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Unified analysis of $BMAP/G/1$ cyclic polling model

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Abstract This paper presents an analysis of the $BMAP/G/1$ cyclic polling model with gated and exhaustive service disciplines. The applied methodology is based on the separation of the analysis into service discipline independent and dependent parts. New expressions are derived for the vector generating function of the stationary number of customers and for its mean in terms of vector quantities depending on the service discipline. They are valid for a broad class of service disciplines and both for zero- and nonzero-switchover-times polling models.

We present the service discipline specific solution for the nonzero-switchover-times model with gated and exhaustive service disciplines. We set up the governing equations of the system by using Kronecker product notation. They can be numerically solved by means of a system of linear equations. The resulted vectors are used to compute the service discipline specific vector quantities.

Keywords queueing theory · polling model · BMAP · stationary relationship · service discipline

Mathematics Subject Classification (2000) 60K25

1 Introduction

The classical polling model is a single-server queuing model, in which the server attends the N stations in cyclic manner. For the various analysis meth-

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ods of polling systems we refer to the excellent survey of Takagi [1]. Among the numerous model variants the one with (batch) Poisson arrival process is dominant.

The natural generalization of the batch Poisson arrival process is the batch Markovian arrival process ($BMAP$) introduced by Lucantoni [2].

Most of the analysis of $BMAP/G/1$ queueing models are based on matrix analytic-methods pioneered by Neuts [3], utilizing the $M/G/1$-type structure of the embedded Markov chain at customer departure epochs. It provides the vector probability-generating function (PGF or vector GF) of the number of customers at departure or at an arbitrary epoch. Determination of the waiting time distribution requires more effort, see Kasahara, Takine, Takahashi and Hasegawa [4].

Chang, Takine, Chae and Lee [5] established a factorization property for the $BMAP/G/1$ queues with generalized vacations. The vector GF of the stationary queue length is factored into two PGFs of proper random variables. One of them is the vector GF of the conditional stationary queue length given that the server is on vacation (or idle). Chang and Takine [6] applied the factorization property together with matrix analytic-method to get analytical results for several fundamental $BMAP/G^B/1$ queueing models with generalized vacations and with exhaustive discipline.

In most of the $BMAP/G/1$ vacation models the number of customers and the phase of $BMAP$ at customer departure epochs does not form an $M/G/1$-type Markov chain. To form a Markov chain at least one additional discrete variable is needed, which describes the service discipline. An example for this behavior is the gated discipline, for which this variable is the number of customers not yet served from those present at the start of the service of the particular station. Hence for the analysis of an important class of $BMAP/G/1$ vacation models the standard matrix analytic-method is not appropriate in its current form. One of the other method used in the studies of vacation queues is the supplementary variable technique, see Banik et al. [7], who studied the $BMAP/G/1/N$ queue with vacations and E-limited service discipline.

Takine [8] has analyzed the nonpreemptive priority queue with MAP arrivals. In this work he also pointed out, that the matrix analytic-method allows only one random variable having countable infinite space to describe the system dynamics. However the priority queueing model requires mutually dependent random variables, each of which is defined in the countable infinite space, i.e. the number of customers in each priority class. Consequently the matrix analytic-method alone is not enough to analyze such a model. Hence for the analysis of this model he also utilized a relation among the stationary PGF of the queue length at the embedded customer departure epochs and of the time-average queue length, which was established by Takine and Takahashi [9]. Nishimura [10] gave a spectral method for the nonpreemptive $BMAP/G/1$ priority queue with two priority classes. The nonpreemptive priority queue models of [9] and [10] are slightly different from the zero-swithover times polling model with exhaustive service, since they have a common underlying arrival process forming a finite state continuous-time Markov chain (CTMC).

Summarizing all these so far, mainly different single-server vacation and priority queueing models have been analyzed among $BMAP/G/1$ queueing models, but polling models have not been yet.

The principal goal of this paper is to provide a unified methodology for analyzing polling models with $BMAP$s and applying it for the most important gated and exhaustive disciplines as examples.

The $BMAP/G/1$ cyclic polling models can be seen as the generalization of both the corresponding vacation and priority queueing models. Hence they inherit both of the above-mentioned analytical difficulties, i.e., for most of these models the standard matrix analytic-method is not appropriate in its current form and these models require the analysis of dependent random variables. To overcome these drawbacks we separate the analysis into two parts treating only one of the above mentioned difficulties in both of them:

- Service discipline independent part :
  In this part we establish the relation of the vector GF of the stationary number of customers at a particular station in terms of the vector GFs of the stationary number of customers at server arrival $\widehat{\mathbf{f}}_i(z)$ and departure epochs $\widehat{\mathbf{m}}_i(z)$, i.e., $\widehat{\mathbf{q}}_i(z) = \mathcal{F}\left(\widehat{\mathbf{f}}_i(z), \widehat{\mathbf{m}}_i(z)\right)$. In its derivation we rely on the above mentioned stationary relationship from Takine and Takahashi [9] ($\widehat{\mathbf{q}}_i(z) = \mathcal{H}\left(\widehat{\mathbf{q}}_i^d(z)\right)$). To relate $\widehat{\mathbf{q}}_i^d(z)$ to $\widehat{\mathbf{f}}_i(z)$ and $\widehat{\mathbf{m}}_i(z)$ we utilize only the evolution of the system from the i-polling epoch to the i-departure epoch of the same cycle. During this period the service process is completely independent of the other stations. Therefore in this part we consider only the number of customers at the particular station without representing the state of the other stations in the notation. This way the inter-dependency of stations is not needed and hence it is not visible in this part of the paper (sections 3 and 4) .
- Service discipline dependent part :
  The solution of the service discipline independent part enables to describe the system dynamics for a concrete discipline by mutually dependent discrete random variables (number of customers and the phase of the arrival BMAP at the stations) at server arrival and departure epochs. We setup the governing equations of the system at these epochs in terms of the joint PGFs of the stationary number of customers and the phases of the BMAPs at every stations, i.e., $\widehat{\mathbf{f}}_i(z_1, \ldots, z_N) \to \widehat{\mathbf{m}}_i(z_1, \ldots, z_N)$ and $\widehat{\mathbf{m}}_i(z_1, \ldots, z_N) \to \widehat{\mathbf{f}}_{i+1}(z_1, \ldots, z_N)$. The required quantities at server arrival and departure epochs are computed from the solution of these governing equations. In this part of the paper (section 5) the inter-dependencies of stations are also captured due to the treatment of the mutually dependent discrete random variables.

This methodology can be also seen as a generalization of the one, which has also been used for analyzing cyclic polling models with Poisson arrival in [11].

The contributions of this paper are twofold. The first contribution is the new expressions for the vector GF of the stationary number of customers and a new formula for its mean in terms of the vector GFs of the stationary number of customers at server arrival and departure epochs and their factorial moments, respectively. The derivation of the new vector GF formula is based on the generalization of the former result of Eisenberg [12], while newly established properties of model specific key matrices are utilized in the derivation of the formula for its mean. The new expressions hold for a wide variety of service disciplines (service discipline independent part) and both for zero-switchover-times and nonzero-switchover-times polling models. The new formulas enable the application of the above described two step methodology and thus they open the way for analyzing polling models with $BMAP$s and several disciplines.

The second one is the analysis of the $BMAP/G/1$ cyclic nonzero-switchover-times polling model with gated and exhaustive disciplines. By using Kronecker product notation we generalize the buffer occupancy method for our model to set up the governing equations of the system in terms of joint PGFs of the stationary number of customers and the phases of the $BMAP$s at server arrival and departure epochs. The description of the system by mutually dependent random variables at server arrival and departure epochs results in a simpler mathematical structure comparing to the possible descriptions at other system epochs (like e.g. at customer departure times) or to the application of other methods (like e.g. the supplementary variable technique). The governing equations of the system can be numerically solved by means of system of linear equations and afterwards the required quantities at the server arrival and departure epochs can be computed.

The rest of this paper is organized as follows. In section 2 we introduce the model and the notations. In section 3 we derive the vector GF of the stationary number of customers. The new formula for the mean of the stationary number of customers is established in section 4. In section 5 the analysis of the nonzero-switchover-times polling model with gated and exhaustive service disciplines follows. Numerical considerations and illustrative numerical examples are provided in Section 6. We give our final remarks in section 7. Finally the Appendix with properties of model specific key matrices and with the proof of the new mean formula closes the paper.

2 Model and Notation

2.1 $BMAP$ process

We give a brief summary on the $BMAP$ related definitions and notations. For more details we refer to [2].

In the $BMAP$ the arrivals are governed by a background CTMC, which is referred to as phase process. The state of the phase process is called phase. $\Lambda(t)$ is the count of the number of arrivals in $(0,t]$ and $J(t)$ is the phase at time t. The $BMAP$ batch arrival process is characterized by the $\{(\Lambda(t), J(t))\,; t \geq 0\}$ bivariate CTMC on the state space $\{0,1,\ldots\} \times \{1,2,\ldots,L\}$ where $(\Lambda(t) \in \{0,1,\ldots\}, J(t) \in \{1,2,\ldots,L\})$. It has the following infinitesimal generator:

$$\begin{pmatrix} \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \mathbf{D}_3 & \ldots \\ \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \mathbf{D}_2 & \ldots \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_0 & \mathbf{D}_1 & \ldots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\mathbf{0}$ and $\mathbf{D}_k (k \geq 0)$ are $L \times L$ matrices.

$\mathbf{D}_0$ and $\mathbf{D}_k (k \geq 1)$ govern the transitions corresponding to no arrivals and to batch arrivals with size k, respectively.

The irreducible infinitesimal generator of the phase process is $\mathbf{D} = \sum_{k=0}^{\infty} \mathbf{D}_k$. Let $\boldsymbol{\pi}$ be the stationary probability vector of the phase process. Then $\boldsymbol{\pi}\mathbf{D} = \mathbf{0}$ and $\boldsymbol{\pi}\mathbf{e} = 1$ uniquely determine $\boldsymbol{\pi}$, where $\mathbf{e}$ is the column vector having all elements equal to one. It follows that

$$rank\,(\mathbf{D}) = L - 1. \tag{1}$$

The diagonal matrix $\mathbf{D}_0^d$ is composed by the diagonal elements of $\mathbf{D}_0$. Utilizing the non-singularity of $\mathbf{D}_0^d$, matrix $\mathbf{D}$ can be expressed as follows:

$$\mathbf{D} = \mathbf{D}_0^d + \left(\mathbf{D} - \mathbf{D}_0^d\right) = \left(\mathbf{D}_0^d\right) \left(\mathbf{I} - \left(-\mathbf{D}_0^d\right)^{-1} \left(\mathbf{D} - \mathbf{D}_0^d\right)\right), \tag{2}$$

where $\mathbf{I}$ denotes the identity matrix. Matrix $\boldsymbol{\Psi}$ is defined as

$$\boldsymbol{\Psi} = \left(-\mathbf{D}_0^d\right)^{-1} \left(\mathbf{D} - \mathbf{D}_0^d\right). \tag{3}$$

$\boldsymbol{\Psi}$ is stochastic, since $\boldsymbol{\Psi}\mathbf{e} = \mathbf{e}$ and its elements are non-negative. It can be interpreted as the transition probability matrix of embedded events of $BMAP$ state changes (either phase change or arrival). Applying (3) $\mathbf{D}$ can be expressed in terms of $\boldsymbol{\Psi}$ by

$$\mathbf{D} = \left(\mathbf{D}_0^d\right) (\mathbf{I} - \boldsymbol{\Psi}). \tag{4}$$

The matrix generating function (matrix GF) of $\mathbf{D}_k$, $\widehat{\mathbf{D}}(z)$ is defined as

$$\widehat{\mathbf{D}}(z) = \sum_{k=0}^{\infty} \mathbf{D}_k z^k, \quad |z| \leq 1. \tag{5}$$

The stationary arrival rate of a BMAP is

$$\lambda = \boldsymbol{\pi} \left. \frac{d\widehat{\mathbf{D}}(z)}{dz} \right|_{z=1} \mathbf{e} = \boldsymbol{\pi} \sum_{k=0}^{\infty} k\mathbf{D}_k \mathbf{e}. \tag{6}$$

We assume that λ is finite, hence $0 < \lambda < \infty$.

$P_{j,l}(k,t)$ denotes the probability $Pr\{\Lambda(t) = k, J(t) = l | \Lambda(0) = 0, J(0) = j\}$ and it is the (j,l)-th element of an $L \times L$ matrix $\mathbf{P}(k,t)$. It is shown in [2] that

$$\widehat{\mathbf{P}}(z,t) = \sum_{k=0}^{\infty} \mathbf{P}(k,t) z^k = e^{\widehat{\mathbf{D}}(z)t}. \tag{7}$$

So far we used the conventional $BMAP$ notations. From now on the first subscript stands for the station index, i.e. $\mathbf{D}_i$ denotes matrix $\mathbf{D}$ of the $BMAP$ at station i. In the rest of the paper we use only $\widehat{\mathbf{D}}_i(z)$, $\mathbf{D}_i$, λ_i and $\widehat{\mathbf{P}}_i(z,t)$ among the $BMAP$ notations.

