{I}-\mathbf{A}^{(1)}\right)\right)\mathbf{e}=0$ from (15):

$$\mathbf{q}^{(1)}\mathbf{e} = \frac{1}{2(1-\rho)}\left(\mathbf{r}^{(2)}\mathbf{e}+\boldsymbol{\pi}\mathbf{A}^{(2)}\mathbf{e}\right) \tag{21}$$
$$+\frac{1}{(1-\rho)}\left(\mathbf{r}^{(1)}-\boldsymbol{\pi}\left(\mathbf{I}-\mathbf{A}^{(1)}\right)\right)(\mathbf{I}-\mathbf{A}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{A}^{(1)}\mathbf{e}.$$

Substituting (21) into (17) leads to:

$$\mathbf{q}^{(1)} = \frac{\mathbf{r}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2(1-\rho)} + \mathbf{r}^{(1)} \quad \frac{1}{1-\rho}(\mathbf{I}-\mathbf{A}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi} + (\mathbf{I}-\mathbf{A}+\mathbf{e}\boldsymbol{\pi})^{-1} \tag{22}$$
$$+\boldsymbol{\pi} \quad \frac{\mathbf{A}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2(1-\rho)}$$
$$-\boldsymbol{\pi} \quad \frac{1}{1-\rho} \quad \mathbf{I}-\mathbf{A}^{(1)} \quad (\mathbf{I}-\mathbf{A}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{A}^{(1)}\mathbf{e}\boldsymbol{\pi} + \quad \mathbf{I}-\mathbf{A}^{(1)} \quad (\mathbf{I}-\mathbf{A}+\mathbf{e}\boldsymbol{\pi})^{-1} \quad .$$

Substituting matrix $\mathbf{C_1}$ into (22) results in the statement. Q.E.D.

**Lemma 2.** *The terms* $\mathbf{r}^{(1)}$ *and* $\mathbf{r}^{(2)}\mathbf{e}$ *can be expressed from (25) in terms of* $\mathbf{t}^{(1)}$, $\mathbf{t}^{(2)}\mathbf{e}$, $\mathbf{t}^{(2)}$ *and* $\mathbf{t}^{(3)}\mathbf{e}$ *as follows:*

$$\mathbf{r}^{(1)} = \frac{\mathbf{t}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2\lambda} + \mathbf{t}^{(1)}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\left(\mathbf{I}-\frac{\mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi}}{\lambda}\right), \tag{23}$$

$$\mathbf{r}^{(2)}\mathbf{e} = \frac{\mathbf{t}^{(3)}\mathbf{e}}{3\lambda} - \frac{\mathbf{t}^{(2)}}{\lambda}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{D}^{(1)}\mathbf{e} + \frac{\mathbf{t}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2\lambda}\frac{2\mathbf{C_2}\mathbf{e}}{\lambda} + \tag{24}$$
$$\mathbf{t}^{(1)}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\left(\mathbf{I}-\frac{\mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi}}{\lambda}\right)\frac{2\mathbf{C_2}\mathbf{e}}{\lambda},$$

*where vector* $\widehat{\mathbf{t}}(z)$ *is defined as*

$$\widehat{\mathbf{t}}(z) = \widehat{\mathbf{r}}(z)\,\widehat{\mathbf{D}}(z). \tag{25}$$

*Proof.* We apply again the same method as in Lemma 1, but now utilizing that $(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})$ is nonsingular. Setting $z=1$ in (14) we get:

$$\mathbf{r} = \boldsymbol{\pi}(\mathbf{I}-\mathbf{A}) = 0. \tag{26}$$

Taking the first three derivatives of (25) and applying (26) results in

$$\mathbf{r}^{(1)}\mathbf{D} = \mathbf{t}^{(1)}, \tag{27}$$
$$\mathbf{r}^{(2)}\mathbf{D} = \mathbf{t}^{(2)} - 2\mathbf{r}^{(1)}\mathbf{D}^{(1)}, \tag{28}$$
$$\mathbf{r}^{(3)}\mathbf{D} = \mathbf{t}^{(3)} - 3\mathbf{r}^{(2)}\mathbf{D}^{(1)} - 3\mathbf{r}^{(1)}\mathbf{D}^{(2)}. \tag{29}$$

