Fluid polling system with Markov modulated load and gated discipline

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Abstract. In this paper we provide an analysis for fluid polling models with Markov modulated load and gated discipline. The fluid arrival to the stations is modulated by a common continuous-time Markov chain. The fluid is removed at the stations during the service period by a station dependent constant rate.
We build partly on the methods used previously in the analysis of fluid vacation models with gated discipline. We establish steady-state relationships on Laplace transform level regarding the joint distribution of the fluid levels at the stations and the state of the modulating Markov chain among different characteristic epochs including start and end of the service at each station. We derive the steady-state vector Laplace transform of the fluid levels at the stations at arbitrary epoch and its mean.

Keywords: queueing theory, fluid model, polling system, gated discipline

1 Introduction

In fluid queueing models the work arrives on a continuous manner, i.e., fluid flows into the buffer instead of customer arrivals. Such models can be used as the limit for the workload in the analysis of regular queueing systems, for example in Heavy-Traffic (HT) analysis or stability analysis [1], [2].

The Markov modulated fluid queues have been analyzed by several authors using matrix analytic methods, see, e.g., [3], [4].

The first paper relevant to fluid polling model is the paper from Czerniak and Yechiali [5]. They analyzed a fluid polling model with constant load and service rate. The only non-deterministic part of their model is the switchover time.

Fluid vacation models with Markov modulated load have been analyzed in the subsequent papers [6], [7], [8]. The authors studied the fluid vacation models with gated discipline and with exhaustive discipline under negative fluid rate during service. The analysis of the exhaustive fluid vacation model has been extended to the case of the non-negative fluid rate during service in [9].

This work is a natural continuation of the above research line on fluid vacation models in which we extend the analysis of fluid gated vacation model to the corresponding fluid polling system. The contribution of this work is the extension of the analysis of fluid gated vacation model with Markov modulated load to the fluid polling system. However, we build only partly on the methods used in the analysis of fluid vacation model with gated discipline. We establish steady-state relationships on Laplace transform (LT) level regarding the joint distribution of the fluid levels at the stations and the state of the modulating Markov chain among different characteristic epochs, like start and end of the service at each station. We derive the steady-state vector LT of the fluid levels at the stations at arbitrary epoch and its mean.

The rest of the paper is organized as follows. Section 2 gives the description and the stability criterion of the model. The analysis of the steady-state fluid levels at characteristic epochs follows in section 3. Section 4 provides the analysis of the steady-state fluid levels at arbitrary epoch and its mean. .

2 Model and Notation

2.1 Model description

We consider a fluid polling model with Markov modulated load and gated discipline. The polling system consists of N stations. Each station has an infinite fluid buffer.

A common continuous-time Markov chain (CTMC) ($\Omega(t)$ for $t \geq 0$) with state space $\Omega = \{1, \ldots, L\}$ modulates the arriving fluid flows at the station. The generator of this background CTMC is denoted by $\mathbf{Q}$. The input fluid rates at station i are specified by diagonal fluid input rate matrix $\mathbf{R}_i$, for $i \in \{1, \ldots, N\}$. If the background CTMC is in state j ($\Omega(t) = j$) then fluid flows into the buffer of station i at rate $r_i(j)$ for $j \in \{1, \ldots, L\}$ and $i \in \{1, \ldots, N\}$. When the server visits station i it removes fluid from its fluid buffer at finite rate $d_i > 0$ for $i \in \{1, \ldots, N\}$. Consequently, when the server visits station i and the overall Markov chain is in state j ($\Omega(t) = j$) then the fluid level of the buffer of station i changes at rate $r_i(j) - d_i$ otherwise it changes at rate $r_i(j)$ due to the lack of service. The length of the server's visit at station i in the polling model is determined by the service discipline applied at that station. In this work we consider the gated discipline. Under gated discipline only the fluid is removed during the server visit at station i, which is present at the station already upon the server arrival. The cycle time (or simple cycle) is the time between two consecutive visits of the server to the same station. In this paper, if not stated otherwise then we understand the station index i as $mod(N)$, i.e. whenever it reaches N it continues by 1. The switchover time from station i to the next station in the consecutive cycles is independent and identically distributed (i.i.d.). The probability distribution function (pdf) of the switchover time from station i, the corresponding Laplace transform (LT) and mean is denoted by $\sigma_i(t)$ and $\sigma_i^*(s)$, σ_i, respectively. We consider non-zero switchover-times model, and we use the notation $\sigma = \sum_{i=1}^N \sigma_i$. We set the following assumptions on the fluid polling model:

- **A.1** The generator matrix $\mathbf{Q}$ of the modulating CTMC is irreducible.
- **A.2** The fluid rates $r_i(j)$ are positive and finite, i.e. $r_i(j > 0$ for $j \in \{1, \dots, L\}$ and $i \in \{1, \dots, N\}$.