2.2 The $BMAP/G/1$ cyclic polling model

We consider a continuous-time asymmetric polling model with N stations [1]. A single server attends the stations in a cyclic manner. Each station has an infinite buffer queue, which is served when the server attends that station. If no customer is present at a station at server arrival, the server immediately attends the next station. In this model, each station can have different service discipline (mixed-discipline system). At each station batch of customers arrive according to $BMAP$ process. We call the $BMAP$ arrival process at station i as i-th $BMAP$ arrival process and λ_i denotes its stationary arrival rate. The customer who arrives to station i is called i-customer. The customer service times at station i are general independent and identically distributed. B_i stands for the customer service time at station i and $\widetilde{B}_i(s)$, $B_i(t)$, b_i, $b_i^{[2]}$ denote its Laplace-Stieljes transform (LST), its cumulated distribution function and its first two moments, respectively. The mean service time is positive and finite at each station, $0 < b_i < \infty$. The model enables both zero-switchover-times and nonzero-switchover-times. R_i denotes the switchover time from station i to the next one. The R_i switchover times of the consecutive cycles are general independent and identically distributed. $\widetilde{R}_i(s)$, $R_i(t)$ and r_i are its LST, cumulated distribution function and its mean, respectively. The arrival processes, the service times and the switchover times are mutually independent. The server utilization at station i and the overall utilization are $\rho_i = \lambda_i b_i$ and $\rho = \sum_{i=1}^{N} \rho_i$, respectively. We assume that all stations of the polling system is stable (for stability condition see Saffer and Telek [13]).

Definition 1 Polling epoch, departure epoch: The arrival of the server to a station and the departure of the server from a station are called *polling epoch* and *departure epoch*, respectively. We call the polling epoch of station i as i-polling epoch. Similarly the departure epoch of station i is an i-departure epoch.

Definition 2 Station time: The *station time* of a given station is defined as the time elapsed from the arrival of the server to station i until its next departure. The station time of station i is called i-station time.

Definition 3 Intervisit time: The *intervisit time* of a given station is defined as the time elapsed from the departure of the server from station i until its next arrival to the same station. The intervisit time of station i is called i-intervisit time.

Definition 4 Cycle time: The *cycle time* of a given station is defined as the time elapsed from the server visit to station i in the actual cycle to the server visit to the same station in the next cycle. It is also called as polling cycle. The polling cycle of station i is called i-polling cycle.

$adj\mathbf{Y}$, $det\mathbf{Y}$ and $Tr(\mathbf{Y})$ denote the adjugate, the determinant and the trace of matrix $\mathbf{Y}$, respectively. Furthermore $[\mathbf{Y}]_{j,l}$ stands for the j,l-th element of matrix $\mathbf{Y}$. Similarly $[\mathbf{y}]_j$ denotes the j-th element of vector $\mathbf{y}$.

When $\widehat{\mathbf{Y}}(z)$, $|z| \leq 1$ is a matrix GF, $\mathbf{Y}^{(k)}$ denotes its k-th ($k \geq 1$) factorial moment, i.e., $\mathbf{Y}^{(k)} = \frac{d^k}{dz^k}\widehat{\mathbf{Y}}(z)|_{z=1}$ and $\mathbf{Y}$ denotes its value at $z = 1$, i.e., $\mathbf{Y} = \widehat{\mathbf{Y}}(1)$. Similarly when $\widehat{\mathbf{y}}(z)$, $|z| \leq 1$ is a vector GF, $\mathbf{y}^{(k)}$ denotes its k-th ($k \geq 1$) factorial moment, i.e., $\mathbf{y}^{(k)} = \frac{d^k}{dz^k}\widehat{\mathbf{y}}(z)|_{z=1}$ and $\mathbf{y}$ denotes its value at $z = 1$, i.e., $\mathbf{y} = \widehat{\mathbf{y}}(1)$. We also use notation $y^{(k)} = \mathbf{y}^{(k)}\mathbf{e}$.

2.3 The number of arrivals during a customer service time

Let $\Lambda_i(t)$ and $J_i(t)$ be the number of arrivals of the i-th $BMAP$ in interval $(0, t]$ and the phase of the i-th $BMAP$ at time t, respectively. $\Lambda_i(t)$ and $J_i(t)$ are right continuous, and thus every trajectories of every quantities based on them are also right continuous. t_i^s and t_i^{d-} denote the instants at the start of the service and just before the departure of any i-customer, respectively.

We define matrix $\mathbf{A}_i(k)$, whose (j, l)-th element, for $k \geq 0$, $1 \leq j, l \leq L$, is given by

$$[\mathbf{A}_i(k)]_{j,l} = P\left\{\Lambda_i(t_i^{d-}) = k, J_i(t_i^{d-}) = l | \Lambda_i(t_i^s) = 0, J_i(t_i^s) = j\right\},$$

and it is interpreted as the conditional probability that during an i-customer service time the number of i-th $BMAP$ arrivals is k and the final phase of the i-th $BMAP$ is l given that the initial phase of the i-th $BMAP$ is j. Matrix GF $\widehat{\mathbf{A}}_i(z)$ is defined as

$$\widehat{\mathbf{A}}_i(z) = \sum_{k=0}^{\infty} \mathbf{A}_i(k)z^k, \quad |z| \leq 1. \tag{8}$$

By applying the definition of $\widehat{\mathbf{P}}_i(z,t)$ and using (7) $\widehat{\mathbf{A}}_i(z)$ can be explicitly expressed by

$$\widehat{\mathbf{A}}_i(z) = \int_{t=0}^{\infty} e^{\widehat{\mathbf{D}}_i(z)t} dB_i(t) = E[e^{\widehat{\mathbf{D}}_i(z)B_i}], \tag{9}$$

see [2].

The elements of $\mathbf{A}_i$ are interpreted as the probabilities of the phase transitions of the i-th $BMAP$ from the start to the end of an i-customer service time, and therefore $\mathbf{A}_i$ is stochastic. It follows that the structures of matrix $(\mathbf{I} - \mathbf{A}_i)$ and matrix $\mathbf{D}_i$ (see (4)) are similar. Utilizing the non-singularity of the diagonal matrix in (4) and applying (1) yields

$$rank\,(\mathbf{I} - \mathbf{A}_i) = L - 1. \tag{10}$$

2.4 Service discipline

Definition 5 The *service discipline* gives the condition on the beginning and on the end of the service at a given station.

The most commonly known disciplines are, e.g., exhaustive, gated, binomial-exhaustive, binomial-gated, non-exhaustive, semi-exhaustive, limited-N and nonpreemptive limited-T. For more details see [1].

The allowed service disciplines have the following properties:

P.1 Memoryless property: In general the service discipline is independent of the history of the system.

P.2 Work-conservation property: If the service begins at the actual station, then it is work conserving up to the end of the service at that station according to the used discipline.

P.3 Nonpreemptive service property: The service is nonpreemptive. Hence the service of the customer under service, if any, is finished before the server departs from a station.

P.4 Determination property: If the service discipline, the arrival process and the customer service time of station i are given, then the number of i-customers and the phase of the i-th $BMAP$ arrival process at the i-polling epoch completely determine the number of i-customers served at that station, the duration of that service as well as the number of i-customers and the phase of the i-th $BMAP$ at the i-departure epoch in stochastic sense. Additionally for each $k \neq i$ the number of k-customers, the number of i-customers and the phase of the k-th and i-th $BMAP$ arrival processes at the i-polling epoch determine the number of k-customers and the phase of the k-th $BMAP$ at the i-departure epoch in stochastic sense.

A numerous service disciplines satisfy the properties **P.1-P.4**. For example all the above mentioned examples fulfill these properties.

3 Stationary number of i-customers

In this section first we present a general stationary relationship called "Fundamental relation", from which we derive a relation for the vector GF of the stationary number of i-customers in terms of the vector GF of the stationary number of i-customers at the i-polling and i-departure epochs. Both relations are valid for a class of service disciplines satisfying the properties **P.1-P.4**.

3.1 Fundamental relationship

Let $G_i(\ell)$ denote the number of i-customer services during the ℓ-th cycle, for $\ell \geq 1$. We define the mean stationary number of i-customers served during a polling cycle as $g_i = \lim_{k\to\infty} \frac{E\left[\sum_{\ell=1}^{k} G_i(\ell)\right]}{k}$.

Let $N_i(t)$ be the right continuous number of i-customers in the system at time t. Furthermore, $t_i^f(\ell)$ and $t_i^m(\ell)$ denote the i-polling epoch and the i-departure epoch in the ℓ-th cycle, respectively.

We define the $1 \times L$ vector $\mathbf{c}_i^f\,(k,n)$, whose j-th element represents the number of i-polling epochs in the first k polling cycle, at which the number of i-customers is n, and the phase of the i-th $BMAP$ is j. That is, for $k \geq 1$, $n \geq 0$ and $1 \leq j \leq L$,

$$[\mathbf{c}_i^f\,(k,n)]_j = \sum_{\ell=1}^{k} 1_{\left(N_i(t_i^f(\ell))=n\right)} 1_{\left(J_i(t_i^f(\ell))=j\right)},$$

where $1_{(\mathrm{con})}$ denotes the indicator of condition "con". The corresponding stationary vector GF, $\widehat{\mathbf{f}}_i\,(z)$ is defined as

$$\widehat{\mathbf{f}}_i\,(z) = \sum_{n=0}^{\infty} \lim_{k\to\infty} \frac{E[\mathbf{c}_i^f\,(k,n)]}{k} z^n, \quad |z| \leq 1.$$

Similarly we define the $1\times L$ vector $\mathbf{c}_i^m\,(k,n)$, whose j-th element represents the number of i-departure epochs in the first k polling cycle, at which the number of i-customers is n, and the phase of the i-th $BMAP$ equals j. That is, for $k \geq 1$, $n \geq 0$ and $1 \leq j \leq L$,

$$[\mathbf{c}_i^m\,(k,n)]_j = \sum_{\ell=1}^{k} 1_{\left(N_i(t_i^m(\ell))=n\right)} 1_{\left(J_i(t_i^m(\ell))=j\right)},$$

and the corresponding stationary vector GF, $\widehat{\mathbf{m}}_i\,(z)$ is defined as

$$\widehat{\mathbf{m}}_i\,(z) = \sum_{n=0}^{\infty} \lim_{k\to\infty} \frac{E[\mathbf{c}_i^m\,(k,n)]}{k} z^n, \quad |z| \leq 1.$$

Additionally $t_i^s(\ell, r)$ and $t_i^d(\ell, r)$ denote the instants at the service start and at the departure of the r-th i-customer in the ℓ-th cycle, for $\ell \geq 1$ and $1 \leq r \leq G_i(\ell)$, respectively.

We define the $1 \times L$ vector $\mathbf{c}_i^s(k, n)$, whose j-th element represents the number of i-customer service starts in the first k polling cycle, at which the number of i-customers is n, and the phase of the i-th $BMAP$ equals j. That is, for $k \geq 1$, $n \geq 0$ and $1 \leq j \leq L$,

$$[\mathbf{c}_i^s(k, n)]_j = \sum_{\ell=1}^{k} \sum_{r=1}^{G_i(\ell)} 1_{\left(N_i(t_i^s(\ell,r))=n\right)} 1_{\left(J_i(t_i^s(\ell,r))=j\right)},$$

and the corresponding stationary vector GF, $\widehat{\mathbf{q}}_i^s(z)$ is defined as

$$\widehat{\mathbf{q}}_i^s(z) = \sum_{n=0}^{\infty} \lim_{k \to \infty} \frac{E[\mathbf{c}_i^s(k, n)]}{E[\sum_{\ell=1}^{k} G_i(\ell)]} z^n, \quad |z| \leq 1.$$

We also define the $1 \times L$ vector $\mathbf{c}_i^d(k, n)$, whose j-th element represents the number of i-customer departures in the first k polling cycle, at which the number of i-customers is n, and the phase of the i-th $BMAP$ equals j. That is, for $k \geq 1$, $n \geq 0$ and $1 \leq j \leq L$,

$$[\mathbf{c}_i^d(k, n)]_j = \sum_{\ell=1}^{k} \sum_{r=1}^{G_i(\ell)} 1_{\left(N_i(t_i^d(\ell,r))=n\right)} 1_{\left(J_i(t_i^d(\ell,r))=j\right)},$$

and the corresponding stationary vector GF, $\widehat{\mathbf{q}}_i^d(z)$ is defined as

$$\widehat{\mathbf{q}}_i^d(z) = \sum_{n=0}^{\infty} \lim_{k \to \infty} \frac{E[\mathbf{c}_i^d(k, n)]}{E[\sum_{\ell=1}^{k} G_i(\ell)]} z^n, \quad |z| \leq 1.$$

Theorem 1 *(Fundamental relationship.) In the stable $BMAP/G/1$ cyclic polling model with service disciplines satisfying properties* **P.1** - **P.4**. *the following relation holds among the vector GFs of the stationary number of i-customers at station i, at different instants:*

$$\widehat{\mathbf{q}}_i^s(z) - \widehat{\mathbf{q}}_i^d(z) = \frac{1}{g_i} \left(\widehat{\mathbf{f}}_i(z) - \widehat{\mathbf{m}}_i(z) \right). \tag{11}$$

Proof. The proof utilizes an early observation of Eisenberg [12]. We adopt and slightly modify his argument, generalize it to vector quantities and apply it for our polling model. The crucial observation is that each time an i-polling epoch or i-customer departure occurs, this coincides with either an i-customer service start or an i-departure epoch (see Figure 1).