Adding $\mathbf{r}^{(1)}\mathbf{e}\boldsymbol{\pi}$ to both sides of (27) and using $\boldsymbol{\pi}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1} = \boldsymbol{\pi}$ we obtain

$$\mathbf{r}^{(1)} = \left(\mathbf{r}^{(1)}\mathbf{e}\right)\boldsymbol{\pi} + \mathbf{t}^{(1)}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}. \tag{30}$$

Post-multiplying (28) by $\mathbf{e}$ and (30) by $\mathbf{D}^{(1)}\mathbf{e}$ after rearranging yields

$$\mathbf{r}^{(1)}\mathbf{D}^{(1)}\mathbf{e} = \frac{1}{2}\mathbf{t}^{(2)}\mathbf{e}, \tag{31}$$

$$\mathbf{r}^{(1)}\mathbf{D}^{(1)}\mathbf{e} = \left(\mathbf{r}^{(1)}\mathbf{e}\right)\boldsymbol{\pi}\mathbf{D}^{(1)}\mathbf{e} + \mathbf{t}^{(1)}\left(\mathbf{D}+\mathbf{e}\boldsymbol{\pi}\right)^{-1}\mathbf{D}^{(1)}\mathbf{e}, \tag{32}$$

respectively. Combining (31) and (32) and applying $\boldsymbol{\pi}\mathbf{D}^{(1)}\mathbf{e} = \lambda$ results in:

$$\mathbf{r}^{(1)}\mathbf{e} = \frac{1}{2\lambda}\mathbf{t}^{(2)}\mathbf{e} - \frac{1}{\lambda}\mathbf{t}^{(1)}\left(\mathbf{D}+\mathbf{e}\boldsymbol{\pi}\right)^{-1}\mathbf{D}^{(1)}\mathbf{e}. \tag{33}$$

Substituting (33) into (30) results in the first statement.

Adding $\mathbf{r}^{(2)}\mathbf{e}\boldsymbol{\pi}$ to both sides of (28) gives:

$$\mathbf{r}^{(2)} = \left(\mathbf{r}^{(2)}\mathbf{e}\right)\boldsymbol{\pi} + \left(\mathbf{t}^{(2)} - 2\mathbf{r}^{(1)}\mathbf{D}^{(1)}\right)\left(\mathbf{D}+\mathbf{e}\boldsymbol{\pi}\right)^{-1}. \tag{34}$$

Post-multiplying (29) by $\mathbf{e}$ and (34) by $\mathbf{D}^{(1)}\mathbf{e}$ after rearranging leads to

$$\mathbf{r}^{(2)}\mathbf{D}^{(1)}\mathbf{e} = \frac{1}{3}\mathbf{t}^{(3)}\mathbf{e} - \mathbf{r}^{(1)}\mathbf{D}^{(2)}\mathbf{e}, \tag{35}$$

$$\mathbf{r}^{(2)}\mathbf{D}^{(1)}\mathbf{e} = \left(\mathbf{r}^{(2)}\mathbf{e}\right)\boldsymbol{\pi}\mathbf{D}^{(1)}\mathbf{e} + \left(\mathbf{t}^{(2)} - 2\mathbf{r}^{(1)}\mathbf{D}^{(1)}\right)\left(\mathbf{D}+\mathbf{e}\boldsymbol{\pi}\right)^{-1}\mathbf{D}^{(1)}\mathbf{e}, \tag{36}$$

respectively. Combining (35) and (36) and applying $\boldsymbol{\pi}\mathbf{D}^{(1)}\mathbf{e} = \lambda$ results in:

$$\mathbf{r}^{(2)}\mathbf{e} = \frac{1}{3\lambda}\mathbf{t}^{(3)}\mathbf{e} - \frac{1}{\lambda}\mathbf{r}^{(1)}\mathbf{D}^{(2)}\mathbf{e} + \frac{1}{\lambda}\left(2\mathbf{r}^{(1)}\mathbf{D}^{(1)} - \mathbf{t}^{(2)}\right)\left(\mathbf{D}+\mathbf{e}\boldsymbol{\pi}\right)^{-1}\mathbf{D}^{(1)}\mathbf{e}. \tag{37}$$