Remark 1. The case of independent fluid inputs is also included by the approach with one common modulating CTMC as special case. In that case $\mathbf{Q} = \oplus_{i=1}^N \hat{\mathbf{Q}}_i$ and $\mathbf{R}_i = (\otimes_{k=1}^{i-1}\mathbf{I}) \otimes \hat{\mathbf{R}}_i \otimes (\otimes_{k=i+1}^N \mathbf{I})$, where $\hat{\mathbf{Q}}_i$ and $\hat{\mathbf{R}}_i$ denote the independent generator and the fluid input rate matrix of station i, for $i \in \{1, \dots, N\}$, and $\otimes$ and $\oplus$ denote the Kronecker product and Kronecker sum operations, respectively.

Let π be the stationary probability vector of the modulating Markov chain. Due to assumption **A.1**, $\pi\mathbf{Q} = 0$ and $\pi\mathbf{e} = 1$ uniquely determine π, where $\mathbf{e}$ is the $L \times 1$ unit column vector. The stationary fluid flow rate and the utilization at station i, λ_i and ρ_i, respectively, can be given for $i \in \{1, \dots, N\}$ as

$$\lambda_i = \pi\mathbf{R}_i\mathbf{e} \text{ and } \rho_i = \frac{\lambda_i}{d_i}, \tag{1}$$

and the total utilization is

$$\rho = \sum_{i=1}^N \rho_i. \tag{2}$$

The arrival instant of the server to station i is called i-polling epoch. Similarly, the time instant when the server departs from station i is called i-departure epoch.

For the j, l element of the matrix $\mathbf{Z}$ the notation $\mathbf{Z}_{j,l}$ is used. Furthermore, $[\mathbf{z}_i]_j$ denote the j-th element of vector $\mathbf{z}_i$. When there is a set of random variables characterized by one (two) parameters, e.g., Y_n $(Y_{k,n})$, then the n (k, n) element of its vector (matrix) LT is $E(e^{-sY_n})$ $(E(e^{-sY_{k,n}}))$. When $\mathbf{X}^*(v)$, $Re(v) \geq 0$ is a matrix LT, $\mathbf{X}^{(k)}$ denotes its k-th $(k \geq 1)$ moment, i.e., $\mathbf{X}^{(k)} = (-1)^k \frac{d^k}{ds^k}\mathbf{X}^*(v)|_{v=0}$ and $\mathbf{X}$ denotes its value at $s = 0$, i.e., $\mathbf{X} = \mathbf{X}^*(0)$. Similarly when $\mathbf{x}^*(v)$, $Re(v) \leq 0$ is a vector LT, $\mathbf{x}^{(k)}$ denotes its k-th $(k \geq 1)$ moment, i.e., $\mathbf{x}^{(k)} = (-1)^k \frac{d^k}{ds^k}\mathbf{x}^*(v)|_{v=0}$ and $\mathbf{x}$ denotes its value at $s = 0$, i.e., $\mathbf{x} = \mathbf{x}^*(0)$.

2.2 Stability

We apply a workload argument to get a necessary condition of the stability. The amount of work flowing to station i during a time unit is equal to its utilization, ρ_i. The necessary condition of the stability is that the total amount of work flowing to all stations during a time unit must be less than the work-amount of that time unit, which is 1. Therefore the necessary condition of the stability is given as

$$\rho < 1. \tag{3}$$

Remark 2. If the system would limit the work which could be done on average, i.e., when less then 1 work-amount could be done during a time unit, then further restrictions were needed for the sufficiency. However, the gated discipline is "unlimited", since it does not set any load-independent limit on the work-amount, which could be performed during a service period. Therefore the above necessary condition is also a sufficient one for the stability of the system.

3 The steady-state fluid levels at polling epochs

3.1 Transient analysis of the accumulated fluid

In this section, we consider the joint distribution of the accumulated amount of fluid entering into the individual stations during time $t \geq 0$. We derive the joint LT of the accumulated fluid levels flowed into the stations and the state of the common modulated Markov chain as a function of time.

Let $Y_i(t) \in \mathbb{R}^+$ denote the accumulated amount of fluid entering into station i until time t for $i \in \{1, \ldots, N\}$. Using the notation $\bar{y} = (y_1, \ldots, y_N)$ let the transition density matrix $\mathbf{A}(t, \bar{y})$ be composed by its elements $\mathbf{A}_{j,k}(t, \bar{y})$ as

$$\mathbf{A}_{j,k}(t, \bar{y}) = \frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial y_N}$$
$$Pr(\Omega(t) = k, Y_1(t) < y_1, \ldots Y_N(t) < y_N | \Omega(0) = j, Y_1(0) = \ldots = Y_N(0) = 0).$$