Therefore the following relation holds for the ℓ-th cycle, for $\ell \geq 1$:

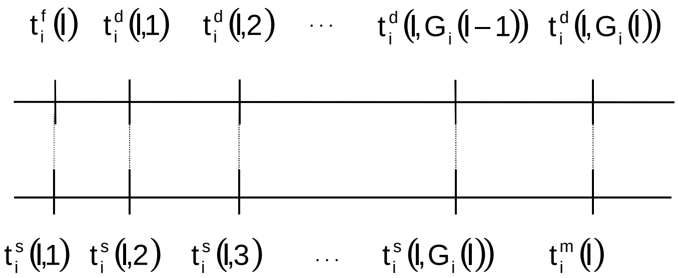

Fig. 1 Coincidence of characteristic epochs at station i in the l-th polling cycle

$$\begin{aligned} & 1_{\left(N_i\left(t_i^f(\ell)\right)=n\right)} 1_{\left(J_i\left(t_i^f(\ell)\right)=j\right)} + \sum_{r=1}^{G_i(\ell)} 1_{\left(N_i\left(t_i^d(\ell,r)\right)=n\right)} 1_{\left(J_i\left(t_i^d(\ell,r)\right)=j\right)} \\ & = \sum_{r=1}^{G_i(\ell)} 1_{\left(N_i\left(t_i^s(\ell,r)\right)=n\right)} 1_{\left(J_i\left(t_i^s(\ell,r)\right)=j\right)} + 1_{\left(N_i\left(t_i^m(\ell)\right)=n\right)} 1_{\left(J_i\left(t_i^m(\ell)\right)=j\right)}. \end{aligned} \quad (12)$$

Summing from $\ell = 1$ to k on both sides of (12) and applying the definitions of $\mathbf{c}_i^f(k,n)$, $\mathbf{c}_i^d(k,n)$, $\mathbf{c}_i^s(k,n)$ and $\mathbf{c}_i^m(k,n)$ leads to

$$\mathbf{c}_i^f(k,n) + \mathbf{c}_i^d(k,n) = \mathbf{c}_i^s(k,n) + \mathbf{c}_i^m(k,n). \quad (13)$$

We take the expectations of all four terms in (13), divide them by the expectation of the total number of i-customer departures in the first k polling cycle ($E[\sum_{\ell=1}^k G_i(\ell)]$) and take the limit for $k \to \infty$. Thus we get a relation among the four stationary probabilities for each j phase of the i-th $BMAP$. In terms of vector generating functions, this yields

$$\gamma_i \widehat{\mathbf{f}}_i(z) + \widehat{\mathbf{q}}_i^d(z) = \widehat{\mathbf{q}}_i^s(z) + \gamma_i \widehat{\mathbf{m}}_i(z). \quad (14)$$

Here $\gamma_i = \lim_{k\to\infty} \frac{k}{E[\sum_{\ell=1}^k G_i(\ell)]}$ is the long-term ratio of the number of i-polling epochs to the number of i-customer departures, which equals $\frac{1}{g_i}$. Substituting it into (14) and rearranging it results in the statement. □

Let $S_i(\ell)$ be the i-station time in the ℓ-th polling cycle, $\ell \geq 1$. $S_i(\ell) = \sum_{r=1}^{G_i(\ell)} B_i$ and $G_i(\ell)$ is stopping time, since it does not depend on the i-th customer service times after the i-departure epoch in the ℓ-th polling cycle. Hence Wald's equation can be applied, which yields

$$E[S_i(\ell)] = E[G_i(\ell)]b_i. \quad (15)$$

The number of arriving i-customers during $S_i(\ell)$ is given by $\Lambda_i(t_i^m(\ell)) - \Lambda_i(t_i^f(\ell))$. The stationary arrival rate of the i-th $BMAP$ during the i-station time, λ_i^S is defined as

$$\lambda_i^S = \lim_{k\to\infty} \frac{E\left[\sum_{\ell=1}^{k}(\Lambda_i(t_i^m(\ell)) - \Lambda_i(t_i^f(\ell)))\right]}{E\left[\sum_{\ell=1}^{k} S_i(\ell)\right]}.$$

Furthermore we define

$$\rho_i^S = \lambda_i^S b_i. \tag{16}$$

Corollary 1 *(Mean equilibrium relationship.) In the stable $BMAP/G/1$ cyclic polling model with service disciplines satisfying properties* **P.1** - **P.4** *the following relation holds for the mean stationary quantities:*

$$f_i^{(1)} - m_i^{(1)} = \left(1 - \rho_i^S\right) g_i. \tag{17}$$

Proof. Let $[\mathbf{q}_i^s]^{(1)}$ and $[\mathbf{q}_i^d]^{(1)}$ denote the first derivatives of $\widehat{\mathbf{q}}_i^s(z)$ and $\widehat{\mathbf{q}}_i^d(z)$ at $z = 1$, respectively, i.e., $[\mathbf{q}_i^s]^{(1)} = \left.\frac{d(\widehat{\mathbf{q}}_i^s(z))}{dz}\right|_{z=1}$ and $[\mathbf{q}_i^d]^{(1)} = \left.\frac{d\left(\widehat{\mathbf{q}}_i^d(z)\right)}{dz}\right|_{z=1}$. Taking the first derivative of (11) at $z = 1$ and post-multiplying it by $\mathbf{e}$ yields

$$[\mathbf{q}_i^s]^{(1)}\mathbf{e} - [\mathbf{q}_i^d]^{(1)}\mathbf{e} = \frac{1}{g_i}\left(f_i^{(1)} - m_i^{(1)}\right). \tag{18}$$

The number of i-customers at i-customer departure equals the number of i-customers at the previous i-customer service start plus those who arrived in between minus the one who left the system at the current i-customer departure. Using it and also (15) we have

$$\begin{aligned}
[\mathbf{q}_i^d]^{(1)}\mathbf{e} &= \lim_{k\to\infty} \frac{E\left[\sum_{\ell=1}^{k}\sum_{r=1}^{G_i(\ell)} N_i(t_i^d(\ell,r))\right]}{E\left[\sum_{\ell=1}^{k} G_i(\ell)\right]} = \lim_{k\to\infty} \frac{E\left[\sum_{\ell=1}^{k}\sum_{r=1}^{G_i(\ell)} N_i(t_i^s(\ell,r))\right]}{E\left[\sum_{\ell=1}^{k} G_i(\ell)\right]} \\
&+ \lim_{k\to\infty} \frac{E\left[\sum_{\ell=1}^{k}\sum_{r=1}^{G_i(\ell)}(\Lambda_i(t_i^d(\ell,r)) - \Lambda_i(t_i^s(\ell,r)))\right] b_i}{E\left[\sum_{\ell=1}^{k} G_i(\ell)\right] b_i} - \lim_{k\to\infty} \frac{E\left[\sum_{\ell=1}^{k}\sum_{r=1}^{G_i(\ell)} 1\right]}{E\left[\sum_{\ell=1}^{k} G_i(\ell)\right]} \\
&= [\mathbf{q}_i^s]^{(1)}\mathbf{e} + \lim_{k\to\infty} \frac{E\left[\sum_{\ell=1}^{k}(\Lambda_i(t_i^m(\ell)) - \Lambda_i(t_i^f(\ell)))\right]}{E\left[\sum_{\ell=1}^{k} S_i(\ell)\right]} b_i - 1.
\end{aligned} \tag{19}$$

Applying the defining equation of λ_i^S in (19) leads to

$$[\mathbf{q}_i^d]^{(1)}\mathbf{e} = [\mathbf{q}_i^s]^{(1)}\mathbf{e} + \lambda_i^S b_i - 1. \tag{20}$$

Substituting (20) into (18) results in

$$1 - \lambda_i^S b_i = \frac{1}{g_i}\left(f_i^{(1)} - m_i^{(1)}\right). \tag{21}$$

Applying (16) in (21) and rearranging gives the statement. □

Remark 1 (Interpretation of the mean equilibrium relationship.) The number of arriving i-customers during the i-intervisit time is $f_i^{(1)} - m_i^{(1)}$. The number of arriving i-customers during the i-station time is $\lambda_i^S b_i g_i$. Adding it to both sides of (17) the relation leads to the mean equilibrium relationship: the mean stationary number of arriving and served i-customers are the same during an i-polling cycle.

3.2 Vector GF of the stationary number of i-customers

We define vector GF of the stationary number of i-customers $\widehat{\mathbf{q}}_i(z)$ by its j-element, $1 \le j \le L$, as

$$[\widehat{\mathbf{q}}_i(z)]_j = \lim_{t\to\infty} \sum_{n=0}^{\infty} P\{N_i(t) = n, J_i(t) = j\} z^n, \quad |z| \le 1.$$

Theorem 2 *(Expression of $\widehat{\mathbf{q}}_i(z)$.) In the stable $BMAP/G/1$ cyclic polling model with service disciplines satisfying properties* **P.1** - **P.4** *the following relation holds for vector GF of the stationary number of i-customers:*

$$\widehat{\mathbf{q}}_i(z)\widehat{\mathbf{D}}_i(z)\left(z\mathbf{I} - \widehat{\mathbf{A}}_i(z)\right) = \lambda_i\left(1-\rho_i^S\right)(z-1)\frac{\widehat{\mathbf{f}}_i(z) - \widehat{\mathbf{m}}_i(z)}{f_i^{(1)} - m_i^{(1)}}\widehat{\mathbf{A}}_i(z). \quad (22)$$

Proof. The vector GF of the number of i-customers just before the departure of the i-customer is $z\widehat{\mathbf{q}}_i^d(z)$. For this vector GF the following $BMAP$ specific relation holds:

$$z\widehat{\mathbf{q}}_i^d(z) = \widehat{\mathbf{q}}_i^s(z)\widehat{\mathbf{A}}_i(z). \quad (23)$$

Post-multiplying the fundamental relationship (11) by $\widehat{\mathbf{A}}_i(z)$, applying (23) and using the mean equilibrium relationship (17) yields

$$\widehat{\mathbf{q}}_i^d(z)\left(z\mathbf{I} - \widehat{\mathbf{A}}_i(z)\right) = \left(1-\rho_i^S\right)\frac{\left(\widehat{\mathbf{f}}_i(z) - \widehat{\mathbf{m}}_i(z)\right)}{f_i^{(1)} - m_i^{(1)}}\widehat{\mathbf{A}}_i(z). \quad (24)$$

Multiplying (24) by $\lambda_i(z-1)$ leads to

$$\lambda_i(z-1)\widehat{\mathbf{q}}_i^d(z)\left(z\mathbf{I} - \widehat{\mathbf{A}}_i(z)\right) = \lambda_i\left(1-\rho_i^S\right)(z-1)\frac{\widehat{\mathbf{f}}_i(z) - \widehat{\mathbf{m}}_i(z)}{f_i^{(1)} - m_i^{(1)}}\widehat{\mathbf{A}}_i(z). \quad (25)$$

Takine and Takahashi proved the following stationary relationship between $\widehat{\mathbf{q}}_i(z)$ and $\widehat{\mathbf{q}}_i^d(z)$ under more general setting in [9]:

$$\widehat{\mathbf{q}}_i(z)\widehat{\mathbf{D}}_i(z) = \lambda_i(z-1)\widehat{\mathbf{q}}_i^d(z), \quad (26)$$

which is identical with the one for $BMAP/G/1$ queue in [2]. Applying (26) in (25) gives the theorem. □

Let $I_i(\ell)$ be the i-intervisit time in the ℓ-th polling cycle, $\ell \geq 1$. The number of i-customers arriving during $I_i(\ell)$ is given by $\Lambda_i(t_i^f(\ell+1)) - \Lambda_i(t_i^m(\ell))$. The stationary arrival rate of the i-th $BMAP$ during the i-intervisit time, λ_i^I is defined as

$$\lambda_i^I = \lim_{k\to\infty} \frac{E\left[\sum_{\ell=1}^{k} (\Lambda_i(t_i^f(\ell+1)) - \Lambda_i(t_i^m(\ell)))\right]}{E\left[\sum_{\ell=1}^{k} I_i(\ell)\right]}.$$

At a randomly chosen instant the model is in service of station i with probability ρ_i and it is in the intervisit time of station i with probability $(1-\rho_i)$. Therefore we have

$$\lambda_i = \rho_i \lambda_i^S + (1-\rho_i)\,\lambda_i^I. \tag{27}$$

Applying (16) in (27) and rearranging it yields

$$\lambda_i \left(1-\rho_i^S\right) = \lambda_i^I\,(1-\rho_i)\,. \tag{28}$$

(28) implies that the dependency on λ_i^S in (22) is equivalent with the dependency on λ_i^I.

Remark 2 (Characteristic difference between the $BMAP/G/1$ and $M/G/1$ polling models.) In the corresponding formula of the $M/G/1$ polling model (see Borst and Boxma [11]) there was no need to introduce λ_i^S or λ_i^I. This is due to the memoryless property of the Poisson process. Hence the appearance of these terms in the expression of $\widehat{\mathbf{q}}_i\,(z)$ shows a characteristic difference between the $BMAP/G/1$ and $M/G/1$ polling models.

By applying the mean equilibrium relationship (17) in (22) $\frac{\left(1-\rho_i^S\right)}{\left(f_i^{(1)}-m_i^{(1)}\right)}$ can be computed as $\frac{\left(1-\rho_i^S\right)}{\left(f_i^{(1)}-m_i^{(1)}\right)} = \frac{1}{g_i}$. For the nonzero-switchover-times polling model $g_i = \lambda_i \frac{\sum_{i=1}^{N} r_i}{1-\rho}$ also holds, and hence for this model $\frac{\left(1-\rho_i^S\right)}{\left(f_i^{(1)}-m_i^{(1)}\right)}$ can be computed in a discipline independent way.

Thus the contribution of the concrete service discipline in (22) is represented by $\widehat{\mathbf{f}}\,(z) - \widehat{\mathbf{m}}\,(z)$.

4 The mean of the stationary number of *i*-customers

This section presents the new vector mean formula for the stationary number of i-customers for the class of service disciplines satisfying properties **P.1** - **P.4**.

Matrix $\widehat{\mathbf{T}}_i(z)$ is defined as:

$$\widehat{\mathbf{T}}_i(z) = \widehat{\mathbf{D}}_i(z)\left(z\mathbf{I} - \widehat{\mathbf{A}}_i(z)\right).$$

Let $[det\mathbf{T}_i]^{(k)}$ denote the k-th ($k \geq 1$) derivative of matrix $det\widehat{\mathbf{T}}_i(z)$ at $z = 1$, i.e., $[det\mathbf{T}_i]^{(k)} = \left.\frac{d^k\left(det\widehat{\mathbf{T}}_i(z)\right)}{dz^k}\right|_{z=1}$. Similarly $[adj\mathbf{T}_i]^{(k)}$ denotes the k-th ($k \geq 1$) derivative of matrix $adj\widehat{\mathbf{T}}_i(z)$ at $z = 1$, i.e., $[adj\mathbf{T}_i]^{(k)} = \left.\frac{d^k\left(adj\widehat{\mathbf{T}}_i(z)\right)}{dz^k}\right|_{z=1}$.

