Exhaustive fluid vacation model with Markov modulated load ☆

Zsolt Saffer[a], Miklós Telek[b,c]

[a] No affiliation currently
[b] Budapest University of Technology and Economics, Department of Networked Systems and Services, Budapest, Hungary
[c] MTA-BME Information Systems Research Group, Budapest, Hungary

Abstract

In this paper we analyze stable fluid vacation models with exhaustive discipline, in which the fluid source is modulated by a background continuous-time Markov chain and the fluid is removed at constant rate during the service period. Due to the continuous nature of the fluid the state space of the model becomes continuous, which is the major novelty and challenge of the analysis. We adapt the descendant set approach used in polling models to the fluid vacation model. We provide steady-state vector Laplace Transform and mean of the fluid level at arbitrary epoch. First we consider the case when the fluid input rate is less than the fluid service rate during service and later we study the case when the fluid input rate is larger than the fluid service rate in some states of the model.

Keywords: Markov fluid queue, Service vacation, Exhaustive discipline.

1. Introduction

Fluid vacation model is an extension of the classical vacation model (see in [4], [10]), in which fluid takes the role of the customer of the classical model. Due to the continuous nature of the fluid, the flow in and the removal of fluid are characterized by rates. Hence the state space becomes continuous, which is a challenge in the analysis comparing to that of the discrete state space of the classical vacation model. This requires different analysis techniques.

In this paper we investigate a fluid vacation model with exhaustive service when the fluid source is modulated by a background Markov chain. The main idea of the analysis is the extension of the descendant set approach (see in [2]) to the continuous fluid model context. This together with the transient analysis of the input fluid flow enable to describe the evolution of the joint fluid level and the state of the background Markov chain between the vacation end and vacation start epochs - on the Laplace transform (LT) level. The resulting relations are called the governing equations. From them we determine the steady-state probability vector of the background Markov chain at the vacation start epochs. In the course of the analysis we derive a relation for the steady-state vector LT and vector mean of the fluid level at arbitrary epoch in terms of the previously mentioned steady-state probability vector. We also derive the steady-state LT of the service time, which is the counterpart of the busy period analysis in the classical vacation queue.

This paper is an extended version of [8]. One of the two main additional contributions of the current work compared to [8] is the explicit expression for the embedded vector at service completion and the results which are built on that (e.g. Corollary 3), the second additional contribution is the extension of the model to the cases when the effective fluid rates (i.e., the fluid input rate minus the fluid service rate) can be positive during the service period. This extension makes the majority of the results obtained for the restricted model with strictly negative effective fluid rates invalid. We apply a new methodology based on the matrix analytic analysis of Markov fluid queues [1, 3].

☆ This work was supported by the Hungarian research project OTKA K101150.
Email addresses: saffer@webspn.hit.bme.hu (Zsolt Saffer), telek@hit.bme.hu (Miklós Telek)

The rest of the paper is organized as follows. In section 2 we present the fluid vacation model and the concept of embedding matrix LTs, which is needed for the extension of the descendant set approach to fluid model. In section 3 we establish the governing equations of the model. The derivation of steady-state results follows in section 4. Section 5 discusses the case with positive effective fluid rate during service and Section 6 presents numerical examples. Finally, section 7 concludes the paper.

2. Model and Notation

2.1. Model description

We consider a fluid vacation model with Markov modulated load and exhaustive discipline. The model has an infinite fluid buffer.

The input fluid flow of the buffer is determined by a modulating CTMC ($\Omega(t)$ for $t \geq 0$) with state space $\mathcal{S} = \{1, \ldots, L\}$ and generator $\mathbf{Q}$. When this Markov chain is in state j ($\Omega(t) = j$) then fluid flows to the buffer at rate r_j for $j \in \{1, \ldots, L\}$. We define the diagonal matrix $\mathbf{R} = diag(r_1, \ldots, r_L)$. During the service period the server removes fluid from the fluid buffer at finite rate $d > 0$. Consequently, when the overall Markov chain is in state j ($\Omega(t) = j$) then the fluid level of the buffer during the service period changes at rate $r_j - d$, otherwise during the vacation periods it changes at rate r_j, because there is no service. In the vacation model the length of the service period is determined by the applied discipline. In this work we consider the exhaustive discipline. Under exhaustive discipline the fluid is removed during the service period until the buffer becomes empty. Each time the buffer becomes empty the server takes a vacation period. During vacation periods there is no service thus the fluid level of the buffer is increasing by the actual flowing rates. The consecutive vacation times are independent and identically distributed (i.i.d.). The random variable of the vacation time, its probability distribution function (pdf), its Laplace transform (LT) and its mean are denoted by $\widetilde{\sigma}$, $\sigma(t) = \frac{d}{dt}Pr(\widetilde{\sigma} < t)$ and $\sigma^*(s) = E(e^{-s\widetilde{\sigma}})$, $\sigma = E(\widetilde{\sigma})$, respectively. We define the cycle time (or simply cycle) as the time between just after the starts of two consecutive service periods.

We set the following assumptions on the fluid vacation model:

- **A.1** The generator matrix $\mathbf{Q}$ of the modulating CTMC is irreducible.
- **A.2** The fluid rates are positive and finite, i.e. $r_j > 0$ for $j \in \{1, \ldots, L\}$.
- **A.3** The fluid level strictly decreases during the service period, i.e., $r_j < d$ for $j \in \{1, \ldots, L\}$.

Let $\boldsymbol{\pi}$ be the stationary probability vector of the modulating Markov chain. Due to assumption **A.1** the equations

$$\boldsymbol{\pi}\mathbf{Q} = 0, \quad \boldsymbol{\pi}\mathbf{e} = 1. \tag{1}$$

uniquely determine $\boldsymbol{\pi}$, where $\mathbf{e}$ is the $L \times 1$ column vector of ones. The stationary fluid flow rate, λ, and the utilization ρ, is given as

$$\lambda = \boldsymbol{\pi}\mathbf{R}\mathbf{e}, \quad \rho = \frac{\lambda}{d}, \tag{2}$$

respectively. The necessary condition of the stability of the fluid vacation model is that the mean fluid arrival rate $\lambda = \boldsymbol{\pi}\mathbf{R}\mathbf{e}$ is less than d, which is equivalent with $\rho < 1$.

If the amount of fluid served during a service period were limited, like e.g. in case of a model with time-limited discipline, then further restriction would be needed for the sufficiency. However the model with the exhaustive discipline does not have any load-independent limitation for a service period, therefore the above necessary condition is also a sufficient one for the stability of the system.

2.2. *Notational conventions and embedded matrix LTs*

For the i,j-th element of the matrix $\mathbf{Z}$ the notations $\mathbf{Z}_{ij}$ or $[\mathbf{Z}]_{ij}$ are used. Similarly $\mathbf{z}_j$ and $[\mathbf{z}]_j$ denote the j-th element of vector $\mathbf{z}$. When $\mathbf{X}^*(s)$, $Re(s) \geq 0$ is a matrix LT, $\mathbf{X}^{(k)}$ denotes its k-th ($k \geq 1$) derivative at $s = 0$, i.e., $\mathbf{X}^{(k)} = \frac{d^k}{ds^k}\mathbf{X}^*(s)|_{s=0}$ and $\mathbf{X}^{(0)}$ denotes its value at $s = 0$, i.e., $\mathbf{X}^{(0)} = \mathbf{X}^*(0)$. Similar notations are applied for vector LT $\mathbf{x}^*(s)$ and scalar LT $x^*(s)$.

Let $\mathbf{Z}$ be an $L \times L$ rate matrix which has the following properties:

- the diagonal elements are negative ($\mathbf{Z}_{i,i} < 0$) and the other elements are non-negative ($\mathbf{Z}_{i,j} \geq 0$, for $i \neq j$),
- the row sums are zero.

For $Re(v) \geq 0$ let

$$\mathbf{H}(v) = \mathbf{D}v - \mathbf{Z}, \tag{3}$$

be a linear $L \times L$ matrix function of the complex variable v, where $\mathbf{Z}$ is a rate matrix and $\mathbf{D}$ is diagonal and its diagonal elements are positive, i.e. $[\mathbf{D}]_{j,j} > 0$ for $j \in \{1, \dots, L\}$. That is $\mathbf{Z}$ and $\mathbf{D}$ are real. The matrix function $-\mathbf{H}(v)$ has the following properties:

- **P.1** it is analytic for $Re(v) \geq 0$,
- **P.2** it is a rate matrix when $v = 0$,
- **P.3** the real part of its diagonal elements are negative for $Re(v) \geq 0$, i.e. $(Re(-\mathbf{H}_{j,j}(v)) < 0)$,
- **P.4** it is a diagonal dominant matrix for $Re(v) \geq 0$, i.e., $|Re(-\mathbf{H}_{j,j}(v))| \geq \sum_{k,k\neq j} |-\mathbf{H}_{j,k}(v))|$.

We define the operator $\mathcal{O}()$ on a complex variable v and on a linear matrix function $\mathbf{G}(v) = \mathbf{G_1}v + \mathbf{G_2}$ as the operator performing the substitution $v \to \mathbf{H}(v)$. That is $\mathcal{O}(v) = \mathbf{H}(v) = \mathbf{D}v - \mathbf{Z}$ and $\mathcal{O}(\mathbf{G}(v)) = \mathbf{G_1}\mathbf{H}(v) + \mathbf{G_2} = \mathbf{G_1}\mathbf{D}v - \mathbf{G_1}\mathbf{Z} + \mathbf{G_2}$, which are linear matrix functions as well. The order of non-commuting matrices are kept according to this definition. The multifold operator $\mathcal{O}^k(\bullet)$ is defined recursively as $\mathcal{O}^k(\bullet) = \mathcal{O}(\mathcal{O}^{k-1}(\bullet)), \quad k \geq 2$, where $\mathcal{O}^1(\bullet) = \mathcal{O}(\bullet)$ by definition. Additionally, we introduce $\mathcal{O}^0(\bullet) = \bullet$. Starting from (3), the $L \times L$ linear matrix function $\mathcal{O}^k(v)$ can be expressed recursively as

$$\mathcal{O}^k(v) = \mathbf{D}^k v - \sum_{i=0}^{k-1} \mathbf{D}^i \mathbf{Z}, \tag{4}$$

The matrix function $-\mathcal{O}^k(v)$ has the following properties:

- ⋄ $-\mathcal{O}^k(v)$ is analytic for $Re(v) \geq 0$ (due to **P.1** of $\mathbf{H}(v)$),
- ⋄ $-\mathcal{O}^k(v)\big|_{v=0}$ is also a rate matrix (**P.2**), since multiplying rate matrix $\mathbf{Z}$ any times by positive diagonal matrices from left results in a rate matrix and the sum of $L \times L$ rate matrices is also an $L \times L$ rate matrix,
- ⋄ $-\mathcal{O}^k(v)$ has also the properties **P.3** and **P.4**.