The fluid level is zero at each station i at $t = 0$ ($Y_i(0) = 0$) with probability 1. Hence the transition density matrix for $t = 0$ is given as

$$\mathbf{A}(0, y_1, \ldots, y_N) = \delta(y_1) \ldots \delta(y_N)\mathbf{I}, \tag{4}$$

where $\delta(y)$ denotes the unit impulse function at y=0, whose LT is 1. Furthermore the accumulated amount of fluids are greater than zero for $t > 0$ at every stations ($Y_i(t) > 0$, for $i \in \{1, \ldots, N\}$) due to assumption **A.2**. It follows that

$$\mathbf{A}(t, y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_N) = \mathbf{0}, \quad t > 0, i \in \{1, \ldots, N\}, \tag{5}$$

where $\mathbf{0}$ denotes the $L \times L$ zero matrix. We also use the notation $\bar{v} = (v_1, \ldots, v_N)$ and we define several LTs of matrix $\mathbf{A}(t, \bar{y})$ as

$$\mathbf{A}^*(s, \bar{y}) = \int_{t=0}^{\infty} \mathbf{A}(t, y_1, \ldots, y_N) e^{-st} dt,$$
$$\mathbf{A}^{N*}(t, \bar{v}) = \int_{y_1=0}^{\infty} \ldots \int_{y_N=0}^{\infty} \mathbf{A}(t, y_1, \ldots, y_N) e^{-v_1 y_1} \ldots e^{-v_N y_N} \, dy_N \ldots dy_1,$$
$$\mathbf{A}^{(N+1)*}(s, \bar{v}) = \int_{y_1=0}^{\infty} \ldots \int_{y_N=0}^{\infty} \mathbf{A}^*(s, y_1, \ldots, y_N) e^{-v_1 y_1} \ldots e^{-v_N y_N} \, dy_N \ldots dy_1,$$

and

$$\mathbf{A}^{(N)*}(s, v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_N) = \int_{y_1=0}^{\infty} \ldots \int_{y_{i-1}=0}^{\infty} \int_{y_{i+1}=0}^{\infty} \ldots \int_{y_N=0}^{\infty} \mathbf{A}^*(s, y_1, \ldots, y_{i-1}, 0, y_{i+1} \ldots y_N) \\ e^{-v_1 y_1} \ldots e^{-v_{i-1} y_{i-1}} e^{-v_{i+1} y_{i+1}} \ldots e^{-v_N y_N} \, dy_N \ldots dy_{i+1} dy_{i-1} \ldots dy_1,$$

where the coefficients of $*$ in the superscript of matrix $\mathbf{A}$ denotes the number of LTs.

Proposition 1. *In the fluid polling model the joint matrix LT of the accumulated amount of fluid entering in interval* $(0, t]$ *can be expressed as*

$$\mathbf{A}^{(N)*}(t, \bar{v}) = e^{-t\left(\sum_{i=1}^{N} \mathbf{R}_i v_i - \mathbf{Q}\right)}. \tag{6}$$

Proof. The Markov process $\{\Omega(t), Y_1(t), \ldots, Y_N(t)\}$ describes a homogenous first order fluid model. Its transient behavior can be characterized by forward Kolmogorov equations as

$$\frac{\partial}{\partial t}\mathbf{A}(t, \bar{y}) + \frac{\partial}{\partial y_1}\mathbf{A}(t, \bar{y})\mathbf{R}_1 + \ldots + \frac{\partial}{\partial y_N}\mathbf{A}(t, \bar{y})\mathbf{R}_N = \mathbf{A}(t, \bar{y})\mathbf{Q}. \tag{7}$$

and with initial conditions (4) and (5). Taking the LT of (7) with respect to t yields

$$\mathbf{A}^*(s, \bar{y})s - \mathbf{A}(0, \bar{y}) + \frac{\partial}{\partial y_1}\mathbf{A}^*(s, \bar{y})\mathbf{R}_1 + \ldots + \frac{\partial}{\partial y_N}\mathbf{A}^*(s, \bar{y})\mathbf{R}_N = \mathbf{A}^*(s, \bar{y})\mathbf{Q}. \tag{8}$$

Now taking the LT of (8) with respect to $y_1, \ldots, y_N$ we have

$$\begin{aligned} \mathbf{A}^{(N+1)*}(s, \bar{v})s - \mathbf{A}^{(N)*}(0, \bar{v}) & \\ + \Big(\mathbf{A}^{(N+1)*}(s, \bar{v})v_1 - \mathbf{A}^{(N)*}(s, 0, v_2, \ldots, v_N)\Big)\, \mathbf{R}_1 + \ldots & \\ + \Big(\mathbf{A}^{(N+1)*}(s, \bar{v})v_N - \mathbf{A}^{(N)*}(s, v_1, \ldots, v_{N-1}, 0)\Big)\, \mathbf{R}_N & \\ = \mathbf{A}^{(N+1)*}(s, \bar{v})\mathbf{Q}. & \end{aligned} \tag{9}$$