Substituting (23) into (37) leads to:

$$\begin{aligned}\mathbf{r}^{(2)}\mathbf{e} = {} & \frac{\mathbf{t}^{(3)}\mathbf{e}}{3\lambda} - \frac{\mathbf{t}^{(2)}}{\lambda}\left(\mathbf{D}+\mathbf{e}\boldsymbol{\pi}\right)^{-1}\mathbf{D}^{(1)}\mathbf{e} + \frac{\mathbf{t}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2\lambda} \quad \frac{2\mathbf{D}^{(1)}}{\lambda}\left(\mathbf{D}+\mathbf{e}\boldsymbol{\pi}\right)^{-1}\mathbf{D}^{(1)}\mathbf{e} - \frac{\mathbf{D}^{(2)}\mathbf{e}}{\lambda} \\ & + \mathbf{t}^{(1)}\left(\mathbf{D}+\mathbf{e}\boldsymbol{\pi}\right)^{-1} \quad \mathbf{I} - \frac{\mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi}}{\lambda} \qquad \frac{2\mathbf{D}^{(1)}}{\lambda}\left(\mathbf{D}+\mathbf{e}\boldsymbol{\pi}\right)^{-1}\mathbf{D}^{(1)}\mathbf{e} - \frac{\mathbf{D}^{(2)}\mathbf{e}}{\lambda} \quad .\end{aligned} \tag{38}$$

Inserting matrix $\mathbf{C_2}$ into (38) results in the second statement. Q.E.D.

*Proof.* PROOF OF THEOREM 3

Due to the fact that $\widehat{\mathbf{D}}(z)$ and $\widehat{\mathbf{A}}(z)$ commute $\widehat{\mathbf{t}}(z)$ equals to the left hand size of (10)

$$\widehat{\mathbf{t}}\left(z\right) = \widehat{\mathbf{q}}\left(z\right)\widehat{\mathbf{D}}\left(z\right)\left(z\mathbf{I} - \widehat{\mathbf{A}}(z)\right). \tag{39}$$

We apply Lemma 1 and 2 to get $\mathbf{q}^{(1)}$ from (39). Substituting (23) and (24) into (13) gives the expression of $\mathbf{q}^{(1)}$ in terms of $\mathbf{t}^{(1)}$, $\mathbf{t}^{(2)}\mathbf{e}$, $\mathbf{t}^{(2)}$ and $\mathbf{t}^{(3)}\mathbf{e}$:

$$\begin{aligned}\mathbf{q}^{(1)} = & \frac{\mathbf{t}^{(3)}\mathbf{e}\boldsymbol{\pi}}{6\lambda(1-\rho)} - \frac{\mathbf{t}^{(2)}}{2\lambda(1-\rho)}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi} \\ & + \frac{\mathbf{t}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2\lambda(1-\rho)}\left(\frac{\mathbf{C}_2\mathbf{e}\boldsymbol{\pi}}{\lambda} + (1-\rho)\mathbf{C}_1\right) \\ & + \frac{\mathbf{t}^{(1)}}{(1-\rho)}(\mathbf{D}+\mathbf{e}\boldsymbol{\pi})^{-1}\left(\mathbf{I} - \frac{\mathbf{D}^{(1)}\mathbf{e}\boldsymbol{\pi}}{\lambda}\right)\left(\frac{\mathbf{C}_2\mathbf{e}\boldsymbol{\pi}}{\lambda} + (1-\rho)\mathbf{C}_1\right) \\ & + \boldsymbol{\pi}\left(\frac{\mathbf{A}^{(2)}\mathbf{e}\boldsymbol{\pi}}{2(1-\rho)} - \left(\mathbf{I}-\mathbf{A}^{(1)}\right)\mathbf{C}_1\right).\end{aligned} \quad (40)$$

Substituting (10) into (39) yields:

$$\widehat{\mathbf{t}}(z) = \frac{\widehat{\mathbf{m}}(z)\left(\widehat{\mathbf{U}}(z) - \mathbf{I}\right)}{v}(1-\rho)(z-1)\widehat{\mathbf{A}}(z). \quad (41)$$