Theorem 3 *(The vector mean formula.) In the stable $BMAP/G/1$ cyclic polling model with service disciplines satisfying properties* **P.1** - **P.4** *the mean of the stationary number of i-customers at an arbitrary instant is given by:*

$$\begin{aligned}
\mathbf{q}_i^{(1)} = & \frac{\mathbf{f}_i^{(2)} - \mathbf{m}_i^{(2)}}{\left(f_i^{(1)} - m_i^{(1)}\right)}\lambda_i\left(1-\rho_i^S\right)\mathbf{A}_i adj\mathbf{T}_i\frac{1}{[det\mathbf{T}_i]^{(2)}} \\
& + 2\frac{\mathbf{f}_i^{(1)} - \mathbf{m}_i^{(1)}}{\left(f_i^{(1)} - m_i^{(1)}\right)}\lambda_i\left(1-\rho_i^S\right)\mathbf{A}_i^{(1)} adj\mathbf{T}_i\frac{1}{[det\mathbf{T}_i]^{(2)}} \\
& + 2\frac{\mathbf{f}_i^{(1)} - \mathbf{m}_i^{(1)}}{\left(f_i^{(1)} - m_i^{(1)}\right)}\lambda_i\left(1-\rho_i^S\right)\mathbf{A}_i [adj\mathbf{T}_i]^{(1)}\frac{1}{[det\mathbf{T}_i]^{(2)}} \\
& + \frac{\mathbf{f}_i - \mathbf{m}_i}{\left(f_i^{(1)} - m_i^{(1)}\right)}\lambda_i\left(1-\rho_i^S\right)\mathbf{A}_i^{(2)} adj\mathbf{T}_i\frac{1}{[det\mathbf{T}_i]^{(2)}} \\
& + 2\frac{\mathbf{f}_i - \mathbf{m}_i}{\left(f_i^{(1)} - m_i^{(1)}\right)}\lambda_i\left(1-\rho_i^S\right)\mathbf{A}_i^{(1)} [adj\mathbf{T}_i]^{(1)}\frac{1}{[det\mathbf{T}_i]^{(2)}} \\
& + \frac{\mathbf{f}_i - \mathbf{m}_i}{\left(f_i^{(1)} - m_i^{(1)}\right)}\lambda_i\left(1-\rho_i^S\right)\mathbf{A}_i [adj\mathbf{T}_i]^{(2)}\frac{1}{[det\mathbf{T}_i]^{(2)}} \\
& - \frac{1}{3}\pi_i [det\mathbf{T}_i]^{(3)}\frac{1}{[det\mathbf{T}_i]^{(2)}}.
\end{aligned} \tag{29}$$

The essential statement of the theorem is that the mean of the stationary number of i-customers ($\mathbf{q}_i^{(1)}$) can be expressed explicitly by the first two factorial moments of the number of i-customers and the phase probability vectors of the i-th $BMAP$ at i-polling and i-departure epochs (($\mathbf{f}_i^{(2)} - \mathbf{m}_i^{(2)}$), ($\mathbf{f}_i^{(1)} - \mathbf{m}_i^{(1)}$) and ($\mathbf{f}_i - \mathbf{m}_i$), which are service discipline dependent.

Proof. The proof of the theorem can be found in the Appendix.

4.1 Special case of Poisson arrival process

For the case of Poisson arrival process $\rho_i^S = \rho_i$, $\widehat{\mathbf{D}}_i(z) = -(\lambda_i - \lambda_i z)$, $\widehat{\mathbf{A}}_i(z) = \widetilde{B}_i(\lambda_i - \lambda_i z)$, $\mathbf{A}_i^{(1)} = \rho_i$, $\mathbf{A}_i^{(2)} = \lambda_i^2 b_i^{[2]}$, $\widehat{\mathbf{T}}_i(z) = -(\lambda_i - \lambda_i z)\left(z - \widetilde{B}_i(\lambda_i - \lambda_i z)\right)$, $adj\widehat{\mathbf{T}}_i(z) = 1$. In addition $\frac{1}{[det\mathbf{T}_i]^{(2)}} = \frac{1}{2\lambda_i(1-\rho_i)}$ and $[det\mathbf{T}_i]^{(3)} = -3\lambda_i^3 b_i^{[2]}$. Substituting them into (29) leads to the expression of the mean stationary number of i-customers in the classical cyclic $M/G/1$ polling model:

$$q_i^{(1)} = \rho_i + \frac{\lambda_i^2 b_i^{[2]}}{2(1-\rho_i)} + \frac{f_i^{(2)} - m_i^{(2)}}{2\left(f_i^{(1)} - m_i^{(1)}\right)}.$$

Remark 3 (Dependency of the mean stationary number of i-customers on service discipline specific terms in polling model with Poisson arrivals and in polling model with $BMAP$s.) In the classical cyclic polling model with Poisson arrivals $q_i^{(1)}$ depends on the concrete service discipline via $f_i^{(2)} - m_i^{(2)}$ and $f_i^{(1)} - m_i^{(1)}$. In the polling model with $BMAP$s the corresponding relation is (29), in which $\mathbf{q}_i^{(1)}$ depends on $(\mathbf{f}_i^{(2)} - \mathbf{m}_i^{(2)})$, $(\mathbf{f}_i^{(1)} - \mathbf{m}_i^{(1)})$ and $(\mathbf{f}_i - \mathbf{m}_i)$, where $(\mathbf{f}_i - \mathbf{m}_i)\mathbf{e} = 0$.

5 Nonzero-switchover-times polling model with gated and exhaustive disciplines

In this section we turn to the service discipline specific part of our analysis approach. We restrict the $BMAP/G/1$ cyclic polling model to the nonzero-switchover-times polling model with gated and exhaustive disciplines.

To apply formula (29), the vector moments of the stationary number of i-customers at i-polling and i-departure epochs have to be determined. These moments are computed by the help of the joint probabilities of the stationary number of customers and the phases of the $BMAP$s at i-polling and i-departure epochs.

We describe the joint probabilities of the stationary number of customers and the phases of the BMAPs at i-polling and i-departure epochs as hypervectors. Notation $\otimes$ stands the Kronecker product and $\mathbf{e}_j = (0, \ldots, 1, \ldots, 0)$ denotes the $1 \times L$ vector with 1 at the j-th position. Then the $1 \times L^N$ stationary probability hypervector $\mathbf{p}_i^f(n_1, \ldots, n_N)$ is defined as

$$\mathbf{p}_i^f(n_1, \ldots, n_N) = \lim_{k\to\infty} \sum_{j_1=1}^{L} \ldots \sum_{j_N=1}^{L} \mathbf{e}_{j_1} \otimes \ldots \otimes \mathbf{e}_{j_N}$$
$$Pr\left\{N_1(t_i^f(k)) = n_1, \ldots, N_N(t_i^f(k)) = n_N, J_1(t_i^f(k)) = j_1, \ldots, J_N(t_i^f(k)) = j_N\right\},$$
$$n_1, \ldots, n_N \in \{0, 1, \ldots\}; \quad i = 1, \ldots, N.$$

Similarly the $1 \times L^N$ stationary probability hypervector $\mathbf{p}_i^m\,(n_1,\ldots,n_N)$ is defined as:

$$\begin{aligned}
&\mathbf{p}_i^m\,(n_1,\ldots,n_N) = \lim_{k\to\infty} \sum_{j_1=1}^{L} \ldots \sum_{j_N=1}^{L} \mathbf{e}_{j_1} \otimes \ldots \otimes \mathbf{e}_{j_N} \\
&Pr\Big\{ N_1(t_i^m(k)) = n_1, \ldots, N_N(t_i^m(k)) = n_N, J_1(t_i^m(k)) = j_1, \ldots, J_N(t_i^m(k)) = j_N \Big\}, \\
&n_1,\ldots,n_N \in \{0,1,\ldots\}; \quad i = 1,\ldots,N.
\end{aligned}$$

The hypervector GFs of the stationary number of customers at i-polling and i-departure epochs are defined as

$$\begin{aligned}
\widehat{\mathbf{f}}_i(z_1,\ldots,z_N) &= \sum_{n_1=0}^{\infty} \ldots \sum_{n_N=0}^{\infty} \mathbf{p}_i^f\,(n_1,\ldots,n_N)\, z_1^{n_1} \ldots z_N^{n_N}, \\
\widehat{\mathbf{m}}_i(z_1,\ldots,z_N) &= \sum_{n_1=0}^{\infty} \ldots \sum_{n_N=0}^{\infty} \mathbf{p}_i^m\,(n_1,\ldots,n_N)\, z_1^{n_1} \ldots z_N^{n_N}, \\
&\qquad i = 1,\ldots,N; \quad |z_1| \le 1,\ldots,|z_N| \le 1.
\end{aligned}$$

We remark here that the marginal vector GFs $\widehat{\mathbf{f}}_i(1,\ldots,z_i,\ldots,1)\mathbf{e} \otimes \ldots \otimes \mathbf{I} \otimes \ldots \otimes \mathbf{e}$ and $\widehat{\mathbf{m}}_i(1,\ldots,z_i,\ldots,1)\mathbf{e} \otimes \ldots \otimes \mathbf{I} \otimes \ldots \otimes \mathbf{e}$ equal to the quantities $\widehat{\mathbf{f}}_i(z)$, and $\widehat{\mathbf{m}}_i(z)$ introduced in subsection 3.1. Here $\mathbf{I}$ is at the i-th position. In spite of the fact that they are defined on slightly different way, this holds due to the finite memory properties of the Markov chains, which describe the number of customers and the phases of the $BMAP$s at the embedded i-polling and i-departure epochs, respectively.

We define a notation also for substituting an $L^N \times L^N$ hypermatrix $\boldsymbol{A}$ into the defining series of the $1 \times L^N$ hypervector GF $\widehat{\mathbf{f}}_i(z_1,\ldots,z_N)$:

$$\begin{aligned}
\widehat{\mathbf{f}}_i(z_1,\ldots,z_{i-1},\bullet\boldsymbol{A},z_{i+1},\ldots,z_N) = &\sum_{n_1=0}^{\infty} \ldots \sum_{n_N=0}^{\infty} \mathbf{p}_i^f\,(n_1,\ldots,n_N) \\
&z_1^{n_1} \ldots z_{i-1}^{n_{i-1}} \boldsymbol{A}^{n_i} z_{i+1}^{n_{i+1}} \ldots z_N^{n_N}, \quad i = 1,\ldots,N,
\end{aligned}$$

This results in also an $1 \times L^N$ hypervector.

We use notation $\oplus$ for the Kronecker sum. Additionally $\oplus_{k=1}^N \widehat{\mathbf{D}}_k(z_k)$ stands for $\widehat{\mathbf{D}}_1(z_1) \oplus \ldots \oplus \widehat{\mathbf{D}}_N(z_N)$ and we introduce further notations as follows:

$$\widehat{\mathbf{A}}_i(z_1,\ldots,z_N) = \int_0^{\infty} e^{t \oplus_{k=1}^N \widehat{\mathbf{D}}_k(z_k)} dB_i(t),$$

$$\widehat{\mathbf{C}}_i(z_1,\ldots,z_N) = \int_0^{\infty} e^{t \oplus_{k=1}^N \widehat{\mathbf{D}}_k(z_k)} dR_i(t).$$

Theorem 4 *(Governing equations of the system.) The governing equations of the stable $BMAP/G/1$ cyclic nonzero-switchover-times polling model with gated service discipline are given in terms of the hypervector GFs $\widehat{\mathbf{f}}_i(z_1,\ldots,z_N)$ and $\widehat{\mathbf{m}}_i(z_1,\ldots,z_N)$ for $i=1,\ldots,N$ as*

$$\begin{aligned}\widehat{\mathbf{m}}_i\left(z_1,\ldots,z_N\right) &= \widehat{\mathbf{f}}_i\left(z_1,\ldots,z_{i-1},\bullet\widehat{\mathbf{A}}_i(z_1,\ldots,z_N),z_{i+1},\ldots,z_N\right),\\ \widehat{\mathbf{f}}_{i+1}\left(z_1,\ldots,z_N\right) &= \widehat{\mathbf{m}}_i\left(z_1,\ldots,z_N\right)\widehat{\mathbf{C}}_i(z_1,\ldots,z_N).\end{aligned} \tag{30}$$

Proof. We generalize the buffer occupancy method (see in [1]) for our model with $BMAP$s. Under gated discipline only those i-customers are served during the service of station i, which already present at i-polling epoch. Hence those i-customers present at i-departure epoch who arrived during the service of station i. The matrix GF of the number of i-customers arriving during a service of one i-customer is given as $\widehat{\mathbf{A}}_i(z_i)$. Assuming, that the number of i-customers present at i-polling epoch is n_i, the PGF of the number of i-customers present at i-departure epoch can be expressed by:

$$\left(\widehat{\mathbf{A}}_i(z_i)\right)^{n_i}=\left(\int_0^\infty e^{\widehat{\mathbf{D}}_i(z_i)t}dB_i(t)\right)^{n_i}.$$

To describe the evolution of the k-customers ($k\neq i$), we have to take into account also the number of k-customers present at the i-polling epoch. The number of k-customers ($k\neq i$) at i-departure epoch equals the number of k-customers present at the i-polling epoch plus the number of k-customers arriving during the service of station i, which are independent once the number of i-customers and k-customers present at the i-polling epoch are given. Hence assuming, that the number of k-customers present at i-polling epoch is n_k, and the number of i-customers present at i-polling epoch is n_i, the matrix GF of the number of k-customers present at i-departure epoch can be expressed by:

$$z_k^{n_k}\left(\int_0^\infty e^{\widehat{\mathbf{D}}_k(z_k)t}dB_i(t)\right)^{n_i}.$$

To describe the evolution of both the i-customers and the k-customers ($k\neq i$) at the same time, we need the hypermatrix GF of the number of i-customers and k-customers, which arrive during the service of one i-customer.