Applying (4) and (5) in (9) gives

$$\begin{aligned} \mathbf{A}^{(N+1)*}(s, \bar{v})s - \mathbf{I} + \mathbf{A}^{(N+1)*}(s, \bar{v})\mathbf{R}_1 v_1 + \ldots + \mathbf{A}^{(N+1)*}(s, \bar{v})\mathbf{R}_N v_N & \\ = \mathbf{A}^{(N+1)*}(s, \bar{v})\mathbf{Q}. & \end{aligned} \tag{10}$$

After rearranging (10) we get

$$\mathbf{A}^{(N+1)*}(s, \bar{v}) = \left(\mathbf{I}s + \mathbf{R}_1 v_1 + \ldots + \mathbf{R}_N v_N - \mathbf{Q}\right)^{-1}. \tag{11}$$

Taking the inverse Laplace transform of (11) with respect to s results in the statement of the proposition.

3.2 The governing equations of the system at polling and departure epochs

Let $X_i(t) \in \mathbb{R}^+$ denote the actual level of the fluid buffer at station i at time t for $i \in \{1,\ldots,N\}$. Let $t_i^f(\ell)$ be the time of the i-polling epoch in the ℓ-th cycle for $\ell \geq 1$ and $i = \{1,\ldots,N\}$. We use the notation $\bar{x} = (x_1,\ldots,x_N)$. We define the joint densities of the fluid levels at the stations and the state of the modulating Markov chain at the i-polling epoch in the ℓ-th cycle, for $\ell \geq 1$ and $i = \{1,\ldots,N\}$, the $1 \times L$ vector $\mathbf{f}_i(\ell,\bar{x})$ by its elements as

$$[\mathbf{f}_i(\ell,\bar{x})]_j = \frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_N} \\ Pr(\Omega(t_i^f(\ell)) = j, X_1(t_i^f(\ell)) < x_1, \ldots X_N(t_i^f(\ell)) < x_N),\ j \in \Omega.$$

The steady-state counterpart of the vector $\mathbf{f}_i(\ell,\bar{x})$ is defined as

$$\mathbf{f}_i(\bar{x}) = \lim_{\ell\to\infty} \mathbf{f}_i(\ell,\bar{x}),$$

and its LT is given as

$$\mathbf{f}_i^{(N)*}(\bar{v}) = \int_{x_1=0}^{\infty}\cdots\int_{x_N=0}^{\infty} \mathbf{f}_i(\bar{x}) e^{-v_1x_1}\ldots e^{-v_Nx_N} dx_N \ldots dx_1.$$

Analogously let $t_i^m(\ell)$ be the time of the i-departure epoch in the ℓ-th cycle for $\ell \geq 1$ and $i = \{1,\ldots,N\}$. We define the joint densities of the fluid levels at the stations and the state of the modulating Markov chain at the i-departure epoch in the ℓ-th cycle, for $\ell \geq 1$ and $i = \{1,\ldots,N\}$, the $1 \times L$ vector $\mathbf{m}_i(\ell,\bar{x})$ by its elements as

$$[\mathbf{m}_i(\ell,\bar{x})]_j = \frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_N} \\ Pr(\Omega(t_i^m(\ell)) = j, X_1(t_i^m(\ell)) < x_1, \ldots X_N(t_i^m(\ell)) < x_N),\ j \in \Omega.$$

The steady-state joint densities of the fluid levels at the stations and the state of the modulating Markov chain at the i-departure epoch are defined as

$$\mathbf{m}_i(\bar{x}) = \lim_{\ell\to\infty} \mathbf{m}_i(\ell,\bar{x}),$$

and its LT is given as

$$\mathbf{m}_i^{(N)*}(\bar{v}) = \int_{x_1=0}^{\infty}\cdots\int_{x_N=0}^{\infty} \mathbf{m}_i(\bar{x}) e^{-v_1x_1}\ldots e^{-v_Nx_N} dx_N \ldots dx_1.$$

We define a notation for substituting the multivariate $L \times L$ matrix function $\mathbf{H}(\bar{v})$ into the defining integral of the LT $\mathbf{f}_i^{(N)*}(\bar{v})$ as

$$\mathbf{f}_i^{(N)*}(v_1,\ldots,v_{i-1},\mathbf{H}(\bar{v}),v_{i+1},\ldots,v_N) = \\ \int_{x_1=0}^{\infty}\cdots\int_{x_N=0}^{\infty} \mathbf{f}_i(\bar{x}) e^{-v_1x_1}\ldots e^{-v_{i-1}x_{i-1}} e^{-\mathbf{H}(\bar{v})x_i} e^{-v_{i+1}x_{i+1}} \ldots e^{-v_Nx_N} dx_N \ldots dx_1. \quad (12)$$

Theorem 1. *The governing equations of the stable fluid polling model with gated discipline in terms of the steady-state joint vector LTs of the fluid levels at the stations at the i-polling and i-departure epochs for $i \in \{1, \ldots, N\}$ are given as*