Taking the first three derivatives of $\widehat{\mathbf{t}}(z)$ at $z=1$:

$$\mathbf{t}^{(1)} = (1-\rho)\frac{\mathbf{m}}{v}(\mathbf{U}-\mathbf{I})\mathbf{A}, \quad (42)$$

$$\mathbf{t}^{(2)} = 2(1-\rho)\frac{\mathbf{m}^{(1)}}{v}(\mathbf{U}-\mathbf{I})\mathbf{A} + 2(1-\rho)\frac{\mathbf{m}}{v}\left(\mathbf{U}^{(1)}\mathbf{A} + (\mathbf{U}-\mathbf{I})\mathbf{A}^{(1)}\right), \quad (43)$$

$$\mathbf{t}^{(2)}\mathbf{e} = 2(1-\rho)\frac{\mathbf{m}}{v}\left(\mathbf{U}^{(1)}\mathbf{A}\mathbf{e} + (\mathbf{U}-\mathbf{I})\mathbf{A}^{(1)}\mathbf{e}\right), \quad (44)$$

$$\begin{aligned}\mathbf{t}^{(3)}\mathbf{e} = & 6(1-\rho)\frac{\mathbf{m}^{(1)}}{v}\left(\mathbf{U}^{(1)}\mathbf{e} + (\mathbf{U}-\mathbf{I})\mathbf{A}^{(1)}\mathbf{e}\right) \\ & + 3(1-\rho)\frac{\mathbf{m}}{v}\left(\mathbf{U}^{(2)}\mathbf{e} + (\mathbf{U}-\mathbf{I})\mathbf{A}^{(2)}\mathbf{e} + 2\mathbf{U}^{(1)}\mathbf{A}^{(1)}\mathbf{e}\right).\end{aligned} \quad (45)$$

Substituting (42), (43), (44) and (45) into (40) gives the theorem. Q.E.D.

Note that in (12) the impact of the concrete service discipline on the mean of the stationary number of customers is expressed by the quantities $\mathbf{m}^{(1)}$ and $\mathbf{m}$.

## 5 Vacation models of non-M/G/1-type

Let $t_k^f$ denotes the end of vacation (the instant just before the start of service) in the $k$-th cycle. The vectors $\mathbf{f}_n$ and $\mathbf{m}_n$, $n \geq 0$, are defined by their elements as

$$[\mathbf{f}_n]_j = \lim_{k\to\infty} P\left\{N(t_k^f) = n, J(t_k^f) = j\right\},$$
$$[\mathbf{m}_n]_j = \lim_{k\to\infty} P\left\{N(t_k^m) = n, J(t_k^m) = j\right\},$$

To get the unknown quantities $\mathbf{m}$, $\mathbf{m}^{(1)}$ in (12), we compute the stationary probability vectors $\mathbf{m}_n$, $n \geq 0$. For doing that we setup a system of linear equations for each studied discipline.

## 5.1 Vacation model with gated discipline

**Theorem 4.** *In the vacation model with gated discipline the probability vectors* $\mathbf{m}_n$*,* $n \geq 0$ *are determined by the following system equation:*

$$\sum_{n=0}^{\infty} \mathbf{m}_n \sum_{k=0}^{\infty} \mathbf{U}_k \left(\widehat{\mathbf{A}}(z)\right)^k \left(\widehat{\mathbf{A}}(z)\right)^n = \sum_{n=0}^{\infty} \mathbf{m}_n z^n. \tag{46}$$

*Proof.* Each customer, who is present at the end of the vacation, generates a random population of customers arriving during its service time. The GF of number of customers in this random population is $\widehat{\mathbf{A}}(z)$. Hence the governing relation for transition $f \rightarrow m$ of the vacation model with gated discipline is given by

$$\sum_{n=0}^{\infty} \mathbf{f}_n \left(\widehat{\mathbf{A}}(z)\right)^n = \sum_{n=0}^{\infty} \mathbf{m}_n z^n. \tag{47}$$

The number of customers at the end of the vacation equals the sum of those present at the beginning of the vacation and those who arrived during the vacation period. Applying the phase independent vacation property **A.4**, we get discipline independent governing relation for transition $m \rightarrow f$ of the vacation model

$$\sum_{k=0}^{n} \mathbf{m}_k \mathbf{U}_{n-k} = \mathbf{f}_n. \tag{48}$$

Combining (47) and (48) and rearranging results in the statement. Q.E.D.