It follows from the recursive definition of the multifold operator $\mathcal{O}^k()$ that the matrix LT $\int_{x=0}^{\infty} p(x)e^{-\mathcal{O}^k(v)x}dx$ is created by consecutive embedding of the matrix $\mathbf{H}(v)$ in the previous matrix LT and therefore we call this matrix LT as *embedded matrix LT*. If $p(x) \geq 0$ for $x \geq 0$ and v is the complex argument of the LT $\int_{x=0}^{\infty} p(x)e^{-vx}dx$ then

$$\int_{x=0}^{\infty} p(x)e^{-\mathcal{O}(v)x}dx = \int_{x=0}^{\infty} p(x)e^{-\mathbf{H}(v)x}dx \tag{5}$$

is an $L \times L$ matrix LT. According to the Gerschgorin Circle Theorem [5] each eigenvalue of $-\mathbf{H}(v)$ is in one of the disks $\{z : |z - (-\mathbf{H}_{j,j}(v))| \leq \sum_{k \neq j} |-\mathbf{H}_{j,k}(v)|\}$ (i.e. disks in complex z-plane with center at $(-\mathbf{H}_{j,j}(v))$ and radius $\sum_{k \neq j} |-\mathbf{H}_{j,k}(v)|$), for $\forall j \in \{1, \ldots, L\}$. This together with properties **P.3** and **P.4** imply that the eigenvalues of $-\mathbf{H}(v)$ have negative or zero real part for $Re(v) \geq 0$. The matrix function $e^{-\mathbf{H}(v)x}$ can be written in Lagrange matrix polynomial form as

$$(e^x)^{-\mathbf{H}(v)} = \sum_{k=1}^{K} (e^x)^{\gamma_k} L_k(-\mathbf{H}(v)), \tag{6}$$

where K is the number of different eigenvalues of matrix $(-\mathbf{H}(v))$, γ_k and $L_k(v)$, for $k = 1, \ldots, K$ denotes the different eigenvalues of matrix $(-\mathbf{H}(v))$ and the finite Lagrange-polynomials belonging to the roots of the minimal-polynomial of matrix $(-\mathbf{H}(v))$, respectively. Applying (6) in (5) and rearranging it yields

$$\int_{x=0}^{\infty} p(x) e^{-\mathcal{O}(v)x} dx = \sum_{k=1}^{K} L_k(-\mathbf{H}(v)) \int_{x=0}^{\infty} p(x) e^{\gamma_k x} dx$$

Recall that the eigenvalues γ_k have negative or zero real part for $Re(v) \geq 0$. Consequently (5) is finite when the LT $\int_{x=0}^{\infty} p(x) e^{-vx} dx$ is finite for $Re(v) \geq 0$.

The same argument holds also for the matrix LT $\int_{x=0}^{\infty} p(x) e^{-\mathcal{O}^k(v)x} dx$, since the utilized properties **P.3** and **P.4** of $(-\mathbf{H}(v))$ hold also for $-\mathcal{O}^k(v)$. We remark here that the order of matrix and scalar $\mathbf{D}v$ in the definition of $\mathbf{H}(v)$ is crucial in order to ensure the validity of the properties **P.2** and **P.4** for the matrix function $\mathcal{O}^k(v)$.

3. The governing equations of the system

3.1. Transient analysis of the arriving fluid

In this section we consider the accumulated fluid during time $t \geq 0$. More precisely we derive the matrix LT of the fluid flowing into the buffer as a function of time, where the rows and columns of the matrix LT represent the initial and the final states of the modulating Markov chain.

Let $Y(t) \in \mathbb{R}^+$ be the accumulated fluid arrived at the buffer until time t, $\mathbf{A}(t, y)$ be the transition density matrix composed by elements $\mathbf{A}_{j,k}(t, y) = \frac{\partial}{\partial y} Pr(\Omega(t) = k, Y(t) < y | \Omega(0) = j, Y(0) = 0)$. We define the following transforms of matrix $\mathbf{A}(t, y)$:

$$\tilde{\mathbf{A}}^*(s, y) = \int_{t=0}^{\infty} \mathbf{A}(t, y) e^{-st} dt, \quad \mathbf{A}^*(t, v) = \int_{y=0}^{\infty} \mathbf{A}(t, y) e^{-vy} dy, \quad \mathbf{A}^{**}(s, v) = \int_{y=0}^{\infty} \mathbf{A}^*(t, v) e^{-st} dt.$$

Proposition 1. *[9] The matrix LT of the fluid generated by the Markov modulated fluid source in interval $(0, t]$ can be expressed as*

$$\mathbf{A}^*(t, v) = e^{-t(\mathbf{R}v - \mathbf{Q})}. \tag{7}$$

Proof. The fluid level is zero at $t = 0$ ($Y(0) = 0$) with probability 1. It follows that for $t = 0$ and $y = 0$ the initial value of the transition density matrix is given as

$$\mathbf{A}(0, y) = \delta(y)\mathbf{I} \quad \text{and } \mathbf{A}(t, 0) = \mathbf{0}, \quad \forall t > 0, \tag{8}$$

where $\delta(y)$ stands for the unit impulse function at y=0, $\mathbf{0}$ stands for the $L \times L$ zero matrix. The second initial value utilizes that the accumulated fluid is greater than zero for $t > 0$ ($Y(t) > 0$) due to assumption **A.2**. The Markov process $\{\Omega(t), Y(t)\}$ characterizes a homogenous first order fluid model, whose transient behavior can be described by the forward Kolmogorov equations

$$\frac{\partial}{\partial t} \mathbf{A}(t, y) + \frac{\partial}{\partial y} \mathbf{A}(t, y) \mathbf{R} = \mathbf{A}(t, y) \mathbf{Q}, \tag{9}$$

with initial conditions (8). Taking the LT of (9) with respect to t yields

$$\tilde{\mathbf{A}}^*(s,y)s - \tilde{\mathbf{A}}(0,y) + \frac{\partial}{\partial y}\tilde{\mathbf{A}}^*(s,y)\mathbf{R} = \tilde{\mathbf{A}}^*(s,y)\mathbf{Q}. \tag{10}$$

Now taking the LT of (10) with respect to y we have

$$\mathbf{A}^{**}(s,v)s - \mathbf{A}^*(0,v) + \left(\mathbf{A}^{**}(s,v)v - \tilde{\mathbf{A}}^*(s,0)\right)\mathbf{R} = \mathbf{A}^{**}(s,v)\mathbf{Q}, \tag{11}$$

where $\mathbf{A}^*(0,v) = \mathbf{I}$ and $\tilde{\mathbf{A}}^*(s,0) = \mathbf{0}$ according to (8). Applying them in (11) yields

$$\mathbf{A}^{**}(s,v)s - \mathbf{I} + \mathbf{A}^{**}(s,v)v\mathbf{R} = \mathbf{A}^{**}(s,v)\mathbf{Q},$$

from which by rearrangement we get

$$\mathbf{A}^{**}(s,v) = (\mathbf{I}s + \mathbf{R}v - \mathbf{Q})^{-1}. \tag{12}$$

The statement of the theorem comes by taking the inverse Laplace transform of (12) with respect to s. □

3.2. The descendant set based analysis of the fluid level

We extend the concept of descendant set (see in Borst and Boxma [2]) to fluid models and describe the exhaustive service period as consecutive gated service intervals without vacations. In discrete queueing models the (first) descendant set is the set of customers which arrive during the service of the customers which are present at the queue when the server starts the service. Consecutive descendant sets are defined recursively in a similar way. In our fluid queueing model we apply the descendant set concept for the amount of fluid which arrives during the service of the fluid which is in the buffer when the server starts the service. We define the 1-st descendant fluid level of the given fluid amount as the fluid flowing into the buffer during the service of the given fluid amount. This is similar to the descendant set of a customer in the regular vacation model, which consists of the group of customers arrived during the service of the original customer. Similarly we define the k-th descendant fluid level of the given fluid amount recursively as the fluid accumulated during the service of the $(k-1)$-th descendant fluid level. This is the same as the fluid level after k cycles in a gated system without vacation initiated by the given fluid amount. By definition the 0-th descendant fluid level of a given fluid amount is equal to itself. The k-th descendant period is defined as the removal time of the k-th descendant fluid for $k \geq 0$. Moreover we consider the evolution of the fluid level jointly with the evolution of the state of the modulating Markov chain. This joint evolution is described by the help of the matrix LT formalism. When $\mathbf{g}^*(v)$ is the vector LT of a given initial fluid density then the pdf and the vector LT of its k-th descendant fluid level, for $k \geq 1$, are denoted by $\mathbf{g}^{<k>}(x)$ and $\mathbf{g}^{*<k>}(v)$. Furthermore $\mathbf{g}^{<0>}(x) = \mathbf{g}(x)$ and $\mathbf{g}^{*<0>}(v) = \mathbf{g}^*(v)$. Let

$$\mathcal{O}(v) = \mathbf{H}(v) = \frac{\mathbf{R}v - \mathbf{Q}}{d}, \tag{13}$$

that is $\mathbf{D} = \frac{\mathbf{R}}{d}$ and $\mathbf{Z} = \frac{\mathbf{Q}}{d}$ in (3).

Furthermore we introduce a notation for the LT with respect to the $L \times L$ matrix function $\mathbf{H}(v)$ as follows

$$\mathbf{g}^*(\mathbf{H}(v)) \quad = \quad \int_{x=0}^{\infty} \mathbf{g}(x) e^{-\mathbf{H}(v)x} dx, \tag{14}$$

where $\mathbf{g}()$ is an $1 \times L$ vector function.

Proposition 2. *Starting from the initial fluid amount whose vector LT is $\mathbf{g}^*(v)$ the vector LT of the k-th $(k \geq 0)$ descendant fluid can be expressed as*

$$\mathbf{g}^{*<k>}(v) = \mathbf{g}^*(\mathcal{O}^k(v)). \tag{15}$$

Proof. The k-th descendant fluid is defined as the fluid accumulated during the service of the $(k-1)$-th descendant fluid for $k \geq 1$. The fluid density vector of the $(k-1)$-th descendant fluid is $\mathbf{g}^{<k-1>}(\xi)$. When the $(k-1)$-th descendant fluid is ξ, then its service duration is $\frac{\xi}{d}$, from which we can express $[\mathbf{g}^{<k>}(x)]_k$ as

$$[\mathbf{g}^{<k>}(x)]_\ell = \sum_{j=1}^{L} \int_{\xi=0}^{\infty} [\mathbf{g}^{<k-1>}(\xi)]_j \mathbf{A}_{j\ell}(\frac{\xi}{d}, x) d\xi \ , \tag{16}$$

whose vector form is

$$\mathbf{g}^{<k>}(x) = \int_{\xi=0}^{\infty} \mathbf{g}^{<k-1>}(\xi) \mathbf{A}(\frac{\xi}{d}, x) d\xi \ .$$

Applying (7), the LT of $\mathbf{g}^{<k>}(x)$ with respect to x is

$$\mathbf{g}^{*<k>}(v) = \int_{\xi=0}^{\infty} \mathbf{g}^{<k-1>}(\xi) \mathbf{A}^*(\frac{\xi}{d}, v) d\xi = \int_{\xi=0}^{\infty} \mathbf{g}^{<k-1>}(\xi) e^{-\frac{\xi}{d}(\mathbf{R}v - \mathbf{Q})} d\xi \ . \tag{17}$$

Utilizing that the right hand side of (17) is a matrix LT according to (14) and using (13) we have

$$\mathbf{g}^{*<k>}(v) = \mathbf{g}^{*<k-1>}\left(\frac{\mathbf{R}v - \mathbf{Q}}{d}\right) = \mathbf{g}^{*<k-1>}(\mathbf{H}(v)) = \mathbf{g}^{*<k-1>}(\mathcal{O}(v)). \tag{18}$$

Using the definition $\mathbf{g}^{*<0>}(v) = \mathbf{g}^*(v)$ for $k = 1$ we get

$$\mathbf{g}^{*<1>}(v) = \mathbf{g}^*(\mathcal{O}(v)). \tag{19}$$

Applying (19) recursively in (18) gives the proposition for $k \geq 2$. For $k = 0$ the proposition follows from the definitions $\mathbf{g}^{*<0>}(v) = \mathbf{g}^*(v)$ and $\mathcal{O}^0(v) = v$. $\square$

Proposition 3. *If the diagonal elements of* $\frac{\mathbf{R}}{d}$ *are less than one then* $\lim_{k\to\infty} \mathcal{O}^k(v)$ *exists, is finite, independent of* v *and it is*

$$\mathcal{O}^\infty(v) = \lim_{k\to\infty} \mathcal{O}^k(v) \quad = \quad \left(\frac{\mathbf{R}}{d} - \mathbf{I}\right)^{-1} \frac{\mathbf{Q}}{d}. \tag{20}$$

Proof. Applying $\mathbf{D} = \frac{\mathbf{R}}{d}$ and $\mathbf{Z} = \frac{\mathbf{Q}}{d}$ in (4) gives

$$\mathcal{O}^k(v) = \left(\frac{\mathbf{R}}{d}\right)^k v - \sum_{i=0}^{k-1} \left(\frac{\mathbf{R}}{d}\right)^i \frac{\mathbf{Q}}{d}. \tag{21}$$