To this end we need the following property for the Kronecker product of exponential functions of matrices:

$$e^{\widehat{\mathbf{D}}_i(z_i)}\otimes e^{\widehat{\mathbf{D}}_k(z_k)}=e^{\widehat{\mathbf{D}}_i(z_i)\otimes\mathbf{I}+\mathbf{I}\otimes\widehat{\mathbf{D}}_k(z_k)}=e^{\widehat{\mathbf{D}}_i(z_i)\oplus\widehat{\mathbf{D}}_k(z_k)}, \tag{31}$$

Using (31) we can express the hypermatrix GF of the number of i-customers and k-customers, which arrive during the service of one i-customer by

$$\int_0^\infty \left(e^{\widehat{\mathbf{D}}_i(z)t} \otimes e^{\widehat{\mathbf{D}}_k(z)t}\right) dB_i(t) = \int_0^\infty e^{(\widehat{\mathbf{D}}_i(z_i)\otimes\mathbf{I}+\mathbf{I}\otimes\widehat{\mathbf{D}}_k(z_k))t} dB_i(t)$$
$$= \int_0^\infty e^{(\widehat{\mathbf{D}}_i(z_i)\oplus\widehat{\mathbf{D}}_k(z_k))t} dB_i(t) = \widehat{\mathbf{A}}_i(z_i, z_k). \tag{32}$$

Assuming that the number of k-customers present at i-polling epoch is n_k, and the number of i-customers present at i-polling epoch is n_i, using again the previous arguments of the buffer occupancy method for the evolution of both the i-customers and k-customers at the same time and applying (32) results in the number of i-customers and k-customers present at i-departure epoch as

$$z_k^{n_k} \left(\widehat{\mathbf{A}}_i(z_i, z_k)\right)^{n_i}.$$

Repeating the same arguments now for the i-customers and for every k-customers ($k \neq i$) at the same time, after unconditioning we get the relation for the transition $f_i \to m_i$ of the gated polling model with N stations :

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \mathbf{p}_i^f (n_1, \ldots, n_N) z_1^{n_1} \ldots z_{i-1}^{n_{i-1}} \left(\widehat{\mathbf{A}}_i(z_1, \ldots, z_N)\right)^{n_i} z_{i+1}^{n_{i+1}} \ldots z_N^{n_N}$$
$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \mathbf{p}_i^m (n_1, \ldots, n_N) z_1^{n_1} \ldots z_N^{n_N}, \quad i = 1, \ldots, N. \tag{33}$$

Using similar arguments as before, for transition $m_i \to f_{i+1}$ of the nonzero-switchover-times gated polling model with N stations we get

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \mathbf{p}_i^m (n_1, \ldots, n_N) z_1^{n_1} \ldots z_N^{n_N} \widehat{\mathbf{C}}_i(z_1, \ldots, z_N)$$
$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \mathbf{p}_{i+1}^f (n_1, \ldots, n_N) z_1^{n_1} \ldots z_N^{n_N}, \quad i = 1, \ldots, N. \tag{34}$$

Applying notation $\widehat{\mathbf{f}}_i(z_1, \ldots, z_i - 1, \bullet \boldsymbol{A}, z_{i+1}, \ldots, z_N)$ and the defining equations of the hypervector GFs $\widehat{\mathbf{f}}_i(z_1, \ldots, z_N)$ and $\widehat{\mathbf{m}}_i(z_1, \ldots, z_N)$ in (33) and (34), after rearranging we get the stated equations as compact forms of the governing relations of the system. □

To compute the vector moments of the stationary number of i-customers at i-polling and i-departure epochs we need the $1 \times L$ marginal stationary probability vectors $\mathbf{p}_i^f(n_i) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_{i-1}=0}^{\infty} \sum_{n_{i+1}=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \mathbf{p}_i^f(n_1, \ldots, n_N) \mathbf{e} \otimes \ldots \otimes \mathbf{I} \otimes \ldots \otimes \mathbf{e}$ and $\mathbf{p}_i^m(n_i) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_{i-1}=0}^{\infty} \sum_{n_{i+1}=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \mathbf{p}_i^m(n_1, \ldots, n_N) \mathbf{e} \otimes \ldots \otimes \mathbf{I} \otimes \ldots \otimes \mathbf{e}$, where $\mathbf{I}$ is at the i-th position, for $i = 1, \ldots, N$. However the structure of (33) and (34) does not allow the direct determination of these quantities. Instead we can determine the stationary probability hypervectors $\mathbf{p}_i^f(n_1,\ldots,n_N)$ and $\mathbf{p}_i^m(n_1,\ldots,n_N)$ for $i=1,\ldots,N$.

We set an upper limit U for $n_1,\ldots,n_N$ in (33) and (34). Then we take their u_1-th, …, u_N-th derivatives $(u_1,\ldots,u_N \in \{0,\ldots,U\})$ at $z_1 = \ldots = z_N = 1$, respectively. This results in the following system of linear equations for $i=1,\ldots,N$ and $u_1,\ldots,u_N \in \{0,\ldots,U\}$:

$$\sum_{n_1=0}^{U}\cdots\sum_{n_N=0}^{U}\mathbf{p}_i^f(n_1,\ldots,n_N) \left.\frac{d^{u_1}\ldots d^{u_N}\left(z_1^{n_1}\ldots z_{i-1}^{n_{i-1}}\left(\widehat{\mathbf{A}}_i(z_1,\ldots,z_N)\right)^{n_i} z_{i+1}^{n_{i+1}}\ldots z_N^{n_N}\right)}{dz_1{}^{u_1}\ldots dz_N{}^{u_N}}\right|_{\mathbf{z}=1} = \sum_{n_1=u_1}^{U}\cdots\sum_{n_N=u_N}^{U}\mathbf{p}_i^m(n_1,\ldots,n_N)\frac{n_1!}{(n_1-u_1)!}\cdots\frac{n_N!}{(n_N-u_N)!}, \tag{35}$$

$$\sum_{n_1=0}^{U}\cdots\sum_{n_N=0}^{U}\mathbf{p}_i^m(n_1,\ldots,n_N)\left.\frac{d^{u_1}\ldots d^{u_N}\left(z_1^{n_1}\ldots z_N^{n_N}\widehat{\mathbf{C}}_i(z_1,\ldots,z_N)\right)}{dz_1{}^{u_1}\ldots dz_N{}^{u_N}}\right|_{\mathbf{z}=1} = \sum_{n_1=u_1}^{U}\cdots\sum_{n_N=u_N}^{U}\mathbf{p}_{i+1}^f(n_1,\ldots,n_N)\frac{n_1!}{(n_1-u_1)!}\cdots\frac{n_N!}{(n_N-u_N)!}, \tag{36}$$

where $\mathbf{z}=1$ stands for $z_1=\ldots=z_N=1$.

These linear equations relate quantities, which are essentially close to the factorial moments of the stationary number of customers at polling and departure epochs. Based on this system of linear equations a numerical method can be developed for computing the stationary probability hypervectors $\mathbf{p}_i^f(n_1,\ldots,n_N)$ and $\mathbf{p}_i^m(n_1,\ldots,n_N)$ for $i=1,\ldots,N$. In a basic realization U is increased until $\left(1-\sum_{n_1=0}^{U}\cdots\sum_{n_N=0}^{U}\mathbf{p}_i^f(n_1,\ldots,n_N)\right)$ or/and $\left(1-\sum_{n_1=0}^{U}\cdots\sum_{n_N=0}^{U}\mathbf{p}_i^m(n_1,\ldots,n_N)\right)$ becomes less than the allowed error according to the required precision. □

The number of equations and the number of unknowns in the linear system of equations ((35) and (36)) is $2NL^N(U+1)^N$.

5.2 Polling model with exhaustive discipline

We define the homogenous bivariate Markov chain $\left\{(N_i(t_i^d(r)), J_i(t_i^d(r))); r\in\{1,\ldots\}\right\}$ on the state space $(N_i(t_i^d(r)), J_i(t_i^d(r)))$, where $t_i^d(r)$ denotes the r-th i-customer departure epoch for $r\geq 1$. We define matrix $\mathbf{G}_i(t)$, $t\geq 0$, whose (j_i^1, j_i^2)-th element is given as the probability that the first passage starting from state (n_i+1, j_i^1) in the Markov chain to the state (n_i, j_i^2), $n_i \in 0,1,2,\ldots$, $1 \le j_i^1, j_i^2 \le L$, occurs no later than time t, and the first state visited in level n_i is (n_i, j_i^2).

We define an i-dependent rearrangement operator, which rearranges the positions of the elements of an $L^N \times L^N$ hypermatrix $e^{t\oplus_{k=1}^{i-1}\widehat{\mathbf{D}}_k(z_k)} \otimes e^{t\oplus_{k=i+1}^{N}\widehat{\mathbf{D}}_k(z_k)} \otimes \mathbf{G}_i(t)$ as

$$\mathcal{R}_i\left(e^{t\oplus_{k=1}^{i-1}\widehat{\mathbf{D}}_k(z_k)} \otimes e^{t\oplus_{k=i+1}^{N}\widehat{\mathbf{D}}_k(z_k)} \otimes \mathbf{G}_i(t)\right) = e^{t\oplus_{k=1}^{i-1}\widehat{\mathbf{D}}_k(z_k)} \otimes \mathbf{G}_i(t) \otimes e^{t\oplus_{k=i+1}^{N}\widehat{\mathbf{D}}_k(z_k)}.$$

By the help of this rearrangement operator we introduce the notation:

$$\widehat{\mathbf{E}}_i(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_N) = \mathcal{R}_i\left(\int_0^\infty e^{t\oplus_{k=1}^{i-1}\widehat{\mathbf{D}}_k(z_k)} \otimes e^{t\oplus_{k=i+1}^{N}\widehat{\mathbf{D}}_k(z_k)} \otimes d\mathbf{G}_i(t)\right).$$

We remark here that the elements of $\widehat{\mathbf{E}}_i(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_N)$ can be expressed by means of the elements of the matrix LST $\widetilde{\mathbf{G}}_i(s) = \int_0^\infty e^{-st}d\mathbf{G}_i(t)$.

Theorem 5 *(Governing equations of the system.) The governing equations of the stable $BMAP/G/1$ cyclic nonzero-switchover-times polling model with exhaustive service discipline are given in terms of the hypervector GFs $\widehat{\mathbf{f}}_i(z_1,\ldots,z_N)$ and $\widehat{\mathbf{m}}_i(z_1,\ldots,z_N)$ for $i = 1,\ldots,N$ as*

$$\begin{aligned}
&\widehat{\mathbf{m}}_i\left(z_1,\ldots,z_{i-1},1,z_{i+1},\ldots,z_N\right)\\
&= \widehat{\mathbf{f}}_i\left(z_1,\ldots,z_{i-1},\bullet\widehat{\mathbf{E}}_i(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_N),z_{i+1},\ldots,z_N\right),\\
&\widehat{\mathbf{f}}_{i+1}\left(z_1,\ldots,z_N\right) = \widehat{\mathbf{m}}_i\left(z_1,\ldots,z_{i-1},1,z_{i+1},\ldots,z_N\right)\widehat{\mathbf{C}}_i(z_1,\ldots,z_N).
\end{aligned} \quad (37)$$

Proof. The transition $m_i \to f_{i+1}$ is discipline independent. However in case of exhaustive discipline $n_i = 0$ at i-departure epochs. Applying it to the second part of (30) results in the second relation of (37).

To describe the transition $f_i \to m_i$ we apply a similar line of arguments as we used for the model with gated discipline. Concerning the evolution of i-customers only the evolution of the phase of i-th $BMAP$ arises, as due to the exhaustive discipline the number of i-customers is 0 at i-departure epoch.