To compute the probability vectors $\mathbf{m}_n$ a numerical method can be developed by setting a $\rho$ dependent upper limit $X$ for $n$ and $k$ in (46). Taking the $x$-th derivatives of (46) at $z = 1$, where $x = 0, \ldots, X$, leads to a system of linear equations, in which the number of equations and the number of unknowns is $L(X+1)$:

$$\sum_{n=0}^{X} \mathbf{m}_n \sum_{k=0}^{X} \mathbf{U}_k \sum_{l=0}^{x} \binom{x}{l} \left. \frac{d^{(x-l)} \left(\widehat{\mathbf{A}}(z)\right)^k}{dz^{(x-l)}} \right|_{z=1} \left. \frac{d^l \left(\widehat{\mathbf{A}}(z)\right)^n}{dz^l} \right|_{z=1}$$

$$= \sum_{n=x}^{X} \mathbf{m}_n \frac{n!}{(n-x)!}, \quad x = 0, \ldots, X. \tag{49}$$

### 5.2 Vacation model with G-limited discipline

**Theorem 5.** *In the vacation model with G-limited discipline the probability vectors $\mathbf{m}_n$, $n \geq 0$ are determined by the following system equation:*

$$\sum_{n=0}^{K}\sum_{k=0}^{n}\mathbf{m}_k\mathbf{U}_{n-k}\left(\widehat{\mathbf{A}}\left(z\right)\right)^n + \sum_{n=K+1}^{\infty}\sum_{k=0}^{n}\mathbf{m}_k\mathbf{U}_{n-k}\left(\widehat{\mathbf{A}}\left(z\right)\right)^K = \sum_{n=0}^{\infty}\mathbf{m}_n z^n. \tag{50}$$

*Proof.* According to the G-limited discipline, the service is gated up to a maximum number $K$ of customers present at the beginning of service. Hence the governing relation for transition $f \rightarrow m$ of the vacation model with G-limited discipline is given by

$$\sum_{n=0}^{K}\mathbf{f}_n\left(\widehat{\mathbf{A}}\left(z\right)\right)^n + \sum_{n=K+1}^{\infty}\mathbf{f}_n\left(\widehat{\mathbf{A}}\left(z\right)\right)^K = \sum_{n=0}^{\infty}\mathbf{m}_n z^n. \tag{51}$$

Combining (51) with the discipline independent governing relation for transition $m \rightarrow f$ (48) and rearranging leads to the statement. Q.E.D.

Again to compute the probability vectors $\mathbf{m}_n$ a numerical method can be developed by setting a $\rho$ dependent upper limit $X$ for $n$ in (50). Taking the $x$-th derivatives of (50) at $z = 1$, where $x = 0, \ldots, X$, leads to a system of linear equations, in which the number of equations and the number of unknowns is $L\left(X+1\right)$:

$$\sum_{n=0}^{K}\sum_{k=0}^{n}\mathbf{m}_k\mathbf{U}_{n-k}\left.\frac{d^x\left(\widehat{\mathbf{A}}\left(z\right)\right)^n}{dz^x}\right|_{z=1} + \sum_{n=K+1}^{X}\sum_{k=0}^{n}\mathbf{m}_k\mathbf{U}_{n-k}\left.\frac{d^x\left(\widehat{\mathbf{A}}\left(z\right)\right)^K}{dz^x}\right|_{z=1}$$
$$= \sum_{n=x}^{X}\mathbf{m}_n\frac{n!}{(n-x)!}, \quad x = 0, \ldots, X. \tag{52}$$

## 6 Numerical example

We provide a simple numerical example just with illustrative purpose for the case of vacation model with gated discipline.

The arrival process is given by

$$\widehat{\mathbf{D}}(z) = \mathbf{D}_0 + z\mathbf{D}_1,$$

$$\mathbf{D}_0 = \begin{pmatrix} -\lambda_1 - \beta_1 & \lambda_1 \\ 0 & -\lambda_2 - \beta_2 \end{pmatrix}, \quad \mathbf{D}_1 = \begin{pmatrix} 0 & \beta_1 \\ \lambda_2 & \beta_2 \end{pmatrix}.$$

The customer service time is constant with value $B = \tau$, and hence

$$\widehat{\mathbf{A}}(z) = \int_{t=0}^{\infty} e^{(\mathbf{D}_0 + z\mathbf{D}_1)t} dB(t) = e^{(\mathbf{D}_0 + z\mathbf{D}_1)\tau}.$$

The vacation time $V$ is exponential with parameter $\gamma$. It follows

$$\mathbf{U}_k = \left( (-\mathbf{D}_0 + \gamma \mathbf{I})^{-1} \mathbf{D}_1 \right)^k (-\mathbf{D}_0 + \gamma \mathbf{I})^{-1} \gamma \mathbf{I}, \quad k \geq 0.$$