– *for the transition $\mathbf{f}_i \to \mathbf{m}_i$*

$$\mathbf{m}_i^{(N)*}(\bar{v}) = \mathbf{f}_i^{(N)*}(v_1, \ldots, v_{i-1}, \frac{\sum_{i=1}^N \mathbf{R}_i v_i - \mathbf{Q}}{d_i}, v_{i+1}, \ldots, v_N), \qquad (13)$$

– *and for the transition $\mathbf{m}_i \to \mathbf{f}_{i+1}$*

$$\mathbf{f}_{i+1}^{(N)*}(\bar{v}) = \mathbf{m}_i^{(N)*}(\bar{v}))\sigma_i^*(\sum_{i=1}^N \mathbf{R}_i v_i - \mathbf{Q}). \qquad (14)$$

Proof. Due to the gated service discipline the fluid level at station i at i-departure epoch equals the level of the fluid arriving during the service duration of station i. The fluid level at stations $j \neq i$ at i-departure epoch is the sum of the fluid level at the previous i-polling epoch and the fluid arrived in between. If the fluid level at station i at i-polling epoch equals $\xi_i > 0$ then service duration is $\frac{\xi_i}{d_i}$ due to the gated discipline. Accordingly we can express $[\mathbf{m}_i(\bar{x})]_k$ as

$$\begin{aligned} [\mathbf{m}_i(\bar{x})]_k = & \sum_{j=1}^L \int_{\xi_i=0}^\infty \int_{y_1=0}^{x_1} \cdots \int_{y_{i-1}=0}^{x_{i-1}} \int_{y_{i+1}=0}^{x_{i+1}} \cdots \int_{y_N=0}^{x_N} \\ & [\mathbf{f}_i(x_1 - y_1, \ldots, x_{i-1} - y_{i-1}, \xi_i, x_{i+1} - y_{i+1}, \ldots, x_N - y_N)]_j \\ & \mathbf{A}_{jk}(\frac{\xi_i}{d_i}, y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_N) dy_N \ldots dy_{i+1} dy_{i-1} \ldots dy_1 d\xi_i. \end{aligned}$$

Changing to vector and matrix notation results in

$$\begin{aligned} \mathbf{m}_i(\bar{x}) = & \int_{\xi_i=0}^\infty \int_{y_1=0}^{x_1} \cdots \int_{y_{i-1}=0}^{x_{i-1}} \int_{y_{i+1}=0}^{x_{i+1}} \cdots \int_{y_N=0}^{x_N} \\ & \mathbf{f}_i(x_1 - y_1, \ldots, x_{i-1} - y_{i-1}, \xi_i, x_{i+1} - y_{i+1}, \ldots, x_N - y_N) \\ & \mathbf{A}(\frac{\xi_i}{d_i}, y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_N) dy_N \ldots dy_{i+1} dy_{i-1} \ldots dy_1 d\xi_i. \end{aligned}$$

Using the convolution property of the LT, the LT of $\mathbf{m}_i(\bar{x})$ with respect to $\bar{x}$ can be given as

$$\mathbf{m}_i^{(N)*}(\bar{v}) = \int_{\xi_i=0}^\infty \mathbf{f}_i^{(N-1)*}(v_1, \ldots, v_{i-1}, \xi_i, v_{i+1}, \ldots, v_N) \mathbf{A}^{(N)*}(\frac{\xi_i}{d_i}, \bar{v}) d\xi_i. \qquad (15)$$

Applying (6) in (15) yields

$$\mathbf{m}_i^{(N)*}(\bar{v}) = \int_{\xi_i=0}^\infty \mathbf{f}_i^{(N-1)*}(v_1, \ldots, v_{i-1}, \xi_i, v_{i+1}, \ldots, v_N) e^{-\frac{\xi_i}{d_i}\left(\sum_{i=1}^N \mathbf{R}_i v_i - \mathbf{Q}\right)} d\xi_i. \qquad (16)$$

The first statement of the theorem comes by observing that the right hand side of (16) is an LT with respect to ξ_i and applying the notation (12).

The fluid level at any station j at $i+1$-polling epoch is the sum of the fluid level at the previous i-departure epoch and the fluid arrived in between. Therefore we have

$$[\mathbf{f}_{i+1}(\bar{x})]_k = \sum_{j=1}^{L} \int_{t=0}^{\infty} \int_{y_1=0}^{x_1} \dots \int_{y_N=0}^{x_N} [\mathbf{m}_i(x_1 - y_1, \dots, x_N - y_N)]_j \\ \mathbf{A}_{jk}(t, y_1, \dots, y_N) \sigma_i(t) dy_N \dots dy_1 dt. \quad (17)$$