When the diagonal elements of $\frac{\mathbf{R}}{d}$ are less than one then $\lim_{k\to\infty} \left(\frac{\mathbf{R}}{d}\right)^k = \mathbf{0}$ and $\lim_{k\to\infty} \sum_{i=0}^{k-1} \left(\frac{\mathbf{R}}{d}\right)^i = (\mathbf{I} - \frac{\mathbf{R}}{d})^{-1}$. Applying them in (21) results in the proposition. $\square$

Proposition 4. *Starting from the initial fluid amount whose vector LT is* $\mathbf{g}^*(v)$*, the limiting vector LT of the* k*-th descendant fluid as* k *tends to infinity can be expressed as*

$$\lim_{k\to\infty} \mathbf{g}^{*<k>}(v) = \mathbf{g}^*(\mathcal{O}^\infty(v)) = \mathbf{g}^*\left(\left(\frac{\mathbf{R}}{d} - \mathbf{I}\right)^{-1} \frac{\mathbf{Q}}{d}\right). \tag{22}$$

Proof. Applying (15) and the operator limit (20) we have

$$\lim_{k\to\infty} \mathbf{g}^{*<k>}(v) = \mathbf{g}^*(\lim_{k\to\infty} \mathcal{O}^k(v)) = \mathbf{g}^*(\mathcal{O}^\infty(v)) = \mathbf{g}^*\left(\left(\frac{\mathbf{R}}{d} - \mathbf{I}\right)^{-1} \frac{\mathbf{Q}}{d}\right).$$

$\square$

3.3. *The governing equations of the system at vacation start and end epochs*

Let $X(t) \in \mathbb{R}^+$ denote the fluid level in the buffer at time t and $t^f(\ell)$ for $\ell \geq 1$ be the time at the end of the vacation in the ℓ-th cycle. We define the $1 \times L$ row vector $\mathbf{f}(\ell, x)$ by its elements as

$$[\mathbf{f}(\ell, x)]_j = \frac{d}{dx} Pr(\Omega(t^f(\ell)) = j, X(t^f(\ell)) < x), \ j \in \Omega,$$

and its LT as $\mathbf{f}^*(\ell, v) = \int_{x=0}^{\infty} \mathbf{f}(\ell, x) e^{-vx} dx$. We also define the steady-state vector LT of the fluid level at end of vacation, the $1 \times L$ row vector $\mathbf{f}^*(v)$ as $\mathbf{f}^*(v) = \lim_{\ell \to \infty} \mathbf{f}^*(\ell, v)$. Analogously let $t^m(\ell)$ be the time at the start of vacation in the ℓ-th cycle. The $1 \times L$ row vector $\mathbf{m}(\ell, x)$ is defined by its elements as

$$[\mathbf{m}(\ell, x)]_j = \frac{d}{dx} Pr(\Omega(t^m(\ell)) = j, X(t^m(\ell)) < x), \ j \in \Omega,$$

and its LT and embedded steady-state vector are $\mathbf{m}^*(\ell, v) = \int_{x=0}^{\infty} \mathbf{m}(\ell, x) e^{-vx} dx$ and $\mathbf{m}^*(v) = \lim_{\ell \to \infty} \mathbf{m}^*(\ell, v)$.

Due to the exhaustive discipline the buffer is idle at the start of the vacation period, but the state of the environment at this time instant is still an important property of the model. The fact that the buffer is idle at the start of vacation implies that $\mathbf{m}^*(\ell, v)$ is independent of v, that is $\mathbf{m}(\ell) = \mathbf{m}^*(\ell, v)$. The remaining analysis problem is to compute the vector $\mathbf{m}(\ell)$ whose ith element contains the probability that the environment is in state i at the start of the ℓth vacation period. In contrast, the buffer is not idle at the start of the service period and this way $\mathbf{f}^*(\ell, v)$ depends on v.

Theorem 1. *In the fluid vacation model with exhaustive discipline the vector LTs of the fluid level at the end of the ℓth vacation, $\mathbf{f}^*(\ell, v)$, $\ell \geq 0$, and at the start of the ℓth vacation, $\mathbf{m}(\ell)$, $\ell \geq 1$, satisfy*

$$\mathbf{m}(\ell) = \mathbf{f}^*(\ell - 1, \mathcal{O}^\infty(v)) = \mathbf{f}^* \left(\ell - 1, \left(\frac{\mathbf{R}}{d} - \mathbf{I} \right)^{-1} \frac{\mathbf{Q}}{d} \right), \tag{23}$$

$$\mathbf{f}^*(\ell, v) = \mathbf{m}(\ell) \sigma^*(\mathbf{R}v - \mathbf{Q}). \tag{24}$$

Proof. The k-th descendant fluid as $\lim_{k \to \infty}$ gives the fluid at the end of the exhaustive service period. Hence starting a service period with initial joint fluid level and phase distribution $\mathbf{g}^*(v)$ at the end of the service period the joint fluid level and phase distribution is $\lim_{k \to \infty} \mathbf{g}^{*<k>}(v)$. Proposition 4 states that $\mathbf{g}^*(\mathcal{O}^\infty(v)) = \mathbf{g}^* \left(\left(\frac{\mathbf{R}}{d} - \mathbf{I} \right)^{-1} \frac{\mathbf{Q}}{d} \right)$. (23) comes from the fact that the joint fluid level and phase distribution at the end of the $\ell - 1$th vacation period is $\mathbf{f}^*(\ell - 1, v)$.

In the exhaustive system the fluid level at the end of the ℓth vacation equals the fluid flowed into the buffer during the ℓth vacation, since the buffer is idle at the start of the ℓth vacation. Taking into account also the state of the modulating Markov chain we have

$$[\mathbf{f}(\ell, x)]_k = \sum_{j=1}^{L} \int_{t=0}^{\infty} [\mathbf{m}(\ell)]_j \mathbf{A}_{jk}(t, x) \sigma(t) dt. \tag{25}$$

Rearranging (25) to matrix form we get

$$\mathbf{f}^*(\ell, v) = \int_{t=0}^{\infty} \mathbf{m}(\ell) \mathbf{A}^*(t, v) \sigma(t) dt = \mathbf{m}(\ell) \int_{t=0}^{\infty} e^{-t(\mathbf{R}v - \mathbf{Q})} \sigma(t) dt, \tag{26}$$

where the explicit form of $\mathbf{A}^*(t, v)$ is taken from (7). Rearranging (26) results in (24). □

In the rest of the paper we avoid the scalar versions of the equations like (16) and (25) and directly write their matrix versions.

4. The steady-state behavior of the fluid vacation model

The main goal of this section is to compute the time stationary distribution of the fluid vector in the transform domain. To this end we first provide the stationary distribution in service start and end epochs in Sec. 4.1, collect some subsequently used general properties of fluid vacation models in Sec. 4.2, compute the service time distribution in Sec. 4.3 and finally evaluate the time stationary behavior in Sec. 4.4.

4.1. Steady-state behavior at start and end of vacation

We define $\mathbf{m}$ as $\mathbf{m} = \lim_{\ell\to\infty} \mathbf{m}(\ell)$ and $\mathbf{e}$ as the column vector of ones.

Theorem 2. *The stationary behavior of the stable fluid vacation model with exhaustive discipline is characterized by* $\mathbf{m}$ *and* $\mathbf{f}^*(v)$ *where* $\mathbf{m}$ *is the solution of the linear system*

$$\begin{aligned} \mathbf{m} &= \mathbf{m}\sigma^*\left(\left(\frac{\mathbf{R}}{d}-\mathbf{I}\right)^{-1}\mathbf{Q}\right), \qquad (27)\\ \mathbf{me} &= 1, \end{aligned}$$

and

$$\mathbf{f}^*(v) = \mathbf{m}\sigma^*(\mathbf{R}v - \mathbf{Q}). \qquad (28)$$

Proof. Applying (24) in (23) gives

$$\mathbf{m}(\ell) = \mathbf{m}(\ell-1)\sigma^*(\mathbf{R}\mathcal{O}^\infty(v) - \mathbf{Q}). \qquad (29)$$

Taking the limit $\ell \to \infty$ in (29) and rearranging it leads to

$$\mathbf{m} = \mathbf{m}\sigma^*(d\frac{\mathbf{R}\mathcal{O}^\infty(v) - \mathbf{Q}}{d}) = \mathbf{m}\sigma^*(d\mathcal{O}(\mathcal{O}^\infty(v))) = \mathbf{m}\sigma^*(d\mathcal{O}^\infty(v)). \qquad (30)$$

Applying (20) in (30) results in the first equation of (27). The normalizing condition of the system of linear equations comes from the fact that $\mathbf{m}$ is the phase distribution of the background Markov chain of the fluid source at service completion. Finally (28) comes by taking the limit $\ell \to \infty$ in (24). □

The next theorem provides an explicit form of $\mathbf{m}$, as a function of the stationary probability vector $\boldsymbol{\pi}$.

Theorem 3. *The stationary distribution at vacation starts,* $\mathbf{m}$, *can be expressed as*

$$\mathbf{m} = \frac{1}{1-\rho}\ \boldsymbol{\pi}\left(\mathbf{I} - \frac{\mathbf{R}}{d}\right). \qquad (31)$$

Proof. The Laplace transform term in (27) can be expressed as

$$\sigma^*\left(\left(\frac{\mathbf{R}}{d}-\mathbf{I}\right)^{-1}\mathbf{Q}\right) = E\left(e^{-\tilde{\sigma}\left(\frac{\mathbf{R}}{d}-\mathbf{I}\right)^{-1}\mathbf{Q}}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n E(\tilde{\sigma}^n)}{n!}\left(\left(\frac{\mathbf{R}}{d}-\mathbf{I}\right)^{-1}\mathbf{Q}\right)^n. \qquad (32)$$

Using this we have that $\boldsymbol{\pi}\left(\mathbf{I} - \frac{\mathbf{R}}{d}\right)$ is a solution of (27), because

$$\begin{aligned} \boldsymbol{\pi}\left(\mathbf{I}-\frac{\mathbf{R}}{d}\right) &= \boldsymbol{\pi}\left(\mathbf{I}-\frac{\mathbf{R}}{d}\right)\left(\mathbf{I}+\sum_{n=1}^{\infty}\frac{\sigma^{*(n)}}{n!}\boldsymbol{\pi}\left(\mathbf{I}-\frac{\mathbf{R}}{d}\right)\left(\left(\frac{\mathbf{R}}{d}-\mathbf{I}\right)^{-1}\mathbf{Q}\right)^n\right)\\ &= \boldsymbol{\pi}\left(\mathbf{I}-\frac{\mathbf{R}}{d}\right)\left(\mathbf{I}-\sum_{n=1}^{\infty}\frac{\sigma^{*(n)}}{n!}\boldsymbol{\pi}\mathbf{Q}\left(\left(\frac{\mathbf{R}}{d}-\mathbf{I}\right)^{-1}\mathbf{Q}\right)^{n-1}\right)\\ &= \boldsymbol{\pi}\left(\mathbf{I}-\frac{\mathbf{R}}{d}\right), \qquad (33) \end{aligned}$$

where we utilized that $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$. Additionally, (31) is properly normalized because

$$\frac{1}{1-\rho}\ \boldsymbol{\pi}\left(\mathbf{I}-\frac{\mathbf{R}}{d}\right)\mathbf{e} = \frac{1}{1-\rho}\ \left(\boldsymbol{\pi}\mathbf{e}-\boldsymbol{\pi}\frac{\mathbf{R}}{d}\mathbf{e}\right) = \frac{1}{1-\rho}\ \left(1-\frac{\lambda}{d}\right) = 1.$$

□

4.2. Equilibrium relationships

Let $S(\ell)$ be the service time in the ℓ-th cycle, $C(\ell)$ be the cycle time between two consecutive service starts in the ℓ-th cycle, $Z(\ell)$ be the amount of fluid served in the ℓ-th cycle, and $Z_i(\ell)$ be the amount of fluid served in the i-th descendant period of the ℓ-th service period. The related $lim_{\ell\to\infty}$ stationary quantities are S, C, Z, and Z_i, their LTs are $s^*(v)$, $c^*(v)$, $z^*(v)$, $z_i^*(v)$, and their means are s, c, z, z_i, respectively. The definitions imply that $Sd = Z$, $C = S + \widetilde{\sigma}$ and $Z = \sum_{i=0}^{\infty} Z_i$ hold as well as these relations hold for their respective means.