We set $X = 3$ and the following parameter values:

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \beta_1 = 3, \quad \beta_2 = 4, \quad \tau = 0.01, \quad \gamma = 10.$$

Based on (49) these results in $2(X + 1) = 8$ equations, whose solution is

$$\begin{aligned}
\mathbf{m}_0 &= (0.3539940000, \quad 0.6233670000), \\
\mathbf{m}_1 &= (0.0060541700, \quad 0.0156250000), \\
\mathbf{m}_2 &= (0.0002602540, \quad 0.0006635270), \\
\mathbf{m}_3 &= (0.0000101579, \quad 0.0000260429),
\end{aligned}$$

from which

$$\begin{aligned}
\mathbf{m} &= \sum_{n=0}^{3} \mathbf{m}_n = (0.360318, \quad 0.639682), \\
\mathbf{m}^{(1)} &= \sum_{n=0}^{3} n \, \mathbf{m}_n = (0.00660515, \quad 0.0170302).
\end{aligned}$$

The following table illustrates the dependency of $\mathbf{m}$ and $\mathbf{m}^{(1)}$ on the parameter $\gamma$:

| $\gamma$ | $\mathbf{m}$ | $\mathbf{m}^{(1)}$ |
|---|---|---|
| 5 | (0.415355, 0.584645) | (0.00992439, 0.02573280) |
| 10 | (0.360318, 0.639682) | (0.00660515, 0.01703020) |
| 20 | (0.341865, 0.658135) | (0.00368462, 0.00944405) |

## 7 Final remarks

A simple numerical algorithm to solve (46) and (50) can be developed by means of consecutive manyfold solution of the corresponding system of linear equations. Starting with an initial $X$, $X$ is doubled in each iteration until the absolute error becomes less than the prescribed limit.

It is a topic of future work to investigate the numerical solutions of the system equations (46) and (50) and to evaluate the complexity of the above mentioned numerical procedure.

The phase independent vacation property **A.4** can be relaxed, and hence the presented analysis can be extended to the case, when the vacation period depends on at least the phase of the $BMAP$.

Moreover the model can be also extended by handling further quantities like the set-up time or repair time.

## References


1. B. T. Doshi, Queueing systems with vacations - a survey, *Queueing Systems,* **1** (1986), 29-66.
2. H. Takagi, Queueing Analysis - A Foundation of Performance Evaluation, Vacation and Prority Systems, vol.1, *North-Holland, New York,* (1991).
3. D. L. Lucantoni, New results on the single server queue with a batch Markovian arrival process, *Stochastic Models,* **7** (1991), 1-46.
4. M. F. Neuts, Structured stochastic matrices of $M/G/1$ type and their applications, *Marcel Dekker, New York,* (1989).
5. M. F. Neuts, Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach, *The John Hopkins University Press, Baltimore,* (1981).
6. D. L. Lucantoni, The BMAP/G/1 queue: A tutorial, *Models and Techniques for Performance Evaluation of Computer and Communications Systems, L. Donatiello and R. Nelson Editors, Springer Verlag,* (1993).
7. S. H. Chang and T. Takine, Factorization and Stochastic Decomposition Properties in Bulk Queues with Generalized Vacations, *Queueing Systems* **50** (2005), 165-183.
8. S. H. Chang, T. Takine, K. C. Chae and H. W. Lee, A unified queue length formula for $BMAP/G/1$ queue with generalized vacations, *Stochastic Models,* **18** (2002), 369-386.
9. J. M. Ferrandiz, The BMAP/G/1 queue with server set-up times and server vacations, *Adv. Appl. Prob.,* **25** (1993), 235-254.
10. Y. W. Shin and C. E. M. Pearce, The BMAP/G/1 vacation queue with queue-length dependent vacation schedule, *J. Austral. Math. Soc.,* **Ser. B 40** (1998), 207-221.
11. A. D. Banik, U. C. Gupta and S. S. Pathak, BMAP/G/1/N queue with vacations and limited service discipline *Applied Mathematics and Computation,* **180** (2006), 707-721.
12. S. W. Fuhrmann and R. B. Cooper, Stochastic Decompositions in the M/G/1 Queue with Generalized Vacations *Operations Research,* **33** (1985), 1117-1129.