To describe the evolution of both the phase of i-th $BMAP$ and the k-customers ($k \ne i$) at the same time, we need the hypermatrix transform of the phase of i-th $BMAP$ and k-customers, which arrive during the first passage time at station i. For the sake of notation simplicity we assume here that $k < i$. This transform is given by

$$\int_0^\infty e^{\widehat{\mathbf{D}}_k(z_k)t} \otimes d\mathbf{G}_i(t).$$

Assuming that the number of k-customers present at i-polling epoch is n_k, and the number of i-customers present at i-polling epoch is n_i, using again the arguments of the buffer occupancy method for the evolution of both the phase of i-th $BMAP$ and k-customers at the same time results in the phase of i-th $BMAP$ and k-customers present at i-departure epoch as

$$z_k^{n_k} \left(\widehat{\mathbf{E}}_i(z_k)\right)^{n_i}.$$

Repeating the same arguments now for the phase of i-th $BMAP$ and for every k-customers ($k \neq i$) at the same time, after unconditioning we get the relation for the transition $f_i \rightarrow m_i$ of the exhaustive polling model:

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \mathbf{p}_i^f\left(n_1,\ldots,n_N\right) z_1^{n_1} \ldots z_{i-1}^{n_{i-1}} \left(\widehat{\mathbf{E}}_i(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_N)\right)^{n_i} z_{i+1}^{n_{i+1}} \ldots z_N^{n_N}$$
$$= \sum_{n_1=0}^{\infty} \cdots \sum_{n_{i-1}=0}^{\infty} \sum_{n_{i+1}=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \mathbf{p}_i^m\left(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N\right)$$
$$z_1^{n_1} \ldots z_{i-1}^{n_{i-1}} z_{i+1}^{n_{i+1}} \ldots z_N^{n_N}, \quad i = 1,\ldots,N. \tag{38}$$

Applying notation $\widehat{\mathbf{f}}_i(z_1,\ldots,z_i-1,\bullet\boldsymbol{A},z_{i+1},\ldots,z_N)$ and the defining equations of the hypervector GFs $\widehat{\mathbf{f}}_i(z_1,\ldots,z_N)$ and $\widehat{\mathbf{m}}_i(z_1,\ldots,z_N)$ in (38), after rearranging we get the first governing relation of the system. □

To compute the vector moments of the stationary number of i-customers at i-polling and i-departure epochs we follow the same way as before for the gated discipline. We set an upper limit U for $n_1,\ldots,n_N$ in (38) and (34). Then we take their u_1-th, …, u_N-th derivatives ($u_1,\ldots,u_N \in \{0,\ldots,U\}$) at $z_1 = \ldots = z_N = 1$, respectively. This results in the following system of linear equations for $i = 1,\ldots,N$ and $u_1,\ldots,u_N \in \{0,\ldots,U\}$:

$$\sum_{n_1=0}^{U} \cdots \sum_{n_N=0}^{U} \mathbf{p}_i^f\left(n_1,\ldots,n_N\right) \frac{d^{u_1} \ldots d^{u_{i-1}} d^{u_{i+1}} \ldots d^{u_N}}{dz_1{}^{u_1} \ldots dz_{i-1}{}^{u_{i-1}}\, dz_{i+1}{}^{u_{i+1}} \ldots dz_N{}^{u_N}}$$
$$\left(z_1^{n_1} \ldots z_{i-1}^{n_{i-1}} \left(\widehat{\mathbf{E}}_i(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_N)\right)^{n_i} z_{i+1}^{n_{i+1}} \ldots z_N^{n_N}\right)\Big|_{\mathbf{z}=1} \tag{39}$$
$$= \sum_{n_1=0}^{U} \cdots \sum_{n_{i-1}=0}^{U} \sum_{n_{i+1}=0}^{U} \cdots \sum_{n_N=0}^{U} \mathbf{p}_i^m\left(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N\right)$$
$$\frac{n_1!}{(n_1-u_1)!} \cdots \frac{n_{i-1}!}{(n_{i-1}-u_{i-1})!} \frac{n_{i+1}!}{(n_{i+1}-u_{i+1})!} \cdots \frac{n_N!}{(n_N-u_N)!},$$

$$\sum_{n_1=0}^{U} \cdots \sum_{n_{i-1}=0}^{U} \sum_{n_{i+1}=0}^{U} \cdots \sum_{n_N=0}^{U} \mathbf{p}_i^m\left(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N\right)$$
$$\left.\frac{d^{u_1} \ldots d^{u_N} \left(z_1^{n_1} \ldots z_N^{n_N} \widehat{\mathbf{C}}_i(z_1,\ldots,z_N)\right)}{dz_1{}^{u_1} \ldots dz_N{}^{u_N}}\right|_{\mathbf{z}=1} \tag{40}$$
$$= \sum_{n_1=u_1}^{U} \cdots \sum_{n_N=u_N}^{U} \mathbf{p}_{i+1}^f\left(n_1,\ldots,n_N\right) \frac{n_1!}{(n_1-u_1)!} \cdots \frac{n_N!}{(n_N-u_N)!},$$

where $\mathbf{z} = 1$ stands for $z_1 = \ldots = z_N = 1$.

The number of equations and the number of unknowns in the linear system of equation ((39) and (40)) is $NL^N\left((U+1)^N + (U+1)^{N-1}\right)$.

5.3 Symmetric system

In the symmetrical system each station has the same parameters and therefore the behavior of each station is the same. Hence the set of unknowns $\mathbf{p}_i^f(n_1,\ldots,n_N)$ and $\mathbf{p}_i^m(n_1,\ldots,n_N)$ are the same for each station i. Similarly, the set of equations (35) and (36) as well as (39) and (40) for $u_1,\ldots,u_N = 0,\ldots,U$ are also the same for each station i, respectively . It follows, that the sizes of the linear systems of equations reduces to $2L^N(U+1)^N$ for the system with gated discipline and to $L^N\left((U+1)^N + (U+1)^{N-1}\right)$ for the system with exhaustive discipline .

6 Numerical solution

6.1 Numerical considerations

Eliminating quantities $\mathbf{p}_i^f(n_1,\ldots,n_N)$ from (35) and (36) (from (40) and (39) for the system with exhaustive discipline) results in a system of linear equations for stationary probability hypervectors $\mathbf{p}_i^m(n_1,\ldots,n_N)$ for $i = 1,\ldots,N$. This has a form $\mathbf{p}_i^m(n_1,\ldots,n_N) \to \mathbf{p}_{i+1}^m(n_1,\ldots,n_N)$.

For several disciplines including also the gated and the exhaustive one in general the stationary number of i-customers at i-departure epochs is less than at i-polling epochs. This implies that the major part of the distribution of the stationary number of i-customers is much closer to 0 at i-departure epochs than at i-polling epochs. Therefore selecting $\mathbf{p}_i^m(n_1,\ldots,n_N)$-s as unknowns instead of $\mathbf{p}_i^f(n_1,\ldots,n_N)$-s enables a lower value of U reducing the complexity of the corresponding system of linear equations.

The system of linear equations with the form $\mathbf{p}_i^m(n_1,\ldots,n_N) \to \mathbf{p}_{i+1}^m(n_1,\ldots,n_N)$, for each $i = 1,\ldots,N$, can be rearranged into one large hypervector and two large hypermatrices representing the unknowns and the coefficients of the system, respectively.

The total number of required elementary computational steps increases with the power of U and exponentially with N. Hence the solution of (35) and (36) ((39) and (40) for the system with exhaustive discipline) becomes difficult when the server utilization is high or the system is large.

6.1.1 System with gated discipline

The system of linear equations with the form $\mathbf{p}_i^m(n_1,\ldots,n_N) \to \mathbf{p}_{i+1}^m(n_1,\ldots,n_N)$ consists of $L^N(U+1)^N$ equations. Hence after rearrangement it leads to one large $1 \times L^N(U+1)^N$ hypervector and two large

$L^N(U+1)^N \times L^N(U+1)^N$ hypermatrices. The normalization relation $\sum_{n_1=0}^{U} \cdots \sum_{n_N=0}^{U} \mathbf{p}_i^m(n_1,\ldots,n_N) = 1$ makes the system complete.

The computation of the vector moments of the stationary number of i-customers at i-polling and i-departure epochs consists of the following steps:

1. Computation of matrices $\widehat{\mathbf{A}}_i(z_1,\ldots,z_N)$ for every $i=1,\ldots,N$.
2. Building up the large hypermatrices of the system for every $i=1,\ldots,N$, whose elements represents the u_1-th, …, u_N-th derivatives ($u_1,\ldots,u_N \in \{0,\ldots,U\}$) arising in (35) and (36) for n_1, …, n_N ($n_1,\ldots,n_N \in \{0,\ldots,U\}$).
3. Composing relations $\mathbf{p}_i^m(n_1,\ldots,n_N) \to \mathbf{p}_i^m(n_1,\ldots,n_N)$ by recursively applying the relations $\mathbf{p}_i^m(n_1,\ldots,n_N) \to \mathbf{p}_{i+1}^m(n_1,\ldots,n_N)$ for every $i=1,\ldots,N$. This steps involves inverse computation of N large hypermatrices ($N\left(L^{3N}(U+1)^{3N}\right)$ elementary computational steps) and a total of N^2+N times multiplication of large hypermatrices ($(N^2+N)\left(L^{3N}(U+1)^{3N}\right)$ elementary computational steps).
4. Solving the linear system of equations for relation $\mathbf{p}_i^m(n_1,\ldots,n_N) \to \mathbf{p}_i^m(n_1,\ldots,n_N)$. This takes $L^{3N}(U+1)^{3N}$ elementary computational steps for station i.
5. Computation of stationary probability distribution $\mathbf{p}_i^f(n_1,\ldots,n_N)$ from $\mathbf{p}_{i-1}^m(n_1,\ldots,n_N)$ by applying (30).
6. Computation of vector moments of the stationary number of i-customers at i-polling and i-departure epochs from the stationary probability distributions $\mathbf{p}_i^f(n_1,\ldots,n_N)$ and $\mathbf{p}_i^m(n_1,\ldots,n_N)$.

6.1.2 System with exhaustive discipline

The system of linear equations with the form $\mathbf{p}_i^m(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N) \to \mathbf{p}_{i+1}^m(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N)$ consists of $L^N(U+1)^{N-1}$ equations. After rearrangement it yields one large $1 \times L^N(U+1)^{N-1}$ hypervector and two large $L^N(U+1)^{N-1} \times L^N(U+1)^{N-1}$ hypermatrices representing the unknowns and the coefficients of the system, respectively. The normalization relation is $\sum_{n_1=0}^{U} \cdots \sum_{n_{i-1}=0}^{U} \sum_{n_{i+1}=0}^{U} \cdots \sum_{n_N=0}^{U} \mathbf{p}_i^m(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N) = 1$.

Similarly to the case with gated discipline the computation of the vector moments of the stationary number of i-customers at i-polling and i-departure epochs consists of the following steps:

1. Computation of matrices $\mathbf{G}_i(s)$ (from their LST), for every $i=1,\ldots,N$, by applying the method described by Lucantoni ([2], pp. 27-28).
2. Computation of matrices $\widehat{\mathbf{E}}_i(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_N)$ by the help of matrices $\mathbf{G}_i(s)$ for every $i=1,\ldots,N$.
3. Building up the large hypermatrices of the system, whose elements represents the u_1-th, …, u_{i-1}-th, u_{i+1}-th, …, u_N-th derivatives ($u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N \in \{0,\ldots,U\}$) arising in (39) and (40) for n_1, …, n_{i-1}, n_{i+1}, …, n_N ($n_1,\ldots,n_{i-1},n_{i+1},\ldots,n_N \in \{0,\ldots,U\}$).

4. Composing relations $\mathbf{p}_i^m(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N) \to \mathbf{p}_i^m(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N)$ by recursively applying the relations $\mathbf{p}_i^m(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N) \to \mathbf{p}_{i+1}^m(n_1,\ldots,n_i,0,n_{i+2},\ldots,n_N)$ for every $i = 1,\ldots,N$. This steps involves inverse computation of N large hypermatrices ($N\left(L^{3N}(U+1)^{3N-3}\right)$ elementary computational steps) and a total of N^2+N times multiplication of large hypermatrices ($(N^2+N)\left(L^{3N}(U+1)^{3N-3}\right)$ elementary computational steps).
5. Solving the linear system of equations for relation $\mathbf{p}_i^m(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N) \to \mathbf{p}_i^m(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N)$. This takes $L^{3N}(U+1)^{3N-3}$ elementary computational steps for station i.
6. Computation of stationary probability distribution $\mathbf{p}_i^f(n_1,\ldots,n_N)$ from $\mathbf{p}_{i-1}^m(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N)$ by applying (37).
7. Computation of vector moments of the stationary number of i-customers at i-polling and i-departure epochs from the stationary probability distributions $\mathbf{p}_i^f(n_1,\ldots,n_N)$ and $\mathbf{p}_i^m(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_N)$.

6.2 Numerical examples

We provide simple numerical examples to illustrate the numerical solution of the $BMAP/G/1$ nonzero-switchover-times polling model with gated and exhaustive disciplines.

We investigate a polling model with two stations, i.e. $N = 2$.

The form of matrix GFs of the arrival processes at both stations ($\widehat{\mathbf{D}}_1(z)$ and $\widehat{\mathbf{D}}_2(z)$) are given by

$$\begin{pmatrix} -\alpha_1-\beta_1 & \alpha_1 \\ 0 & -\alpha_2-\beta_2 \end{pmatrix} + z\begin{pmatrix} 0 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}.$$

Hence the number of phases of both $BMAP$s is $L = 2$.

6.2.1 Symmetric load

The customer service times B_1, B_2 are exponential with parameter γ. The switchover times R_1, R_2 are also exponential with parameter δ. It follows

$$\widehat{\mathbf{C}}_1(z_1,z_2) = \widehat{\mathbf{C}}_2(z_1,z_2) = \int_0^\infty e^{\left(\widehat{\mathbf{D}}_1(z_1)\oplus\widehat{\mathbf{D}}_2(z_2)\right)t}\delta e^{-\delta t}dt.$$

We set the following parameter values:

$$\alpha_1 = 1,\quad \alpha_2 = 4,\quad \beta_1 = 5,\quad \beta_2 = 0,\quad \gamma = 10,\quad \delta = 40.$$

The stationary arrival rates are normalized to have $\lambda_1 = \lambda_2 = 2$ and hence $\rho_1 = \rho_2 = 0.2$.

Example 1 Polling model with gated discipline - symmetric load

Due to the exponential customer service times

$$\widehat{\mathbf{A}}_1(z_1,z_2)=\widehat{\mathbf{A}}_2(z_1,z_2)=\int_0^{\infty} e^{\left(\widehat{\mathbf{D}}_1(z_1)\oplus\widehat{\mathbf{D}}_2(z_2)\right)t}\gamma e^{-\gamma t}dt.$$

For this example $U=2$ results in a sufficient precision.

Based on ((35) and (36)) we get $(L(U+1))^N=36$ equations for each station. From their solutions we get the factorial moments as

$$\begin{aligned}
\mathbf{f}_1 &= (0.380537,\ \ 0.619463), & \mathbf{m}_1 &= (0.375303,\ \ 0.624697),\\
\mathbf{f}_1^{(1)} &= (0.0838928,\ \ 0.0819299), & \mathbf{m}_1^{(1)} &= (0.0149513,\ \ 0.0186721),\\
\mathbf{f}_1^{(2)} &= (0.0314305,\ \ 0.040742), & \mathbf{m}_1^{(2)} &= (0.00761849,\ \ 0.00889137).
\end{aligned}$$

Applying them in the vector mean formula (29) gives the mean of the stationary number of customers as

$$\mathbf{q}_1^{(1)}=(0.208287,\ \ 0.22916).$$

Example 2 Polling model with exhaustive discipline - symmetric load

For this example $U=2$ results in a sufficient precision.