Let $Y(t)$ be the accumulated fluid flowed into the buffer in interval $(0, t]$ and a be the mean amount of fluid, which flows into the buffer during a cycle in steady state. That is

$$a = \lim_{k\to\infty} \frac{\sum_{\ell=1}^{k} E[Y(t^f(\ell+1)) - Y(t^f(\ell))]}{k},$$

whose right hand side can be rearranged as

$$a = \lim_{k\to\infty} \frac{E[\sum_{\ell=1}^{k} Y(t^f(\ell+1)) - Y(t^f(\ell))]}{E[\sum_{\ell=1}^{k} C(\ell)]} \lim_{k\to\infty} \frac{E[\sum_{\ell=1}^{k} C(\ell)]}{k} = \lambda c. \tag{34}$$

Corollary 1. *In the stable fluid vacation model the steady-state mean cycle time can be expressed as*

$$c = \frac{\sigma}{1-\rho}. \tag{35}$$

Proof. In the stable fluid vacation model the amount of fluid flowing into the buffer during a cycle equals the amount of fluid removed during the service period, that is $a = sd$. From this and $a = \lambda c$ we get $s = \frac{\lambda}{d}c = \rho c$ and $c = \sigma + s = \sigma + \rho c$, which gives the statement. □

4.3. The steady-state distribution of the service time

Theorem 4. *The amount of fluid served in a service period and the length of the service period satisfy*

$$z^*(v) = \mathbf{f}^*\left(\left(\mathbf{I}-\frac{\mathbf{R}}{d}\right)^{-1}\left(v\mathbf{I}-\frac{\mathbf{Q}}{d}\right)\right)\mathbf{e}, \tag{36}$$

$$s^*(v) = z^*\left(v/d\right). \tag{37}$$

Proof. Let $f_{Z_0,Z_1,\ldots,Z_k}(x_0, x_1, \ldots, x_k)$ denote the joint density of $Z_0, Z_1, \ldots, Z_k$. Furthermore let the matrix $\mathbf{A}(\frac{x_i}{d}, x_{i+1})$ stand for the state dependent density of the fluid arrived to the buffer during the $\frac{x_i}{d}$ long i-th descendant period for $i = 0, \ldots, k-1$.

The fluid served in the i-th descendant period is the fluid flowed into the buffer during the $(i-1)$-th descendant period for $i = 1, \ldots, k$. Using it and taking into account also the evolution of the modulating Markov state the joint density $f_{Z_0,Z_1,\ldots,Z_k}(x_0, x_1, \ldots, x_k)$ can be given as

$$f_{Z_0,Z_1,\ldots,Z_k}(x_0, x_1, \ldots, x_k) = \mathbf{f}(x_0)\mathbf{A}(\frac{x_0}{d}, x_1)\ldots\mathbf{A}(\frac{x_{k-1}}{d}, x_k)\mathbf{e}. \tag{38}$$

By the help of (38) the mean $E(e^{-v\sum_{i=0}^{k} Z_i})$ can be expressed as

$$
\begin{aligned}
&E(e^{-v\sum_{i=0}^{k} Z_i}) = \\
&\int_{x_0}\int_{x_1}\dots\int_{x_k} E(e^{-v\sum_{i=0}^{k} Z_i}|Z_0=x_0, Z_1=x_1,\dots,Z_k=x_k) \\
&\times f_{Z_0,Z_1,\dots,Z_k}(x_0,x_1,\dots,x_k)dx_k\dots dx_1 dx_0 = \\
&\int_{x_0}\mathbf{f}(x_0)\int_{x_1}\mathbf{A}(\frac{x_0}{d},x_1)\dots\underbrace{\int_{x_k}\mathbf{A}(\frac{x_{k-1}}{d},x_k)e^{-vx_k}dx_k}_{\mathbf{A}^*(\frac{x_{k-1}}{d},v)}\dots e^{-vx_1}dx_1 e^{-vx_0}dx_0\mathbf{e} = \\
&\int_{x_0}\mathbf{f}(x_0)\int_{x_1}\mathbf{A}(\frac{x_0}{d},x_1)\dots\underbrace{\int_{x_{k-1}}\mathbf{A}(\frac{x_{k-2}}{d},x_{k-1})\underbrace{e^{-x_{k-1}\frac{\mathbf{R}v-\mathbf{Q}}{d}}e^{-vx_{k-1}}}_{e^{-x_{k-1}(\mathcal{O}(v)+\mathbf{I}v)}}dx_{k-1}}_{\mathbf{A}^*(\frac{x_{k-2}}{d},\mathcal{O}(v)+\mathbf{I}v)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\dots\ e^{-vx_1}dx_1 e^{-vx_0}dx_0\mathbf{e} = \\
&\int_{x_0}\mathbf{f}(x_0)\int_{x_1}\mathbf{A}(\frac{x_0}{d},x_1)\dots\underbrace{\int_{x_{k-2}}\mathbf{A}(\frac{x_{k-3}}{d},x_{k-2})e^{-x_{k-2}\frac{\mathbf{R}(\mathcal{O}(v)+\mathbf{I}v)-\mathbf{Q}}{d}}e^{-vx_{k-2}}dx_{k-2}}_{\mathbf{A}^*(\frac{x_{k-3}}{d},\mathcal{O}^2(v)+\frac{\mathbf{R}}{d}v+\mathbf{I}v)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\dots\ e^{-vx_1}dx_1 e^{-vx_0}dx_0\mathbf{e} = \\
&\dots = \mathbf{f}^*\left(\mathcal{O}^k(v)+\sum_{i=0}^{k-1}\left(\frac{\mathbf{R}}{d}\right)^i v\right)\mathbf{e}.
\end{aligned}
$$

This together with (20) yields

$$
\begin{aligned}
z^*(v) = E(e^{-v\sum_{i=0}^{\infty} Z_i}) &= \mathbf{f}^*\left(\mathcal{O}^\infty(v)+\sum_{i=0}^{\infty}\left(\frac{\mathbf{R}}{d}\right)^i v\right)\mathbf{e} = \\
&= \mathbf{f}^*\left(\left(\frac{\mathbf{R}}{d}-\mathbf{I}\right)^{-1}\frac{\mathbf{Q}}{d}+\left(\mathbf{I}-\frac{\mathbf{R}}{d}\right)^{-1}v\right)\mathbf{e},
\end{aligned}
$$

from which rearrangement results in the first statement of the theorem. For the service time distribution we have

$$
s^*(v) = E(e^{-vS}) = E(e^{-vZ/d}) = E(e^{-(v/d)Z}) = z^*(v/d).
$$

□

Sanity check: If the background Markov chain has a single state (i.e. the fluid source is deterministic) and the service period starts with ϑ amount of fluid in the buffer then $\mathbf{Q}=0$, $\mathbf{R}=r_1$ and $f^*(v)=e^{-\vartheta v}$. In this case the fluid level decreases with rate $d-r_1$, the length of service period is $\frac{\vartheta}{d-r_1}$, that is $s^*(v)=e^{-v\frac{\vartheta}{d-r_1}}$, and the amount of fluid served during the service period is $\frac{\vartheta d}{d-r_1}$, that is $z^*(v)=e^{-v\frac{\vartheta d}{d-r_1}}$. At the same time, substituting $\mathbf{Q}=0$, $\mathbf{R}=r_1$ and $f^*(v)=e^{-\vartheta v}$ into (36) gives

$$
z^*(v) = f^*\left(\left(1-\frac{r_1}{d}\right)^{-1}v\right) = f^*\left(\frac{dv}{d-r_1}\right) = e^{-\vartheta\frac{dv}{d-r_1}}.
$$

4.4. The steady-state vector LT of the fluid level

The steady-state joint distribution of the fluid level and the state of the modulating Markov chain at an arbitrary epoch is defined by the $1 \times L$ row vector $\mathbf{q}(x)$ whose j-th element is

$$[\mathbf{q}(x)]_j = \lim_{t\to\infty} \frac{d}{dx} Pr(\Omega(t) = j, X(t) < x), \ j \in \Omega.$$

The LT of $\mathbf{q}(x)$ is $\mathbf{q}^*(v) = \int_{x=0}^{\infty} \mathbf{q}(x) e^{-vx} dx$.

Let d_k be the start time of the kth descendant service period for $k \geq 0$, where $d_0 = 0$. The steady-state joint density of the fluid level and the state of the modulating Markov chain at an arbitrary epoch in the kth ($k \geq 0$) descendant service period, the $1 \times L$ row vector $\mathbf{q}_k(x)$ is defined by its j-th element as

$$[\mathbf{q}_k(x)]_j = \lim_{t\to\infty} \frac{d}{dx} Pr(\Omega(t) = j, X(t) < x \mid t \in (d_k, d_{k+1})), \ j \in \Omega.$$

and corresponding LT is $\mathbf{q}_k^*(v)$.

Let $1_{(\text{con})}$ denote the indicator of condition "con". Furthermore let $\mathbf{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$ be the $1 \times L$ vector with 1 at the j-th position. We define the $1 \times L$ indicator vector $\mathbf{1}_{(\Omega(t))}$ as $\mathbf{1}_{(\Omega(t))} = \sum_{j=1}^{L} 1_{(\Omega(t)=j)} \mathbf{e}_j$.

Proposition 5. *For $k \geq 0$ the LT of the mean fluid level for the kth descendant interval, $E[\int_{t=d_k}^{d_{k+1}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt]$, satisfies*

$$E[\int_{t=d_k}^{d_{k+1}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt] \left((\mathbf{R} - d\mathbf{I})v - \mathbf{Q}\right) = \mathbf{f}^*(\mathcal{O}^k(v)) - \mathbf{f}^*(\mathcal{O}^{k+1}(v)). \tag{39}$$

Proof. If the fluid level at the beginning of the k-th descendant period is x_k then the fluid level after time t in the k-th descendant period is $x_k - td + A(t)$ where $A(t)$ denotes the amount of fluid arrived in $(0, t)$ in the k-th descendant period. The LT of this quantity is $E(e^{-v(x_k - td + A(t))}) = E(e^{-vA(t)}) e^{-v(x_k - td)}$, where the first term is the LT of $A(t)$. Considering the state dependency of fluid level at the beginning of the k-th descendant period and the fluid arrival process for $k > 0$ we have

$$\begin{aligned}
&E[\int_{t=d_k}^{d_{k+1}} e^{-X(t)v} \mathbf{1}_{(\Omega(t))} dt] = \\
&\int_{x_0} \mathbf{f}(x_0) \int_{x_1} \mathbf{A}(\frac{x_0}{d}, x_1) \dots \int_{x_k} \mathbf{A}(\frac{x_{k-1}}{d}, x_k) \int_{t=0}^{x_k/d} \mathbf{A}^*(t, v) e^{-(x_k - td)v} \ dt \ dx_k \dots dx_0 = \\
&\int_{x_0} \mathbf{f}(x_0) \int_{x_1} \mathbf{A}(\frac{x_0}{d}, x_1) \dots \int_{x_k} \mathbf{A}(\frac{x_{k-1}}{d}, x_k) \underbrace{\int_{t=0}^{x_k/d} e^{-t((\mathbf{R} - d\mathbf{I})v - \mathbf{Q})} \ dt}\, e^{-x_k v} \ dx_k \dots dx_0.
\end{aligned}$$