Changing (17) to matrix notation and using the convolution property of LT we get

$$\mathbf{f}_{i+1}^{(N)*}(\bar{v}) = \int_{t=0}^{\infty} \mathbf{m}_i^{(N)*}(\bar{v}) \mathbf{A}^{(N)*}(t, \bar{v}) \sigma_i(t) dt. \quad (18)$$

Applying (6) in (18) and rearrangement leads to

$$\mathbf{f}_{i+1}^{(N)*}(\bar{v}) = \mathbf{m}_i^{(N)*}(\bar{v})) \int_{t=0}^{\infty} e^{-t\left(\sum_{i=1}^{N} \mathbf{R}_i v_i - \mathbf{Q}\right)} \sigma_i(t) dt. \quad (19)$$

The second statement of the theorem comes by observing that on the r.h.s. of (19) there is an LT with respect to t. □

3.3 The steady-state vector moments of the fluid levels at polling epochs

Corollary 1. *The relation for the transition* $\mathbf{f}_i \to \mathbf{f}_{i+1}$*, for* $i \in \{1, \dots, N\}$ *in the stable fluid polling model with gated discipline are given as*

$$\mathbf{f}_{i+1}^{(N)*}(\bar{v}) = \mathbf{f}_i^{(N)*}(v_1, \dots, v_{i-1}, \frac{\sum_{i=1}^{N} \mathbf{R}_i v_i - \mathbf{Q}}{d_i}, v_{i+1}, \dots, v_N) \ \sigma_i^*(\sum_{i=1}^{N} \mathbf{R}_i v_i - \mathbf{Q}), \quad (20)$$

Proof. The corollary comes by applying (13) in (14). □

We define the joint moments of the fluid levels at the stations as

$$\mathbf{f}_i^{(j_1, \dots, j_N)} = (-1)^{\sum_{m=1}^{N} j_m} \frac{\partial^{j_1}}{\partial v_1^{j_1}} \dots \frac{\partial^{j_N}}{\partial v_N^{j_N}} \mathbf{f}_i^{(N)*}(v_1, \dots, v_N) \bigg|_{v_1 = \dots = v_N = 0}.$$

Furthermore, we define the following quantities

$$\mathbf{H}_k^{(j_1, \dots, j_N)} = (-1)^{\sum_{m=1}^{N} j_m} \frac{1}{k!} \frac{\partial^{j_1}}{\partial v_1^{j_1}} \dots \frac{\partial^{j_N}}{\partial v_N^{j_N}} \left(\frac{\mathbf{Q} - \sum_{i=1}^{N} \mathbf{R}_i v_i}{d_i} \right)^k \bigg|_{v_1 = \dots = v_N = 0}$$

$$\sigma_i^{(j_1, \dots, j_N)} = (-1)^{\sum_{m=1}^{N} j_m} \frac{\partial^{j_1}}{\partial v_1^{j_1}} \dots \frac{\partial^{j_N}}{\partial v_N^{j_N}} \sigma_i^*(\sum_{i=1}^{N} \mathbf{R}_i v_i - \mathbf{Q}) \bigg|_{v_1 = \dots = v_N = 0}$$

Corollary 2. *The joint moments of the fluid levels at the stations can be determined from the following approximate system of linear equations*

$$\mathbf{f}_{i+1}^{(j_1,\dots,j_N)} = \sum_{j_{1,1}+\ldots+j_{1,3}=j_1} \binom{j_1}{j_{1,1}, j_{1,2}, j_{1,3}} \cdots \sum_{j_{N,1}+\ldots+j_{N,3}=j_N} \binom{j_N}{j_{N,1}, j_{N,2}, j_{N,3}} \sum_{k=0}^{K-j_{i,1}} \mathbf{f}_i^{(j_{1,1},\dots,j_{i-1,1},j_{i,1}+k,j_{i+1,1},\dots,j_{N,1})} \mathbf{H}_k^{(j_{1,2},\dots,j_{N,2})} \sigma_i^{(j_{1,3},\dots,j_{N,3})}, \quad (21)$$

where $j_1, \dots, j_N = 0, \dots K$ *and* $i \in \{1, \dots, N\}$.

Proof. Taking $(-1)^{\sum_{m=1}^N j_m} \frac{\partial^{j_1}}{\partial v_1^{j_1}} \cdots \frac{\partial^{j_N}}{\partial v_N^{j_N}}$ on (20) and setting $v_1 = \cdots = v_N = 0$ gives

$$\mathbf{f}_{i+1}^{(j_1,\dots,j_N)} = (-1)^{\sum_{m=1}^N j_m} \frac{\partial^{j_1}}{\partial v_1^{j_1}} \cdots \frac{\partial^{j_N}}{\partial v_N^{j_N}} \int_{y_i=0}^{\infty} \mathbf{f}_i^{(N-1)*}(v_1,\dots,v_{i-1},y_i,v_{i+1},\dots,v_N) e^{-y_i \frac{\sum_{i=1}^N \mathbf{R}_i v_i - \mathbf{Q}}{d_i}} dy_i \quad \sigma_i^*(\sum_{i=1}^N \mathbf{R}_i v_i - \mathbf{Q})\Big|_{v_1=\cdots=v_N=0}. \quad (22)$$