Based on ((39) and (40)) we get $(L(U+1))^{N-1}=6$ equations for each station. From their solutions we get the factorial moments as

$$\begin{aligned}
\mathbf{f}_1 &= (0.,\ \ 0.), & \mathbf{m}_1 &= (0.,\ \ 0.),\\
\mathbf{f}_1^{(1)} &= (0.,\ \ 0.),\\
\mathbf{f}_1^{(2)} &= (0.,\ \ 0.).
\end{aligned}$$

Applying them in the vector mean formula (29) gives the mean of the stationary number of customers as

$$\mathbf{q}_1^{(1)}=(0.,\ \ 0.).$$

6.2.2 Asymmetric load

The customer service times B_1, B_2 are constant and exponential with parameters τ_1 and γ_2, respectively. The switchover times R_1, R_2 are both exponential with parameters δ_1 and δ_2, respectively. It follows

$$\begin{aligned}
\widehat{\mathbf{C}}_1(z_1,z_2) &= \int_0^{\infty} e^{\left(\widehat{\mathbf{D}}_1(z_1)\oplus\widehat{\mathbf{D}}_2(z_2)\right)t}\delta_1 e^{-\delta_1 t}dt,\\
\widehat{\mathbf{C}}_2(z_1,z_2) &= \int_0^{\infty} e^{\left(\widehat{\mathbf{D}}_1(z_1)\oplus\widehat{\mathbf{D}}_2(z_2)\right)t}\delta_2 e^{-\delta_2 t}dt.
\end{aligned}$$

We set the following parameter values:

$$\begin{aligned}&\text{Station 1: } \alpha_1 = 1, \quad \alpha_2 = 2, \quad \beta_1 = 3, \quad \beta_2 = 4, \quad \tau_1 = 0.01, \quad \delta_1 = 20,\\ &\text{Station 2: } \alpha_1 = 1, \quad \alpha_2 = 4, \quad \beta_1 = 5, \quad \beta_2 = 0, \quad \gamma_2 = 10, \quad \delta_2 = 30.\end{aligned}$$

The stationary arrival rates are normalized to have $\lambda_1 = 5$, $\lambda_2 = 3$ and hence $\rho_1 = 0.05$ and $\rho_2 = 0.3$.

Example 3 Polling model with gated discipline - asymmetric load

Due to the constant B_1 and exponential B_2

$$\begin{aligned}\widehat{\mathbf{A}}_1(z_1, z_2) &= e^{\left(\widehat{\mathbf{D}}_1(z_1) \oplus \widehat{\mathbf{D}}_2(z_2)\right)\tau_1},\\ \widehat{\mathbf{A}}_2(z_1, z_2) &= \int_0^\infty e^{\left(\widehat{\mathbf{D}}_1(z_1) \oplus \widehat{\mathbf{D}}_2(z_2)\right)t} \gamma_2 e^{-\gamma_2 t} dt.\end{aligned}$$

For this example $U = 3$ results in a sufficient precision.

Based on ((35) and (36)) we get $(L(U+1))^N = 64$ equations for each stations. From their solutions we get the following factorial moments:

$$\begin{aligned}\mathbf{f}_1 \quad &= (0.339098, \quad 0.660902), \qquad \mathbf{m}_1 = (0.341291, \quad 0.658709),\\ \mathbf{f}_1^{(1)} &= (0.184653, \quad 0.444528), \qquad \mathbf{m}_1^{(1)} = (0.00894289, \quad 0.0230997),\\ \mathbf{f}_1^{(2)} &= (0.225154, \quad 0.524511), \qquad \mathbf{m}_1^{(2)} = (0.00101905, \quad 0.00258634),\end{aligned}$$

$$\begin{aligned}\mathbf{f}_2 \quad &= (0.38398, \quad 0.61602), \qquad \mathbf{m}_2 = (0.372321, \quad 0.627679),\\ \mathbf{f}_2^{(1)} &= (0.185588, \quad 0.196402), \quad \mathbf{m}_2^{(1)} = (0.0514999, \quad 0.0635985),\\ \mathbf{f}_2^{(2)} &= (0.119105, \quad 0.153746), \quad \mathbf{m}_2^{(2)} = (0.0423179, \quad 0.0519738).\end{aligned}$$

Applying them in vector mean formula (29) gives the means of the stationary number of customers as

$$\mathbf{q}_1^{(1)} = (0.10492, \quad 0.235217), \qquad \mathbf{q}_2^{(1)} = (0.341397, \quad 0.401698).$$

Example 4 Polling model with exhaustive discipline - asymmetric load

For this example $U = 3$ results in a sufficient precision.

Based on ((39) and (40)) we get $(L(U+1))^{N-1} = 8$ equations for each stations. From their solutions we get the following factorial moments:

$$\begin{aligned}\mathbf{f}_1 \quad &= (0., \quad 0.), \qquad \mathbf{f}_2 = (0., \quad 0.),\\ \mathbf{f}_1^{(1)} &= (0., \quad 0.), \qquad \mathbf{f}_2^{(1)} = (0., \quad 0.),\\ \mathbf{f}_1^{(2)} &= (0., \quad 0.), \qquad \mathbf{f}_2^{(2)} = (0., \quad 0.),\\ \mathbf{m}_1 &= (0., \quad 0.), \qquad \mathbf{m}_2 = (0., \quad 0.).\end{aligned}$$

Applying them in vector mean formula (29) gives the means of the stationary number of customers as

$$\mathbf{q}_1^{(1)} = (0., \quad 0.), \qquad \mathbf{q}_2^{(1)} = (0., \quad 0.).$$

7 Final remarks

It is a topic of future work to investigate the numerical methods in further details for computing the stationary probability hypervectors $\mathbf{p}_i^f(n_1,\ldots,n_N)$ and $\mathbf{p}_i^m(n_1,\ldots,n_N)$ for $i=1,\ldots,N$ based on the system of linear equations (35) and (36) as well as (40) and (39).

The applied method for the service discipline dependent analysis part can be used to analyze also several other service disciplines, like e.g. the limited-N or the semi-exhaustive disciplines.

The presented model can be extended to handle also other quantities like e.g., set-up time or repair time by relaxing the service discipline properties **P.2**, **P.3** and **P.4**. This leads to results having more general form.

Finally we notice that it is straightforward to extend the presented stationary relationships and their derivations for polling model with periodic polling and for polling model with Markovian server routing.

Appendix

In the following we establish the properties of model specific key matrices $\widehat{\mathbf{D}}_i(z)$ and $\left(z\mathbf{I}-\widehat{\mathbf{A}}_i(z)\right)$ at $z=1$ and show the proof of theorem 3.

A Properties of model specific key matrices

Lemma 1 *(First derivative of determinant.) The following relation holds for the determinant of the $L \times L$ matrix $\mathbf{Y}(z)$, which is differentiable when $|z| \leq 1$:*

$$\frac{d\ det\mathbf{Y}(z)}{dz} = Tr(adj\mathbf{Y}(z)\frac{d\mathbf{Y}}{dz}), \quad |z| \leq 1. \tag{41}$$

Proof. Jacobi's formula expresses the differential of the determinant of matrix $\mathbf{Y}(z)$ as

$$d\ det\mathbf{Y}(z) = Tr(adj\mathbf{Y}(z)\,d\mathbf{Y}). \tag{42}$$

Dividing (42) by dz results in the lemma. □

Lemma 2 *(Properties of $adj\mathbf{D}_i$ and $adj(\mathbf{I}-\mathbf{A}_i)$.) Each row of $adj\mathbf{D}_i$ is the same and it differs from $\boldsymbol{\pi}_i$ only in a multiplication constant. Similarly, each row of $adj(\mathbf{I}-\mathbf{A}_i)$ is the same and it differentiates from $\boldsymbol{\pi}_i$ only in a multiplication constant. In other words, there exist real values $c_1 \neq 0, c_2 \neq 0$ so, that*

$$\begin{aligned} adj\mathbf{D}_i &= c_1\mathbf{e}\boldsymbol{\pi}_i, \\ adj(\mathbf{I}-\mathbf{A}_i) &= c_2\mathbf{e}\boldsymbol{\pi}_i. \end{aligned} \tag{43}$$

Proof. $rank\,(\mathbf{D}_i) = L-1$ and so $\mathbf{D}_i$ is singular. Hence $adj\mathbf{D}_i\ \mathbf{D}_i = det\mathbf{D}_i = 0$ and every rows of $adj\mathbf{D}_i$ can differ from $1\times L$ size row vector $\mathbf{x_r}$ only in a multiplication constant, where $\mathbf{x_r}\mathbf{D}_i = 0$. Similarly $\mathbf{D}_i\ adj\mathbf{D}_i = det\mathbf{D}_i = 0$ and every columns of $adj\mathbf{D}_i$ can differ from $L\times 1$ size column vector $\mathbf{x_c}$ only in a multiplication constant, where $\mathbf{D}_i\mathbf{x_c} = 0$. Additionally $\boldsymbol{\pi}_i\mathbf{D}_i = 0$ and $\mathbf{D}_i\mathbf{e} = 0$. It follows that $adj\mathbf{D}_i = c_1\mathbf{e}\boldsymbol{\pi}_i$.

Now we come to the second statement. According to (10) $rank\,(\mathbf{I}-\mathbf{A}_i) = L-1$ and thus $(\mathbf{I}-\mathbf{A}_i)$ is singular. It can be seen from the Taylor expansion of $\mathbf{A}_i$ that $\boldsymbol{\pi}_i\mathbf{A}_i = \boldsymbol{\pi}_i$ and hence $\boldsymbol{\pi}_i\,(\mathbf{I}-\mathbf{A}_i) = 0$. Furthermore $(\mathbf{I}-\mathbf{A}_i)\,\mathbf{e} = 0$, since $\mathbf{A}_i$ is stochastic. Using the same argument as before for matrix $(\mathbf{I}-\mathbf{A}_i)$ results in the second statement. □

Let $[det\mathbf{D}_i]^{(1)}$ denote the first derivative of $det\widehat{\mathbf{D}}_i\,(z)$ at $z = 1$, i.e., $[det\mathbf{D}_i]^{(1)} = \left.\frac{d\left(det\widehat{\mathbf{D}}_i(z)\right)}{dz}\right|_{z=1}$. Similarly $[det\,(\mathbf{I}-\mathbf{A}_i)]^{(1)}$ denotes the first derivative of $det\left(z\mathbf{I}-\widehat{\mathbf{A}}_i(z)\right)$ at $z = 1$, i.e., $[det\,(\mathbf{I}-\mathbf{A}_i)]^{(1)} = \left.\frac{d\left(det\left(z\mathbf{I}-\widehat{\mathbf{A}}_i(z)\right)\right)}{dz}\right|_{z=1}$.

Proposition 1 *(First derivatives of $det\widehat{\mathbf{D}}_i\,(z)$ and $det\left(z\mathbf{I}-\widehat{\mathbf{A}}_i(z)\right)$ at $z = 1$.) The following statements hold for matrices $\widehat{\mathbf{D}}_i\,(z)$ and $\left(z\mathbf{I}-\widehat{\mathbf{A}}_i(z)\right)$ at $z = 1$:*

$$\begin{aligned}&[det\mathbf{D}_i]^{(1)} \neq 0,\\ &[det\,(\mathbf{I}-\mathbf{A}_i)]^{(1)} \neq 0.\end{aligned} \tag{44}$$

Proof. Applying Lemma 1 for determinant $det\mathbf{D}_i\,(z)$ at $z = 1$ leads to

$$[det\mathbf{D}_i]^{(1)} = Tr(adj\mathbf{D}_i\ \mathbf{D}_i^{(1)}). \tag{45}$$

Replacing $adj\mathbf{D}_i$ by $c_1\mathbf{e}\boldsymbol{\pi}_i$ in (45), where $c_1 \neq 0$ (Lemma 2), and applying (6) yields

$$\begin{aligned}&[det\mathbf{D}_i]^{(1)} = Tr(c_1\mathbf{e}\boldsymbol{\pi}_i\ \mathbf{D}_i^{(1)}) = \sum_{\ell=1}^{L}\sum_{j=1}^{L} c_1[\boldsymbol{\pi}_i]_j\ [\mathbf{D}_i^{(1)}]_{j,\ell} \qquad (46)\\ &= \sum_{j=1}^{L} c_1[\boldsymbol{\pi}_i]_j \sum_{\ell=1}^{L}[\mathbf{D}_i^{(1)}]_{j,\ell} = c_1\sum_{j=1}^{L}[\boldsymbol{\pi}_i]_j\mathbf{e}_j\mathbf{D}_i^{(1)}\mathbf{e} = c_1\boldsymbol{\pi}_i\mathbf{D}_i^{(1)}\mathbf{e} = c_1\lambda_i.\end{aligned}$$

Using $c_1 \neq 0$ and $\lambda_i > 0$ the first statement comes from (46).