The underbraced integral can be evaluated by means of the following relation

$$\int_{t=0}^{x} e^{-t\mathbf{Z}} dt \mathbf{Z} = \mathbf{I} - e^{-x\mathbf{Z}}, \tag{40}$$

which leads to

$$
\begin{aligned}
&E[\int_{t=d_k}^{d_{k+1}} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt]\,((\mathbf{R}-d\mathbf{I})v-\mathbf{Q}) = \\
&\int_{x_0}\mathbf{f}(x_0)\int_{x_1}\mathbf{A}(\frac{x_0}{d},x_1)\ldots\int_{x_k}\mathbf{A}(\frac{x_{k-1}}{d},x_k)\left(\mathbf{I}-e^{-\frac{x_k}{d}(\mathbf{R}-d\mathbf{I})v-\mathbf{Q})}\right)e^{-x_kv}\;dx_k\ldots dx_0 = \\
&\int_{x_0}\mathbf{f}(x_0)\int_{x_1}\mathbf{A}(\frac{x_0}{d},x_1)\ldots\int_{x_{k-1}}\mathbf{A}(\frac{x_{k-2}}{d},x_{k-1}) \\
&\quad\left(\int_{x_k}\mathbf{A}(\frac{x_{k-1}}{d},x_k)e^{-x_kv}dx_k-\int_{x_k}\mathbf{A}(\frac{x_{k-1}}{d},x_k)e^{-\frac{x_k}{d}(\mathbf{R}v-\mathbf{Q})}dx_k\right)dx_{k-1}\ldots dx_0 = \\
&\int_{x_0}\mathbf{f}(x_0)\int_{x_1}\mathbf{A}(\frac{x_0}{d},x_1)\ldots\int_{x_{k-1}}\mathbf{A}(\frac{x_{k-2}}{d},x_{k-1}) \\
&\qquad\left(\mathbf{A}^*(\frac{x_{k-1}}{d},v)-\mathbf{A}^*(\frac{x_{k-1}}{d},\mathcal{O}(v))\right)\;dx_{k-1}\ldots dx_0 = \\
&\int_{x_0}\mathbf{f}(x_0)\int_{x_1}\mathbf{A}(\frac{x_0}{d},x_1)\ldots\int_{x_{k-1}}\mathbf{A}(\frac{x_{k-2}}{d},x_{k-1}) \\
&\qquad\left(e^{-x_{k-1}\mathcal{O}(v)}-e^{-x_{k-1}\mathcal{O}^2(v)}\right)\;dx_{k-1}\ldots dx_0 = \\
&\int_{x_0}\mathbf{f}(x_0)\int_{x_1}\mathbf{A}(\frac{x_0}{d},x_1)\ldots\int_{x_{k-2}}\mathbf{A}(\frac{x_{k-3}}{d},x_{k-2}) \\
&\qquad\left(\mathbf{A}^*(\frac{x_{k-2}}{d},\mathcal{O}(v))-\mathbf{A}^*(\frac{x_{k-2}}{d},\mathcal{O}^2(v))\right)\;dx_{k-2}\ldots dx_0 = \ldots = \\
&\int_{x_0}\mathbf{f}(x_0)\left(\mathbf{A}^*(\frac{x_0}{d},\mathcal{O}^{k-1}(v))-\mathbf{A}^*(\frac{x_0}{d},\mathcal{O}^k(v))\right)\;dx_0 = \\
&\int_{x_0}\mathbf{f}(x_0)\left(e^{-x_0/d(\mathbf{R}\mathcal{O}^{k-1}(v)-\mathbf{Q})}-e^{-x_0/d(\mathbf{R}\mathcal{O}^k(v)-\mathbf{Q})}\right)\;dx_0 = \\
&\int_{x_0}\mathbf{f}(x_0)\left(e^{-x_0\mathcal{O}^k(v)}-e^{-x_0\mathcal{O}^{k+1}(v)}\right)\;dx_0 = \\
&\mathbf{f}^*(\mathcal{O}^k(v))-\mathbf{f}^*(\mathcal{O}^{k+1}(v)),
\end{aligned}
$$

where we used in (7) in the last but one step. For the special case of $k=0$ we can write

$$
\begin{aligned}
&E[\int_{t=0}^{d_1} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt] = \int_{x_0}\mathbf{f}(x_0)\int_{t=0}^{x_0/d}\mathbf{A}^*(t,v)e^{-(x_0-td)v}\;dt\;dx_0 = \\
&\int_{x_0}\mathbf{f}(x_0)\underbrace{\int_{t=0}^{x_0/d}e^{-t((\mathbf{R}-d\mathbf{I})v-\mathbf{Q})}\;dt}\,e^{-x_0v}\;dx_0.
\end{aligned}
$$

Substituting the underbraced integral from (40) we have

$$
\begin{aligned}
&E[\int_{t=0}^{d_1} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt]\,((\mathbf{R}-d\mathbf{I})v-\mathbf{Q}) = \\
&\int_{x_0}\mathbf{f}(x_0)\left(\mathbf{I}-e^{-\frac{x_0}{d}(\mathbf{R}-d\mathbf{I})v-\mathbf{Q})}\right)e^{-x_0v}\;dx_0 = \\
&\int_{x_0}\mathbf{f}(x_0)\left(\mathbf{I}-e^{-\frac{x_0}{d}(\mathbf{R}v-\mathbf{Q})}\right)\;dx_0 = \mathbf{f}^*(\mathcal{O}^0(v))-\mathbf{f}^*(\mathcal{O}^1(v)),
\end{aligned}
$$

which completes the proof of the proposition. □

Theorem 5. *In the stable fluid vacation model with exhaustive discipline the following relation holds for the steady-state vector LT of the fluid level at arbitrary epoch*

$$\mathbf{q}^*(v)(\mathbf{R}v-\mathbf{Q})\left((\mathbf{R}-d\mathbf{I})v-\mathbf{Q}\right) \quad = \quad \tfrac{vd}{c}(\mathbf{f}^*(v)-\mathbf{m}) \ . \tag{41}$$

Proof. The fluid level at arbitrary epoch can be expressed by the help of the fluid level at the last service start on LT level by utilizing the transient behavior of the arrived fluid (relation (7)) and taking into account that it can fall either in service or vacation period as well as its position in the actual period. Thus it is enough to average over a cycle for determining the behavior at arbitrary epoch.

$$\begin{aligned} \mathbf{q}^*(v) \quad &= \quad \frac{1}{c}E\left[\int_{t=0}^{C} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt\right] \qquad (42) \\ &= \quad \frac{1}{c}\left(\sum_{k=0}^{\infty}E\left[\int_{t=d_k}^{d_{k+1}} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt\right]+E\left[\int_{t=S}^{C} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt\right]\right). \end{aligned}$$

From Proposition 5 we have

$$\begin{aligned} &E[\int_{t=0}^{S} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt]\left((\mathbf{R}-d\mathbf{I})v-\mathbf{Q}\right)= \\ &\sum_{k=0}^{\infty}E[\int_{t=d_k}^{d_{k+1}} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt]\left((\mathbf{R}-d\mathbf{I})v-\mathbf{Q}\right)= \\ &\sum_{k=0}^{\infty}\mathbf{f}^*(\mathcal{O}^k(v))-\mathbf{f}^*(\mathcal{O}^{k+1}(v))=\mathbf{f}^*(v)-\mathbf{f}^*(\mathcal{O}^{\infty}(v))=\mathbf{f}^*(v)-\mathbf{m}, \end{aligned} \tag{43}$$

where we used the stationary version of (23) in the last step. For the evaluation of $E\left[\int_{t=S}^{C} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt\right]$ it is enough to originate the fluid level from the start of the vacation instead of the last service start. This is because (42) expresses $\mathbf{q}^*(v)$ as sum of expected values. Relying on this and using again (40) we get for the vacation period

$$\begin{aligned} &E\left[\int_{t=S}^{C} e^{-X(t)v}\mathbf{1}_{(\Omega(t))}dt\right](\mathbf{R}v-\mathbf{Q})=\int_{t=0}^{\infty}\int_{x=0}^{t}\mathbf{m}\mathbf{A}^*(x,v)dx\sigma(t)dt\,(\mathbf{R}v-\mathbf{Q})= \\ &\mathbf{m}\int_{t=0}^{\infty}\int_{x=0}^{t} e^{-x(\mathbf{R}v-\mathbf{Q})}dx\sigma(t)dt\,(\mathbf{R}v-\mathbf{Q})=\mathbf{m}\int_{t=0}^{\infty}\left(\mathbf{I}-e^{-t(\mathbf{R}v-\mathbf{Q})}\right)\sigma(t)dt= \\ &\mathbf{m}-\mathbf{m}\sigma^*(\mathbf{R}v-\mathbf{Q})=\mathbf{m}-\mathbf{f}^*(v), \end{aligned} \tag{44}$$

where we used (28) in the last step. Multiplying both sides of (42) by $(\mathbf{R}v-\mathbf{Q})\left((\mathbf{R}-d\mathbf{I})v-\mathbf{Q}\right)$ and substituting (43) and (44) we get statement of the theorem. □

(41) can be further simplified with the use of (31).

Theorem 6. *In the stable exhaustive fluid vacation model the steady-state fluid level at arbitrary epoch is the analytic continuation of*

$$\mathbf{q}^*(v) \quad = \quad \frac{1}{c(1-\rho)}\ \boldsymbol{\pi}\Big(\mathbf{I}-\sigma^*(\mathbf{R}v-\mathbf{Q})\Big)\left(\mathbf{R}v-\mathbf{Q}\right)^{-1} \ . \tag{45}$$

Proof. Using (28), (31) and a similar expansion as in (32) on the right hand side of (41) we have

$$\begin{aligned}
&\frac{vd}{c}(\mathbf{f}^*(v)-\mathbf{m}) = \frac{vd}{c}\ \mathbf{m}\Big(\sigma^*(\mathbf{R}v-\mathbf{Q})-\mathbf{I}\Big) = \\
&\frac{vd}{c(1-\rho)}\quad \boldsymbol{\pi}\left(\mathbf{I}-\frac{\mathbf{R}}{d}\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n E(\widetilde{\sigma}^n)}{n!}\left(\mathbf{R}v-\mathbf{Q}\right)^n-\mathbf{I}\right) = \\
&\frac{v}{c(1-\rho)}\quad \boldsymbol{\pi}\left(d\mathbf{I}-\mathbf{R}\right)\sum_{n=1}^{\infty}\frac{(-1)^n E(\widetilde{\sigma}^n)}{n!}\left(\mathbf{R}v-\mathbf{Q}\right)^n = \\
&\frac{v}{c(1-\rho)}\quad \underbrace{\boldsymbol{\pi}\left(d\mathbf{I}-\mathbf{R}\right)\left(\mathbf{R}v-\mathbf{Q}\right)}\sum_{n=1}^{\infty}\frac{(-1)^n E(\widetilde{\sigma}^n)}{n!}\left(\mathbf{R}v-\mathbf{Q}\right)^{n-1} = \\
&\frac{v}{c(1-\rho)}\quad \underbrace{\boldsymbol{\pi}\mathbf{R}\left(dv\mathbf{I}-v\mathbf{R}+\mathbf{Q}\right)}\sum_{n=1}^{\infty}\frac{(-1)^n E(\widetilde{\sigma}^n)}{n!}\left(\mathbf{R}v-\mathbf{Q}\right)^{n-1}\quad ,
\end{aligned}$$

where, in the underbraced manipulation we used that $\boldsymbol{\pi}\mathbf{Q}=\mathbf{0}$. Due to the fact that $(dv\mathbf{I}-v\mathbf{R}+\mathbf{Q})$ and $(\mathbf{R}v-\mathbf{Q})$ commutes, (41) can be rewritten as

$$\mathbf{q}^*(v)(\mathbf{R}v-\mathbf{Q})\left((\mathbf{R}-d\mathbf{I})v-\mathbf{Q}\right) \quad = \quad \frac{v}{c(1-\rho)}\quad \boldsymbol{\pi}\mathbf{R}\sum_{n=1}^{\infty}\frac{(-1)^n E(\widetilde{\sigma}^n)}{n!}\left(\mathbf{R}v-\mathbf{Q}\right)^{n-1}\left(dv\mathbf{I}-v\mathbf{R}+\mathbf{Q}\right)\ .$$

Multiplying both sides with the inverse of $((\mathbf{R}-d\mathbf{I})v-\mathbf{Q})$, we obtain

$$\begin{aligned}
\mathbf{q}^*(v)(\mathbf{R}v-\mathbf{Q}) \quad &= \quad \frac{-v}{c(1-\rho)}\quad \boldsymbol{\pi}\mathbf{R}\sum_{n=1}^{\infty}\frac{(-1)^n E(\widetilde{\sigma}^n)}{n!}\left(\mathbf{R}v-\mathbf{Q}\right)^{n-1} \\
&= \quad \frac{-1}{c(1-\rho)}\quad \boldsymbol{\pi}\left(v\mathbf{R}-\mathbf{Q}\right)\sum_{n=1}^{\infty}\frac{(-1)^n E(\widetilde{\sigma}^n)}{n!}\left(\mathbf{R}v-\mathbf{Q}\right)^{n-1}\quad ,
\end{aligned}$$

where we extended the expression with a negligible $\boldsymbol{\pi}\mathbf{Q}=\mathbf{0}$ term.