Rearranging (22) leads to

$$\begin{aligned}
\mathbf{f}_{i+1}^{(j_1,\dots,j_N)} &= (-1)^{\sum_{m=1}^N j_m} \frac{\partial^{j_1}}{\partial v_1^{j_1}} \cdots \frac{\partial^{j_N}}{\partial v_N^{j_N}} \int_{y_i=0}^{\infty} \mathbf{f}_i^{(N-1)*}(v_1,\dots,v_{i-1},y_i,v_{i+1},\dots,v_N) \\
&\quad \sum_{k=0}^{\infty} \frac{y_i^k}{k!} \left(\frac{\mathbf{Q} - \sum_{i=1}^N \mathbf{R}_i v_i}{d_i} \right)^k dy_i \quad \sigma_i^*(\sum_{i=1}^N \mathbf{R}_i v_i - \mathbf{Q})\Big|_{v_1=\cdots=v_N=0} \\
&= (-1)^{\sum_{m=1}^N j_m} \frac{\partial^{j_1}}{\partial v_1^{j_1}} \cdots \frac{\partial^{j_N}}{\partial v_N^{j_N}} \sum_{k=0}^{\infty} (-1)^k \frac{\partial^k}{\partial v_i^k} \mathbf{f}_i^{(N)*}(v_1,\dots,v_N) \\
&\quad \frac{1}{k!} \left(\frac{\mathbf{Q} - \sum_{i=1}^N \mathbf{R}_i v_i}{d_i} \right)^k \sigma_i^*(\sum_{i=1}^N \mathbf{R}_i v_i - \mathbf{Q})\Big|_{v_1=\cdots=v_N=0} \\
&= \sum_{j_{1,1}+\ldots+j_{1,3}=j_1} \binom{j_1}{j_{1,1}, j_{1,2}, j_{1,3}} \cdots \sum_{j_{N,1}+\ldots+j_{N,3}=j_N} \binom{j_N}{j_{N,1}, j_{N,2}, j_{N,3}} \\
&\quad \sum_{k=0}^{\infty} \mathbf{f}_i^{(j_{1,1},\dots,j_{i-1,1},j_{i,1}+k,j_{i+1,1},\dots,j_{N,1})} \mathbf{H}_k^{(j_{1,2},\dots,j_{N,2})} \sigma_i^{(j_{1,3},\dots,j_{N,3})}. \quad (23)
\end{aligned}$$

The statement of the corollary comes by applying a truncation at K in the order of the moments. □

The truncation applied in corollary 2 assumes that all the moments $\mathbf{f}_i^{(j_1,\dots,j_N)}$, in which $j_m > K$ at least for one $m = 1, \dots, N$, can be neglected. The number of unknowns and the number of equations in the system of linear equation (21) is $N(K+1)^N$.

4 The steady-state fluid levels at arbitrary epoch

4.1 Equilibrium relationships

Let $\widetilde{s}_i(\ell)$ the service time at station i in the ℓ-th cycle. The steady-state service time at station i and its mean is defined as

$$\widetilde{s}_i = \lim_{k\to\infty} \frac{\sum_{\ell=1}^k \widetilde{s}_i(\ell)}{k} \qquad \text{and} \qquad s_i = \lim_{k\to\infty} \frac{E[\sum_{\ell=1}^k \widetilde{s}_i(\ell)]}{k},$$

respectively. Similarly let $\widetilde{c}_i(\ell)$ the cycle time between two consecutive visit to station i in the ℓ-th cycle. The steady state cycle time at station i, and its mean is defined as

$$\widetilde{c}_i = \lim_{k\to\infty} \frac{\sum_{\ell=1}^k \widetilde{c}_i(\ell)}{k} \qquad \text{and} \qquad c_i = \lim_{k\to\infty} \frac{E[\sum_{\ell=1}^k \widetilde{c}_i(\ell)]}{k},$$

respectively. It follows from the definitions of c_i and s_i that

$$c_i = \sigma + \sum_{j=1}^N s_j, \qquad \text{and hence} \qquad c = c_i, \quad i \in \{1,\dots,N\}. \tag{24}$$

Let $\Lambda_i(t)$ be the accumulated fluid flowed into the buffer of station i in interval $(0,t]$. The steady state mean amount of fluid, which flows into the buffer of station i during one cycle, a_i, is defined as

$$a_i = \lim_{k\to\infty} \frac{E[\sum_{\ell=1}^k \Lambda_i(t_i^f(\ell+1)) - \Lambda_i(t_i^f(\ell))]}{k}.$$