Starting with determinant $det(z\mathbf{I}-\widehat{\mathbf{A}}_i(z))$ at $z = 1$ and using the same argument as before, we get:

$$[det\,(\mathbf{I}-\mathbf{A}_i)]^{(1)} = c_2\boldsymbol{\pi}_i\left(\mathbf{I}-\mathbf{A}_i^{(1)}\right)\mathbf{e}. \tag{47}$$

Rearranging the term $\boldsymbol{\pi}_i\left.\frac{d\widehat{\mathbf{A}}_i(z)}{dz}\right|_{z=1}\mathbf{e}$ and using (9) and $\boldsymbol{\pi}_i\mathbf{D}_i = 0$, we get

$$\begin{aligned}&\boldsymbol{\pi}_i\left.\frac{d\widehat{\mathbf{A}}_i(z)}{dz}\right|_{z=1}\mathbf{e} = \boldsymbol{\pi}_i\left.\frac{dE\left(e^{\widehat{\mathbf{D}}_i(z)B_i}\right)}{dz}\right|_{z=1}\mathbf{e} = \boldsymbol{\pi}_i E\left(\textstyle\sum_{k=0}^{\infty}\left.\frac{d\left(\widehat{\mathbf{D}}_i(z)^k\right)}{dz}\right|_{z=1}\mathbf{e}\ \frac{B_i^k}{k!}\right) =\\ &E\left(\sum_{k=1}^{\infty}\boldsymbol{\pi}_i\mathbf{D}_i^{k-1}\ \left.\frac{d\widehat{\mathbf{D}}_i\,(z)}{dz}\right|_{z=1}\mathbf{e}\ \frac{B_i^k}{k!}\right) = \boldsymbol{\pi}_i\mathbf{D}_i^{(1)}\mathbf{e}\ E\,(B_i) = \ \lambda_i b_i = \rho_i,\end{aligned}$$

from which

$$\boldsymbol{\pi}_i \left(\mathbf{I} - \mathbf{A}_i^{(1)}\right) \mathbf{e} = 1 - \rho_i. \tag{48}$$

Substituting (48) into (47) yields

$$[det\,(\mathbf{I} - \mathbf{A}_i)]^{(1)} = c_2\,(1 - \rho_i)\,. \tag{49}$$

$c_2 \neq 0$ and from stability $(1 - \rho) > 0$ and hence (49) gives the second statement. □

B Proof of theorem 3

We start from $\mathbf{I} = \frac{\widehat{\mathbf{T}}_i(z) adj \widehat{\mathbf{T}}_i(z)}{det \widehat{\mathbf{T}}_i(z)}$. Left multiplying it by $\frac{d\widehat{\mathbf{q}}_i(z)}{dz}$ and taking the limit for $z \to 1$ yields

$$\mathbf{q}_i^{(1)} = lim_{z\to 1} \frac{\frac{d\widehat{\mathbf{q}}_i(z)}{dz} \widehat{\mathbf{T}}_i\,(z)\, adj \widehat{\mathbf{T}}_i\,(z)}{det \widehat{\mathbf{T}}_i\,(z)}. \tag{50}$$

Equations (5) and (8) imply that $\widehat{\mathbf{D}}_i(z)$ and $\widehat{\mathbf{A}}_i(z)$ are continuously differentiable when $|z| \leq 1$. Using the definition of $\widehat{\mathbf{T}}_i\,(z)$ it follows that $\widehat{\mathbf{T}}_i\,(z)$, $adj\widehat{\mathbf{T}}_i\,(z)$ and $det\widehat{\mathbf{T}}_i\,(z)$ are also continuously differentiable. Therefore instead of $lim_{z\to 1}$ we consider the corresponding values at $z = 1$.

In the next we investigate the right hand side (rhs) of (50). First we show that as $lim_{z\to 1}$ both the nominator and the denominator of the rhs of (50) are 0. Furthermore as $lim_{z\to 1}$ also the first derivatives of the nominator and the denominator of the rhs of (50) are 0.

Due to (1) and (10) both $\mathbf{D}_i\,(1)$ and $(\mathbf{I} - \mathbf{A}_i)$ are singular. Hence $det\mathbf{D}_i det(\mathbf{I} - \mathbf{A}_i) = det\mathbf{T}_i = \mathbf{T}_i adj\mathbf{T}_i = 0$, from which follows that as $lim_{z\to 1}$ both the nominator and denominator of the rhs of (50) are 0.

Rearranging the first derivative of the denominator of the rhs of (50) at $z = 1$ and using $det(\mathbf{I} - \mathbf{A}_i) = 0$ and $det\mathbf{D}_i = 0$ we get

$$[det\mathbf{T}_i]^{(1)} = [det\mathbf{D}_i]^{(1)} det(\mathbf{I} - \mathbf{A}_i) + det\mathbf{D}_i [det(\mathbf{I} - \mathbf{A}_i)]^{(1)} = 0,$$

i.e. as $lim_{z\to 1}$ the first derivative of the denominator of the rhs of (50) is 0. Applying $\widehat{\mathbf{T}}_i\,(z)\, adj\widehat{\mathbf{T}}_i\,(z) = det\widehat{\mathbf{T}}_i\,(z)$ in the nominator of the rhs of (50) we get $\frac{d\widehat{\mathbf{q}}_i(z)}{dz} det\widehat{\mathbf{T}}_i\,(z)$. Taking its first derivative as $z \to 1$ and using $[det\mathbf{T}_i]^{(1)} = det\mathbf{T}_i = 0$ leads to

$$\mathbf{q}_i^{(1)} [det\mathbf{T}_i]^{(1)} + \mathbf{q}_i^{(2)} det\mathbf{T}_i = 0,$$

i.e. as $lim_{z\to 1}$ the first derivative of the nominator of the rhs of (50) is also 0.

Now we investigate the second derivative of the denominator of the rhs of (50) as $lim_{z\to 1}$. Taking the second derivative of the denominator of the rhs of (50) as $lim_{z\to 1}$, rearranging and applying $det(\mathbf{I} - \mathbf{A}_i) = 0$ and $det\mathbf{D}_i = 0$ we get

$$\begin{aligned}
[det\mathbf{T}_i]^{(2)} &= \left.\frac{d^2 \left(det\widehat{\mathbf{D}}_i\,(z)\, det(z\mathbf{I} - \widehat{\mathbf{A}}_i(z))\right)}{dz^2}\right|_{z=1} \\
&= 2[det\mathbf{D}_i]^{(1)} [det(\mathbf{I} - \mathbf{A}_i)]^{(1)}.
\end{aligned} \tag{51}$$

Applying proposition 1 in (51) shows that the second derivative of the denominator of the rhs of (50) does not equal 0 as $lim_{z\to 1}$.

Consequently as $lim_{z\to 1}$ the properties of the rhs of (50) can be summarized as follows:

- the nominator and the denominator of the right hand side of (50) and their first derivatives are 0,
- the second derivative of the denominator of the right hand side of (50) differs from 0.

Therefore we apply L'Hospital rule two times on (50). Using also that the terms in (50) are continuously differentiable, we get

$$\mathbf{q}_i^{(1)} = \frac{\left.\frac{d^2\left(\frac{d\widehat{\mathbf{q}}_i(z)}{dz}\widehat{\mathbf{T}}_i(z)adj\widehat{\mathbf{T}}_i(z)\right)}{dz^2}\right|_{z=1}}{[det\mathbf{T}_i]^{(2)}}. \tag{52}$$

We define vector $\widehat{\mathbf{x}}_i(z)$ as

$$\widehat{\mathbf{x}}_i(z) = \widehat{\mathbf{q}}_i(z)\,\widehat{\mathbf{T}}_i(z)\,. \tag{53}$$

Now we evaluate $\frac{d\widehat{\mathbf{q}}_i(z)}{dz}\widehat{\mathbf{T}}_i(z)\,adj\widehat{\mathbf{T}}_i(z)$ in the nominator of (52). Using the first derivative of (53) we have

$$\begin{aligned}\frac{d\widehat{\mathbf{q}}_i(z)}{dz}\widehat{\mathbf{T}}_i(z)\,adj\widehat{\mathbf{T}}_i(z) &= \left(\frac{d\widehat{\mathbf{x}}_i(z)}{dz} - \widehat{\mathbf{q}}_i(z)\frac{d\widehat{\mathbf{T}}_i(z)}{dz}\right)adj\widehat{\mathbf{T}}_i(z) \\ &= \frac{d\widehat{\mathbf{x}}_i(z)}{dz}adj\widehat{\mathbf{T}}_i(z) - \widehat{\mathbf{q}}_i(z)\left(\frac{d\widehat{\mathbf{T}}_i(z)}{dz}adj\widehat{\mathbf{T}}_i(z)\right) \\ &= \frac{d\widehat{\mathbf{x}}_i(z)}{dz}adj\widehat{\mathbf{T}}_i(z) + \widehat{\mathbf{q}}_i(z)\,\widehat{\mathbf{T}}_i(z)\frac{d\left(adj\widehat{\mathbf{T}}_i(z)\right)}{dz} - \widehat{\mathbf{q}}_i(z)\frac{d\left(det\widehat{\mathbf{T}}_i(z)\right)}{dz}.\end{aligned} \tag{54}$$

Applying (53) in (54) and rearranging yields

$$\begin{aligned}\frac{d\widehat{\mathbf{q}}_i(z)}{dz}\widehat{\mathbf{T}}_i(z)\,adj\widehat{\mathbf{T}}_i(z) &= \frac{d\widehat{\mathbf{x}}_i(z)}{dz}adj\widehat{\mathbf{T}}_i(z) + \widehat{\mathbf{x}}_i(z)\frac{d\left(adj\widehat{\mathbf{T}}_i(z)\right)}{dz} \\ -\widehat{\mathbf{q}}_i(z)\frac{d\left(det\widehat{\mathbf{T}}_i(z)\right)}{dz} &= \frac{d\left(\widehat{\mathbf{x}}_i(z)\,adj\widehat{\mathbf{T}}_i(z)\right)}{dz} - \widehat{\mathbf{q}}_i(z)\frac{d\left(det\widehat{\mathbf{T}}_i(z)\right)}{dz}.\end{aligned} \tag{55}$$

Substituting (55) into (52) and using $[det\mathbf{T}_i]^{(1)} = 0$ leads to

$$\begin{aligned}\mathbf{q}_i^{(1)} &= \frac{\left.\frac{d^3\left(\widehat{\mathbf{x}}_i(z)adj\widehat{\mathbf{T}}_i(z)\right)}{dz^3}\right|_{z=1} - \left.\frac{d^2\left(\widehat{\mathbf{q}}_i(z)\frac{d\left(det\widehat{\mathbf{T}}_i(z)\right)}{dz}\right)}{dz^2}\right|_{z=1}}{[det\mathbf{T}_i]^{(2)}} \\ &= \frac{\left.\frac{d^3\left(\widehat{\mathbf{x}}_i(z)adj\widehat{\mathbf{T}}_i(z)\right)}{dz^3}\right|_{z=1} - \mathbf{q}_i[det\mathbf{T}_i]^{(3)} - 2\mathbf{q}_i^{(1)}[det\mathbf{T}_i]^{(2)}}{[det\mathbf{T}_i]^{(2)}}.\end{aligned} \tag{56}$$

Further rearranging (56) and applying $\mathbf{q}_i = \boldsymbol{\pi}_i$ and $\mathbf{x}_i = \mathbf{q}_i\mathbf{T}_i = \boldsymbol{\pi}_i\mathbf{D}_i(\mathbf{I} - \mathbf{A}_i) = 0$ yields

$$\begin{aligned}3\mathbf{q}_i^{(1)}[det\mathbf{T}_i]^{(2)} &= \left.\frac{d^3\left(\widehat{\mathbf{x}}_i(z)\,adj\widehat{\mathbf{T}}_i(z)\right)}{dz^3}\right|_{z=1} - \boldsymbol{\pi}_i[det\mathbf{T}_i]^{(3)} \\ &= \mathbf{x}_i^{(3)}adj\mathbf{T}_i + 3\mathbf{x}_i^{(2)}[adj\mathbf{T}_i]^{(1)} + 3\mathbf{x}_i^{(1)}[adj\mathbf{T}_i]^{(2)} - \boldsymbol{\pi}_i[det\mathbf{T}_i]^{(3)}.\end{aligned} \tag{57}$$

Dividing (57) by $3[det\mathbf{T}_i]^{(2)}$ results in

$$\mathbf{q}_i^{(1)} = \frac{\frac{1}{3}\mathbf{x}_i^{(3)} adj\mathbf{T}_i + \mathbf{x}_i^{(2)}[adj\mathbf{T}_i]^{(1)} + \mathbf{x}_i^{(1)}[adj\mathbf{T}_i]^{(2)} - \frac{1}{3}\boldsymbol{\pi}_i[det\mathbf{T}_i]^{(3)}}{[det\mathbf{T}_i]^{(2)}}. \tag{58}$$

Observe that the left side of (22) is exactly $\widehat{\mathbf{x}}_i\,(z)$. Therefore to get $\mathbf{x}_i^{(1)}$, $\mathbf{x}_i^{(2)}$ and $\mathbf{x}_i^{(3)}$ in (58) we take the first, second and the third derivatives of (22) at $z = 1$:

$$\begin{aligned}
\mathbf{x}_i^{(1)} &= \frac{\mathbf{f}_i - \mathbf{m}_i}{\left(f_i^{(1)} - m_i^{(1)}\right)} \lambda_i \left(1 - \rho_i^S\right) \mathbf{A}_i, \\
\mathbf{x}_i^{(2)} &= 2\frac{\mathbf{f}_i^{(1)} - \mathbf{m}_i^{(1)}}{\left(f_i^{(1)} - m_i^{(1)}\right)} \lambda_i \left(1 - \rho_i^S\right) \mathbf{A}_i + 2\frac{\mathbf{f}_i - \mathbf{m}_i}{\left(f_i^{(1)} - m_i^{(1)}\right)} \lambda_i \left(1 - \rho_i^S\right) \mathbf{A}_i^{(1)}, \\
\mathbf{x}_i^{(3)} &= 3\frac{\mathbf{f}_i^{(2)} - \mathbf{m}_i^{(2)}}{\left(f_i^{(1)} - m_i^{(1)}\right)} \lambda_i \left(1 - \rho_i^S\right) \mathbf{A}_i + 6\frac{\mathbf{f}_i^{(1)} - \mathbf{m}_i^{(1)}}{\left(f_i^{(1)} - m_i^{(1)}\right)} \lambda_i \left(1 - \rho_i^S\right) \mathbf{A}_i^{(1)} \\
&\quad + 3\frac{\mathbf{f}_i - \mathbf{m}_i}{\left(f_i^{(1)} - m_i^{(1)}\right)} \lambda_i \left(1 - \rho_i^S\right) \mathbf{A}_i^{(2)}.
\end{aligned} \tag{59}$$

Substituting (59) into (58) gives the theorem. □

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