Multiplying both sides with $(v\mathbf{R}-\mathbf{Q})^{-1}$, we obtain

$$\begin{aligned}
\mathbf{q}^*(v) \quad &= \quad \frac{-1}{c(1-\rho)}\quad \boldsymbol{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n E(\widetilde{\sigma}^n)}{n!}\left(\mathbf{R}v-\mathbf{Q}\right)^{n}\left(v\mathbf{R}-\mathbf{Q}\right)^{-1} \\
&= \quad \frac{-1}{c(1-\rho)}\quad \boldsymbol{\pi}\Big(\sigma^*(\mathbf{R}v-\mathbf{Q})-\mathbf{I}\Big)\left(v\mathbf{R}-\mathbf{Q}\right)^{-1}\quad ,
\end{aligned}$$

which is the statement of the lemma. □

4.5. Moments of the stationary performance measures

The goal of this section is to obtain computable moments expressions based on the transform domain expressions of the previous section.

Corollary 2. *In the stable fluid vacation model with exhaustive discipline the steady-state vector mean of the fluid level at arbitrary epoch is*

$$\mathbf{q}^{[1]} = -\mathbf{q}^{(1)} = -\bar{q}^{(1)}\boldsymbol{\pi} - \left(\boldsymbol{\pi}\mathbf{R}\mathbf{Q} + \tfrac{d}{c}(\mathbf{f}^{(0)}-\mathbf{m})\right)\left(\mathbf{Q}^2+\mathbf{e}\boldsymbol{\pi}\right)^{-1}\quad , \tag{46}$$

where

$$\bar{q}^{(1)} = \tfrac{1}{\lambda}\left(\tfrac{1}{2}\bar{t}^{(2)} - \left(\boldsymbol{\pi}\mathbf{R}-\mathbf{t}^{(1)}\right)\left(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi}\right)^{-1}\mathbf{R}\mathbf{e}\right), \tag{47}$$

$$\mathbf{t}^{(1)} = \bar{t}^{(1)}\boldsymbol{\pi} - \mathbf{r}^{(1)} \left(\mathbf{Q} + \mathbf{e}\boldsymbol{\pi}\right)^{-1},$$

$$\bar{t}^{(1)} = \tfrac{1}{\lambda - d} \left(\tfrac{1}{2}\mathbf{r}^{(2)}\mathbf{e} + \mathbf{r}^{(1)} \left(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi}\right)^{-1} (\mathbf{R}-d\mathbf{I})\mathbf{e}\right),$$

$$\bar{t}^{(2)} = \tfrac{1}{\lambda - d} \left(\tfrac{1}{3}\mathbf{r}^{(3)}\mathbf{e} - \left(2\mathbf{t}^{(1)}-\mathbf{r}^{(2)}\right) \left(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi}\right)^{-1} (\mathbf{R}-d\mathbf{I})\mathbf{e}\right),$$

and

$$\mathbf{r}^{(1)} = \tfrac{d}{c}\mathbf{m}(\sigma^*(-\mathbf{Q})-\mathbf{I}), \quad \mathbf{r}^{(n)} = \tfrac{nd}{c}\mathbf{m} \left.\tfrac{d^{n-1}\sigma^*(\mathbf{R}v-\mathbf{Q})}{dv^{n-1}}\right|_{v=0} \quad \forall n>1 \tag{48}$$

Proof. The derivative of (41) at $v = 0$ gives

$$\mathbf{q}^{(1)}\mathbf{Q}^2 - \mathbf{q}^{(0)}\mathbf{R}\mathbf{Q} - \mathbf{q}^{(0)}\mathbf{Q}(\mathbf{R} - d\mathbf{I}) = \tfrac{d}{c}(\mathbf{f}^{(0)} - \mathbf{m}).$$

Since $(\mathbf{Q} + \mathbf{e}\boldsymbol{\pi})$ as well as $\left(\mathbf{Q}^2 + \mathbf{e}\boldsymbol{\pi}\right)$ are nonsingular [6, 7], using $\mathbf{q}^{(0)} = \boldsymbol{\pi}$, $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$, $\boldsymbol{\pi} \left(\mathbf{Q}^2 + \mathbf{e}\boldsymbol{\pi}\right)^{-1} = \boldsymbol{\pi}$ and adding and subtracting $\mathbf{q}^{(1)}\mathbf{e}\boldsymbol{\pi}$ we obtain (46), where the only remaining unknown is the scalar $\bar{q}^{(1)} = \mathbf{q}^{(1)}\mathbf{e}$, because according to (28) $\mathbf{f}^{(0)} = \mathbf{m}\sigma^*(-\mathbf{Q})$.

Unfortunately the computation of $\mathbf{q}^{(1)}\mathbf{e}$ is not that straight forward. To derive it we adopt a two step method. According to the structure of (41) we introduce $\mathbf{t}^*(v) = \mathbf{q}^*(v)(\mathbf{R}v - \mathbf{Q})$ and $\mathbf{r}^*(v) = \mathbf{t}^*(v)((\mathbf{R} - d\mathbf{I})v - \mathbf{Q})$ and express the moments of $\mathbf{q}^*(v)$ by the moment of $\mathbf{t}^*(v)$ in the first step and express the moments of $\mathbf{t}^*(v)$ by the moment of $\mathbf{r}^*(v)$ in the second step. Considering $\mathbf{q}^{(0)} = \boldsymbol{\pi}$, the first two derivatives of $\mathbf{t}^*(v) = \mathbf{q}^*(v)(\mathbf{R}v - \mathbf{Q})$ at $v = 0$ are

$$\mathbf{t}^{(1)} = -\mathbf{q}^{(1)}\mathbf{Q} + \boldsymbol{\pi}\mathbf{R}, \tag{49}$$

$$\mathbf{t}^{(2)} = -\mathbf{q}^{(2)}\mathbf{Q} + 2\mathbf{q}^{(1)}\mathbf{R}. \tag{50}$$

Adding and subtracting $\mathbf{q}^{(1)}\mathbf{e}\boldsymbol{\pi}$ to (49) and using $\boldsymbol{\pi}\left(\mathbf{Q} + \mathbf{e}\boldsymbol{\pi}\right)^{-1} = \boldsymbol{\pi}$ leads to

$$\mathbf{q}^{(1)} = \bar{q}^{(1)}\boldsymbol{\pi} + \left(\boldsymbol{\pi}\mathbf{R} - \mathbf{t}^{(1)}\right) \left(\mathbf{Q} + \mathbf{e}\boldsymbol{\pi}\right)^{-1}. \tag{51}$$

Post-multiplying (50) by $\mathbf{e}$, post-multiplying (51) by $\mathbf{Re}$ using $\boldsymbol{\pi}\mathbf{Re} = \lambda$ and $\bar{t}^{(n)} = \mathbf{t}^{(n)}\mathbf{e}$ for $n \geq 1$ and substituting the $\mathbf{q}^{(1)}\mathbf{Re}$ term from one equation to the other we obtain (47), where the unknowns are $\bar{t}^{(2)} = \mathbf{t}^{(2)}\mathbf{e}$ and $\mathbf{t}^{(1)}$. Similarly, the nth derivative of $\mathbf{r}^*(v) = \mathbf{t}^*(v)((\mathbf{R} - d\mathbf{I})v - \mathbf{Q})$ at $v = 0$ is

$$\mathbf{r}^{(n)} = -\mathbf{t}^{(n)}\mathbf{Q} + n\mathbf{t}^{(n-1)}(\mathbf{R} - d\mathbf{I}). \tag{52}$$

Applying the same steps as in the transformation of (49) to (51) we obtain

$$\mathbf{t}^{(n)} = \bar{t}^{(n)}\boldsymbol{\pi} + \left(n\mathbf{t}^{(n-1)}(\mathbf{R} - d\mathbf{I}) - \mathbf{r}^{(n)}\right) \left(\mathbf{Q} + \mathbf{e}\boldsymbol{\pi}\right)^{-1}, \tag{53}$$

and applying the same steps as in the derivation of (47) based on (50) and (51) we obtain

$$\bar{t}^{(n)} = \frac{1}{\lambda - d} \left(\frac{1}{n+1}\mathbf{r}^{(n+1)}\mathbf{e} - \left(n\mathbf{t}^{(n-1)} - \mathbf{r}^{(n)}\right) \left(\mathbf{Q}+\mathbf{e}\boldsymbol{\pi}\right)^{-1} (\mathbf{R}-d\mathbf{I})\mathbf{e}\right). \tag{54}$$

Considering that $\mathbf{t}^{(0)} = -\mathbf{q}^{(0)}\mathbf{Q} = \mathbf{0}$, (53) and (54) allows the consecutive computation of $\bar{t}^{(n)}$and $\mathbf{t}^{(n)}$. Finally, using (28) the derivatives of $\mathbf{r}^*(v) = \frac{vd}{c}(\mathbf{f}^*(v) - \mathbf{m})$ at $v = 0$ gives (48). □

4.6. Mean fluid level

According to Corollary 2 the major difficulty in the analysis of $\mathbf{q}^{[1]}$ is to obtain $\bar{q}^{(1)}$, which is the mean fluid level summed up for all states. The following result allows a simple computation of $\bar{q}^{(1)}$ which can be directly substituted into (46).

Corollary 3. *In the stable fluid vacation model with exhaustive discipline the steady-state mean of the fluid level at arbitrary epoch is*

$$\bar{q}^{(1)} = \frac{\lambda E(\widetilde{\sigma}^2)}{2c(1-\rho)}. \tag{55}$$

Proof. From the proof of Theorem 6 we have

$$\mathbf{q}^*(v) \quad = \quad \frac{-1}{c(1-\rho)} \quad \boldsymbol{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n E(\widetilde{\sigma}^n)}{n!} \left(\mathbf{R}v - \mathbf{Q}\right)^{n-1} \ .$$

Multiplying both sides with $\mathbf{e}$ from the right and taking the derivatives of according to v gives

$$\frac{d}{dv}\ \mathbf{q}^*(v)\mathbf{e} \quad = \quad \frac{-1}{c(1-\rho)} \quad \boldsymbol{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n E(\widetilde{\sigma}^n)}{n!} \sum_{m=0}^{n-2} \left(\mathbf{R}v - \mathbf{Q}\right)^m \mathbf{R} \left(\mathbf{R}v - \mathbf{Q}\right)^{n-m-2} \mathbf{e} \ .$$

Due to $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$ and $\mathbf{Q}\mathbf{e} = \mathbf{0}$, at $v = 0$ the only non-zero term of the right hand side is obtained at $n = 2$, $m = 0$, from which

$$-\bar{q}^{(1)} = \frac{d}{dv}\ \mathbf{q}^*(v)\mathbf{e}\ |_{v=0} \quad = \quad \frac{-1}{c(1-\rho)} \quad \boldsymbol{\pi} \frac{E(\widetilde{\sigma}^2)}{2} \mathbf{R}\mathbf{e} = \frac{-\lambda E(\widetilde{\sigma}^2)}{2c(1-\rho)} \ ,$$

which is the statement of the lemma. □

4.7. Steps of the numerical analysis

The steady-state mean vector $\mathbf{q}^{[1]}$ can be computed by the following steps.

1. Calculation of the steady-state phase distribution of the background Markov chain at start of vacation, $\mathbf{m}$, from the system of linear equations (27).
2. Computation of $\boldsymbol{\pi}$, λ, ρ and c by applying (1), (2) and (35), respectively.
3. Computation of the steady-state mean $\mathbf{q}^{[1]}$ by applying Corollary 2 and Corollary 3.

5. Fluid vacation model with positive effective fluid rates and PH distributed vacation period

In this section, on the one hand we extend the previous analysis with potentially positive effective fluid rate in the service period, which means that we relax assumption **A.3**, and on the other hand we restrict the model with PH distributed vacation time. The extension with positive effective fluid rate in the service period inhibits the application of most results that we obtained in the previous sections. For example, (23), (31) and (55) do not hold anymore. The restriction of the fluid vacation model with PH distributed vacation time makes possible the application of standard fluid queue results for the analysis of the exhaustive fluid vacation model in a reasonably simple way. The next section summarizes the main results borrowed from fluid queue analysis and the subsequent sections present the analysis with potentially positive effective fluid rate in the service period and PH distributed vacation time.

5.1. Fluid queue results

The following summary is mainly based on [1, 3]. Let us consider the Markov fluid queue $(Z(t), U(t)) \in \{\Theta \times \mathbb{R}^+\}$, with generator $\mathbf{T}$ and rate matrix $\mathbf{S}$ (i.e., $\mathbf{S}$ is the diagonal matrix of the fluid rates $s_1, \dots s_{|\Theta|}$).

Here we consider only the cases when the fluid rates are non-zero. Extensions to the cases with zero fluid rate can also be found in [1, 3], but we omit them here in order to avoid complex notations and lost focus.

Θ^+ and Θ^- denote the subset of states with positive and negative fluid rates, respectively. That is, state $i \in \Theta^+$ if the associated fluid rate, s_i, is positive and state $i \in \Theta^-$ if $s_i < 0$. We assume that the states are numbered such that the indices of the states in Θ^+ are lower than the ones in Θ^-. Accordingly, we introduce the following subdivision of matrixes

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}^{++} & \mathbf{T}^{+-} \\ \mathbf{T}^{-+} & \mathbf{T}^{--} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \mathbf{S}^{+} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{-} \end{pmatrix}. \tag{56}$$

Let τ be the stationary distribution of the Markov chain with generator $\mathbf{T}$. τ is the solution of $\tau\mathbf{T} = 0$, $\tau\mathbf{e} = 1$. Hereafter we assume that the fluid buffer is stable, i.e., $\tau\mathbf{S}\mathbf{e} < 0$.

For $i \in \Theta^+$ and $j \in \Theta^-$ the state transition probability during a busy period is

$$\mathbf{\Psi}_{ij} = Pr(Z(\gamma) = j | Z(0) = i, U(0) = 0)$$

where γ ($\gamma > 0$) is the first time when the buffer is idle. From the stability of the buffer we have $\sum_{j\in\Theta^-} \mathbf{\Psi}_{ij} = 1$ ($\forall i \in \Theta^+$). Matrix $\mathbf{\Psi}$ of size $|\Theta^+| \times |\Theta^-|$ is the solution of the Ricatti equation

$$\mathbf{\Psi}(-\mathbf{S}^-)^{-1}\mathbf{T}_{-+}\mathbf{\Psi} + \mathbf{\Psi}(-\mathbf{S}^-)^{-1}\mathbf{T}_{--} + (\mathbf{S}^+)^{-1}\mathbf{T}_{++}\mathbf{\Psi} + (\mathbf{S}^+)^{-1}\mathbf{T}_{+-} = \mathbf{0}.$$

Starting from $i \in \Theta^+$, for $j \in \Theta$ the mean sojourn time at (x, j) during a busy period is defined as

$$\mathbf{G}_{i,j}(x) = \frac{d}{dx}\int_{t=0}^{\infty} Pr(\gamma > t, Z(t) = j, U(t) < x | Z(0) = i, U(0) = 0)dt.$$

$\mathbf{G}_{i,j}(x)$ is the integral of the transition probability matrix from state $i \in \Theta^+$ and level 0 to state $j \in \Theta$ and level x within a busy period. Matrix $\mathbf{G}(x) = \{\mathbf{G}_{i,j}(x)\}$ of size $|\Theta^+| \times |\Theta|$ can be computed as

$$\mathbf{G}(x) = e^{\mathbf{K}x} \left[\mathbf{I} \;\; \mathbf{\Psi}\right] |\mathbf{S}|^{-1}, \tag{57}$$

where $\mathbf{K}$ of size $|\Theta^+| \times |\Theta^+|$ is given by $\mathbf{K} = \mathbf{\Psi}(-\mathbf{S}^-)^{-1}\mathbf{T}_{-+} + (\mathbf{S}^+)^{-1}\mathbf{T}_{++}$. We note that, both, $|\mathbf{S}|$ and $|\mathbf{S}|^{-1}$ are diagonal matrixes with strictly positive diagonal elements.

5.2. Analysis with positive service rates and PH distributed vacation period

During the service period the effective fluid rate in state i is $r_i - d$. We subdivide the set of states $\mathcal{S}$ to $\mathcal{S}^+$ and $\mathcal{S}^-$ according to the sign of $r_i - d$ and (as it is mentioned above) restrict our attention to the cases with non-zero effective rates. Without loss of generality we assume that the indexes of the states in $\mathcal{S}^+$ are lower than the ones in $\mathcal{S}^-$. Accordingly, we introduce the following subdivision of matrixes

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}^{++} & \mathbf{Q}^{+-} \\ \mathbf{Q}^{-+} & \mathbf{Q}^{--} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} \mathbf{R}^{+} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{-} \end{pmatrix}.$$

Let the vacation time be PH distributed with representation $(\alpha, \mathbf{A})$ of size n_{PH}. It means that the vacation time is represented by the time to absorbtion of a Markov chain with n_{PH} transient states and one absorbing state. The density, the Laplace transform and the mean of $\widetilde{\sigma}$ are

$$\sigma(t) = \frac{t}{dt}Pr(\widetilde{\sigma} < t) = \alpha e^{\mathbf{A}t}\mathbf{a}, \quad \sigma^*(s) = \alpha(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{a}, \quad \sigma = E(\widetilde{\sigma}) = \alpha(-\mathbf{A})^{-1}\mathbf{e},$$

where $\mathbf{a} = -\mathbf{A}\mathbf{e}$.

A time period of the fluid vacation model composed by a vacation and a subsequent service period can be represented as a busy period of a Markov fluid queue with an extended state space, $\Theta = \mathcal{S} \times \mathcal{S}_{PH}$, where the state of the Markov fluid queue is defined by the state of the Markov chain of the fluid vacation model, $\mathcal{S}$, and the phase of the phase type vacation time, $\mathcal{S}_{PH}$. As long as the Markov chain characterizing the phase type vacation time, $J(t)$, is in a transient state, the associated fluid vacation model is in vacation and the fluid level increases with rate $\mathbf{R}$, and when the Markov chain characterizing the phase type vacation time moves to the absorbing state the associated fluid vacation model is in service and the fluid level is changing with rate $\mathbf{R} - d\mathbf{I}$. After the end of the first busy period the two models behave differently, but we base our analysis on the identity of the vacation and service cycle of the fluid vacation model and the first busy period of the Markov fluid queue. The matrixes characterizing the Markov fluid queue are

$$\mathbf{T} = \begin{pmatrix} \mathbf{Q} \otimes \mathbf{A} & \mathbf{Q} \otimes \mathbf{a} \\ \mathbf{0} & \mathbf{Q} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \mathbf{R} \otimes \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} - d\mathbf{I} \end{pmatrix}$$

of size $|S|(n_{PH} + 1)$, which are divided into blocks of size $|\mathcal{S}|n_{PH}$ and $|\mathcal{S}|$. Considering the subdivision of matrixes $\mathbf{Q}$ and $\mathbf{R}$ we further have

$$\mathbf{T} = \left(\begin{array}{c|c} \mathbf{T}^{++} & \mathbf{T}^{+-} \\ \hline \mathbf{T}^{-+} & \mathbf{T}^{--} \end{array}\right) = \left(\begin{array}{ccc|c} \mathbf{Q}^{++} \otimes \mathbf{A} & \mathbf{Q}^{+-} \otimes \mathbf{A} & \mathbf{Q}^{++} \otimes \mathbf{a} & \mathbf{Q}^{+-} \otimes \mathbf{a} \\ \mathbf{Q}^{-+} \otimes \mathbf{A} & \mathbf{Q}^{--} \otimes \mathbf{A} & \mathbf{Q}^{-+} \otimes \mathbf{a} & \mathbf{Q}^{--} \otimes \mathbf{a} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}^{++} & \mathbf{Q}^{+-} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{Q}^{-+} & \mathbf{Q}^{--} \end{array}\right),$$

$$\mathbf{S} = \left(\begin{array}{c|c} \mathbf{S}^{+} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{S}^{-} \end{array}\right) = \left(\begin{array}{ccc|c} \mathbf{R}^{+} \otimes \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{-} \otimes \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}^{+} - d\mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}^{-} - d\mathbf{I} \end{array}\right),$$

whose blocks are of size $|\mathcal{S}^{+}|n_{PH}$, $|\mathcal{S}^{-}|n_{PH}$, $|\mathcal{S}^{+}|$ and $|\mathcal{S}^{-}|$. The first three diagonal matrix blocks of $\mathbf{S}$ contains positive fluid rates in the diagonal, while the last block contains negative ones. This way the first three matrix blocks of $\mathbf{T}$ and $\mathbf{S}$ are associated with the states in Θ^{+} and the last one with the states in Θ^{-}. Consequently, $|\Theta^{+}| = |\mathcal{S}^{+}|n_{PH} + |\mathcal{S}^{-}|n_{PH} + |\mathcal{S}^{+}|$ and $|\Theta^{-}| = |\mathcal{S}^{-}|$. The horizontal and vertical lines of the matrixes provide the effective rate based division of $\mathbf{T}$ and $\mathbf{S}$ in (56).

5.3. The system at vacation start epochs and stationary measures

To apply the general fluid queue results from Section 5.1 we still need the stationary distribution at vacation start epoch, $\mathbf{m}$, and the relation of $\mathbf{G}(x)$ with the extended state space Θ and $\mathbf{q}(x)$ with the original state space $\mathcal{S}$. To be more informative about the size of the subsequent matrix expressions, in this subsection, we indicate the size of the unity matrices and the column vector of ones in subscripts, e.g., $\mathbf{e}_{PH}$ and $\mathbf{I}_{|\mathcal{S}^{+}|}$ are the column vector of ones of size n_{PH} and unity matrixes of size $|\mathcal{S}^{+}|$, respectively. Let the row vectors $\mathbf{m}^{+}$ and $\mathbf{m}^{-}$ be the division of $\mathbf{m}$ according to $\mathcal{S}^{+}$ and $\mathcal{S}^{-}$.

Theorem 7. *At stationary vacation start epoch $\mathbf{m}^{+} = \mathbf{0}$ and $\mathbf{m}^{-}$ satisfies*

$$\mathbf{m}^{-} = \mathbf{m}^{-}[\mathbf{0} \;\; \mathbf{I}_{|\mathcal{S}^{-}|} \otimes \alpha \;\; \mathbf{0}]\mathbf{\Psi}, \tag{58}$$

with normalizing condition $\mathbf{m}^{-}\mathbf{e}_{|\mathcal{S}^{-}|} = 1$.

Proof. Since during the service period the effective fluid rates, $r_i - d$, in S^{+} are positive the exhaustive service period can not be completed in S^{+}, only in a state in S^{-}. As a results we have $\mathbf{m}^{+}(\ell) = \mathbf{0}$ for $\ell > 0$. When the ℓth service period is completed with distribution $\mathbf{m}^{-}(\ell)$ the initial state of the Markov fluid queue characterizing the next busy period with extended state space $\Theta = S \times S_{PH}$ is $[\mathbf{0} \;\; \mathbf{m}^{-}(\ell) \otimes \alpha \;\; \mathbf{0} \mid \mathbf{0}]$, out of which the first three blocks belong to Θ^{+} and the last to Θ^{-}, as it is indicated by the vertical line. For the $\mathbf{m}^{-} = \lim_{\ell \to \infty} \mathbf{m}^{-}(\ell)$ limit we obtain the stationary relation defined by (58), because matrix $\mathbf{\Psi}$ is the state transition probability matrix from Θ^{+} to Θ^{-} during a busy period. ☐

Theorem 8. *The steady state cycle time and the vector density of fluid level at arbitrary epoch are*

$$c = \mathbf{m}^-[\mathbf{0} \;\; \mathbf{I}_{|\mathcal{S}^-|} \otimes \alpha \;\; \mathbf{0}](-\mathbf{K})^{-1} \left[\mathbf{I}_{|\Theta^+|} \;\; \boldsymbol{\Psi}\right] |\mathbf{S}|^{-1}\mathbf{e}_{|\Theta|} \;, \tag{59}$$

$$\mathbf{q}(x) = \frac{1}{c} \;\; \mathbf{m}^-[\mathbf{0} \;\; \mathbf{I}_{|\mathcal{S}^-|} \otimes \alpha \;\; \mathbf{0}]e^{\mathbf{K}x} \left[\mathbf{I}_{|\Theta^+|} \;\; \boldsymbol{\Psi}\right] |\mathbf{S}|^{-1} \begin{bmatrix} \mathbf{I}_{|\mathcal{S}|} \otimes \mathbf{e}_{PH} \\ \mathbf{I}_{|\mathcal{S}|} \end{bmatrix}. \tag{60}$$

Proof. The mean cycle time can be computed from

$$\begin{aligned}
c &= \lim_{\ell\to\infty} \sum_{i\in\Theta} E(\gamma | Z(t^m(\ell)) = i, U(t^m(\ell)) = 0) \;\, Pr(Z(t^m(\ell)) = i) \\
&= \lim_{\ell\to\infty} \sum_{i\in\Theta} \int_{t=t^m(\ell)}^{\infty} Pr(\gamma > t - t^m(\ell) | Z(t^m(\ell)) = i, U(t^m(\ell)) = 0)dt \;\, Pr(Z(t^m(\ell)) = i) = \\
&= \lim_{\ell\to\infty} \lim_{x\to\infty} \sum_{i\in\Theta}\sum_{j\in\Theta} \int_{t=t^m(\ell)}^{\infty} Pr(\gamma > t - t^m(\ell), Z(t) = j, U(t) < x | Z(t^m(\ell)) = i, U(t^m(\ell)) = 0)dt \\
&\qquad\qquad Pr(Z(t^m(\ell)) = i) = \\
&= \mathbf{m}^-[\mathbf{0} \;\; \mathbf{I}_{|\mathcal{S}^-|} \otimes \alpha \;\; \mathbf{0}] \int_{x=0}^{\infty} \mathbf{G}(x)\mathbf{e} = \mathbf{m}^-[\mathbf{0} \;\; \mathbf{I}_{|\mathcal{S}^-|} \otimes \alpha \;\; \mathbf{0}](-\mathbf{K})^{-1} \left[\mathbf{I}_{|\Theta^+|} \;\; \boldsymbol{\Psi}\right] |\mathbf{S}|^{-1}\mathbf{e}_{|\Theta|} \;.
\end{aligned}$$

The vector density of fluid level at arbitrary epoch is the normalized fluid level during the stationary cycle which needs to be mapped from the extended state space Θ to the original state space $\mathcal{S}$. Matrix $\begin{bmatrix} \mathbf{I}_{|\mathcal{S}|} \otimes \mathbf{e}_{PH} \\ \mathbf{I}_{|\mathcal{S}|} \end{bmatrix}$ of size $|\mathcal{S}|(n_{PH}+1) \times |\mathcal{S}|$ makes the mapping, from which

$$\mathbf{q}(x) = \frac{1}{c} \;\; \mathbf{m}^-[\mathbf{0} \;\; \mathbf{I}_{|\mathcal{S}^-|} \otimes \alpha \;\; \mathbf{0}]\mathbf{G}(x) \begin{bmatrix} \mathbf{I}_{|\mathcal{S}|} \otimes \mathbf{e}_{PH} \\ \mathbf{I}_{|\mathcal{S}|} \end{bmatrix}$$

□

Corollary 4. *The vector Laplace transform of the stationary fluid level is*

$$\mathbf{q}^*(v) = \frac{1}{c} \;\; \mathbf{m}^-[\mathbf{0} \;\; \mathbf{I}_{|\mathcal{S}^-|} \otimes \alpha \;\; \mathbf{0}] \left(v\mathbf{I}_{|\Theta^+|} - \mathbf{K}\right)^{-1} \left[\mathbf{I}_{|\Theta^+|} \;\; \boldsymbol{\Psi}\right] |\mathbf{S}|^{-1} \begin{bmatrix} \mathbf{I}_{|\mathcal{S}|} \otimes \mathbf{e}_{PH} \\ \mathbf{I}_{|\mathcal{S}|} \end{bmatrix}. \tag{61}$$

Proof. The vector Laplace transform of the stationary fluid level is directly obtained from (60). □

Additionally, the matrix exponential expression in (60) allows to obtain the stationary fluid moments based on the computation of the nth derivative:

$$\begin{aligned}
\mathbf{q}^{(n)} &= \frac{1}{c}E\left[\int_{t=0}^{C} X(t)^n \mathbf{1}_{(\Omega(t))}dt\right] = \int_{x=0}^{\infty} x^n \mathbf{q}(x)dx \\
&= \frac{n!}{c} \;\; \mathbf{m}^-[\mathbf{0} \;\; \mathbf{I}_{|\mathcal{S}^-|} \otimes \alpha \;\; \mathbf{0}](-\mathbf{K})^{-n-1} \left[\mathbf{I}_{|\Theta^+|} \;\; \boldsymbol{\Psi}\right] |\mathbf{S}|^{-1} \begin{bmatrix} \mathbf{I}_{|\mathcal{S}|} \otimes \mathbf{e}_{PH} \\ \mathbf{I}_{|\mathcal{S}|} \end{bmatrix}.
\end{aligned}$$

Finally, we note that matrixes $\mathbf{K}$ and $\boldsymbol{\Psi}$, which are required to compute $\mathbf{q}(x)$ based on (60), can be computed with efficient and numerically stable methods of Markov fluid queue analysis [1, 3].

6. Numerical example

The numerical analysis based on Section 4 has already been demonstrated in [8]. In this section we demonstrate the computability of the analytical results presented in Section 5 and compute the steady-state mean fluid level of an arbitrarily picked exhaustive fluid vacation model. We assume a PH distributed

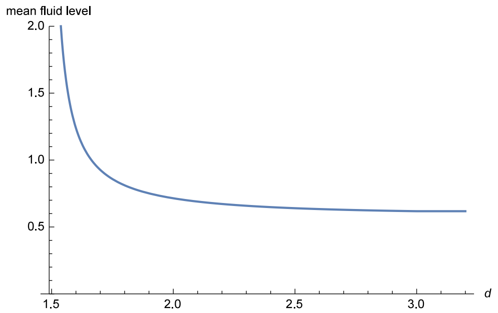

Figure 1: Overall mean fluid level as a function of the service rate

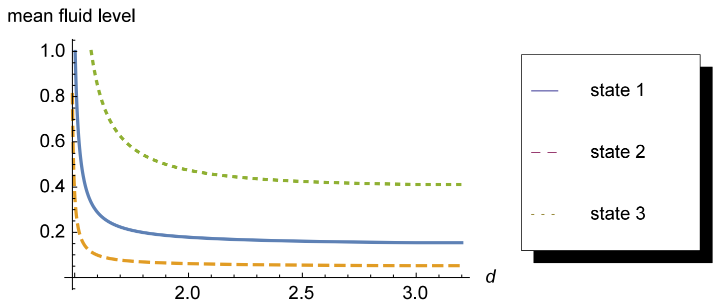

Figure 2: State dependent mean fluid level as a function of the service rate

vacation time with representation $(\alpha, \mathbf{A})$, where $\alpha = \{0.2, 0.8\}$ and $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}$. The fluid source is characterized by

$$\mathbf{Q} = \begin{pmatrix} -8 & 4 & 4 \\ 3 & -12 & 9 \\ 2 & 0 & -2 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From these model parameters we have $\sigma = E(\widetilde{\sigma}) = 0.4$, $\boldsymbol{\pi} = \{0.206897, 0.0689655, 0.724138\}$, $\lambda = \boldsymbol{\pi}\mathbf{Re} = 1.48276$. We study the behavior of this exhaustive fluid vacation model as a function of d. When $d > 3$, all effective fluid rates are negative ($r_i - d < 0$, $\forall i \in \mathcal{S}$ and this way $\mathcal{S}^- = \mathcal{S}$, $\mathcal{S}^+ = \emptyset$) and the results of Sections 3 - 4 are applicable. When $2 < d < 3$, the effective fluid rate of state 1, $3 - d$, is positive ($\mathcal{S}^- = \{2, 3\}$, $\mathcal{S}^+ = \{1\}$) and we need to apply the results of Section 5. When $1.48276 < d < 2$, the effective fluid rate of state 1 and 2 are positive ($\mathcal{S}^- = \{3\}$, $\mathcal{S}^+ = \{1, 2\}$). The vacation model is stable while $d > \lambda = 1.48276$. At $d = 3$ and at $d = 2$ we have a state with zero effective fluid rate and the results of Section 5 are not applicable. These points can be computed only with the extension to the zero fluid rate [1, 3] which is omitted in the current paper. Figure 1 plots the overall mean fluid level and Figure 2 the state dependent mean fluid levels as a function of d. The most involved step of the numerical analysis is the solution of the Ricatti equation. with matrices of size $|\Theta^+| = |\mathcal{S}^+|n_{PH} + |\mathcal{S}^-|n_{PH} + |\mathcal{S}^+|$ and $|\Theta^-| = |\mathcal{S}^-|$, where $n_{PH} = 2$, $|\mathcal{S}^-| + |\mathcal{S}^+| = 3$ and the actual values of $|\mathcal{S}^-|$ and $|\mathcal{S}^+|$ depends on d. In all of the considered cases the solution of the Ricatti equation as well as the whole computation were immediate. We conclude that the numerical analysis based on Section 5 is computationally far less expensive than the one based on Section 4, where the most expensive step is the computation of matrix $\sigma^*\left(\left(\frac{\mathbf{R}}{d} - \mathbf{I}\right)^{-1}\mathbf{Q}\right)$.

7. Conclusion

We considered the analysis of exhaustive fluid vacation models with Markov modulated fluid input. The fluid vacation model with strictly negative effective fluid rate allows the application of a methodology which is a fluid counterpart of the methodology used for discrete vacation models. The case when positive effective fluid rates are also allowed is analyzed with the methodology which is developed for Markov fluid buffers. The paper discusses both cases and presents computational procedures for the (time) stationary measures.

[1] S. Ahn and V. Ramaswami. Efficient algorithms for transient analysis of stochastic fluid flow models. *Journal of Applied Probability*, 42(2):pp. 531–549, 2005.
[2] S. C. Borst and O. J. Boxma. Polling models with and without switchover times. *Operations Research*, 45:536–543, 1997.
[3] A. da Silva Soares and G. Latouche. Matrix-analytic methods for fluid queues with finite buffers. *Performance Evaluation*, 63(4-5):295 – 314, 2006.
[4] B. T. Doshi. Queueing systems with vacations - a survey. *Queueing Systems*, 1:29–66, 1986.
[5] S. Gerschgorin. Über die Abgrenzung der Eigenwerte einer Matrix. *Izv. Akad. Nauk. USSR Otd. Fiz.-Mat. Nauk*, 7:749–754, 1931.
[6] D. L. Lucantoni. New results on the single server queue with a batch markovian arrival process. *Stochastic Models*, 7:1–46, 1991.

[7] M. F. Neuts. *Structured stochastic matrices of M/G/1 type and their applications.* Marcel Dekker, New York, 1989.

[8] Z. Saffer and M. Telek. Fluid vacation model with markov modulated load and exhaustive discipline. In *Computer Performance Engineering, EPEW*, volume 8721 of *LNCS*, pages 59–73, Sept. 2014.

[9] Z. Saffer and M. Telek. Fluid vacation model with markov modulated load and gated discipline. In *QTNA*, Aug. 2014.

[10] H. Takagi. *Queueing Analysis - A Foundation of Performance Evaluation, Vacation and Prority Systems, vol.1.* North-Holland, New York, 1991.