The right hand side of this definition can be rearranged as

$$\lim_{k\to\infty} \frac{E[\sum_{\ell=1}^k \Lambda_i(t_i^f(\ell+1)) - \Lambda_i(t_i^f(\ell))]}{E[\sum_{\ell=1}^k \widetilde{c}_i(\ell)]} \lim_{k\to\infty} \frac{E[\sum_{\ell=1}^k \widetilde{c}_i(\ell)]}{k}$$

and thus we get

$$a_i = \lambda_i c, \quad i \in \{1,\dots,N\}. \tag{25}$$

Corollary 3. *In the stable fluid non-zero switchover-times polling model the steady-state mean cycle time can be expressed as*

$$c = \frac{\sigma}{1-\rho}. \tag{26}$$

Proof. We apply a classical statistical equilibrium argumenting, see e.g. in [10]. The stable model is in statistical equilibrium, which implies that the mean amount of fluid flowing into the buffer of station i during a cycle equals the mean amount of fluid removed at station i during the same cycle, which equals $s_i d_i$. Putting them together yields

$$a_i = s_i d_i. \tag{27}$$

Applying (25) in (27) and expressing s_i from it leads to

$$s_i = \frac{\lambda_i}{d_i} c. \tag{28}$$

Applying (28) in (24) and changing to the notation of utilizations results in

$$c_i = \sigma + \sum_{j=1}^{N} \rho_j c. \tag{29}$$

Rearranging (29) gives the statement. □

Remark 3. The relations (24), (25) and (26) are valid independently of the used service discipline and hence they have more general validity scope.

4.2 The steady-state moments of the service time at station *i*

The steady state pdf of the service time at station i, $s_i(t)$, and the corresponding LT, $s_i^*(v)$, for $t \geq 0$ are defined as

$$s_i(t) = \lim_{k\to\infty} \frac{d}{dt} \frac{E[\sum_{\ell=1}^{k} 1_{(\tilde{s}_i(\ell)<t)}]}{k}, \text{ and } s_i^*(v) = \int_{t=0}^{\infty} s_i(t) e^{-st} dt,$$

where $1_{(\mathrm{con})}$ denotes the indicator of condition "con".

Let $\mathbf{f}_i(x_i)$ and $\mathbf{f}_i^*(v)$ stand for steady-state vector density of the fluid level at station i at i-polling epoch and its LT, respectively. They can be obtained from $\mathbf{f}_i(\bar{x})$ and $\mathbf{f}_i^{(N)*}(\bar{v})$ as

$$\mathbf{f}_i(x_i) = \int_{x_1=0}^{\infty} \dots \int_{x_{i-1}=0}^{\infty} \int_{x_{i+1}=0}^{\infty} \dots \int_{x_N=0}^{\infty} \mathbf{f}_i(\bar{x})\, dx_N \dots dx_{i+1} dx_{i-1} \dots dx_1,$$

$$\mathbf{f}_i^*(v) = \mathbf{f}_i^{(N)*}(\bar{v})\Big|_{v_1=\ldots=v_{i-1}=v_{i+1}=\ldots=v_N=0, v_i=v}.$$

Theorem 2. *In the stable fluid non-zero switchover-times polling model with gated discipline the steady-state LT of the service time at station i can be expressed as*

$$s_i^*(v) = \mathbf{f}_i^*(\frac{v}{d_i})\mathbf{e}, \quad i \in \{1, \ldots, N\}. \tag{30}$$

Proof. If the fluid level at station i is x_i at i-polling epoch then the service time at station i is $\frac{x_i}{d_i}$. Therefore the steady-state LT of the service time at station i can be obtained as

$$s_i^*(v) = \int_{x_i=0}^{\infty} \mathbf{f}_i(x_i) e^{-v\frac{x_i}{d_i}} dx_i \mathbf{e}, \tag{31}$$

which can be rearranged as (30). □

Corollary 4. *In the stable fluid non-zero switchover-times polling model with gated discipline the steady-state moments of the service time at station i are given as*

$$s_i^{(k)} = \frac{1}{d_i^k}\mathbf{f}_i^{(k)}\mathbf{e}, \quad k \geq 1, \quad i \in \{1,\ldots,N\}. \tag{32}$$

Proof. Taking the k-th derivative of (30) with respect to v at $v=0$ and multiplying it by $(-1)^k$ results in the statement. □

4.3 The steady-state joint vector LT of the fluid levels at the stations at arbitrary epoch

The steady-state joint density of the fluid levels at the stations and the state of the modulating Markov chain at an arbitrary epoch, the $1 \times L$ row vector $\mathbf{q}(\bar{x})$ is defined by its j-th element as

$$[\mathbf{q}(\bar{x})]_j = \lim_{t\to\infty} \frac{\partial}{\partial x_1} \ldots \frac{\partial}{\partial x_N} Pr(\Omega(t)=j, X_1(t)<x_1, \ldots X_N(t)<x_N), \; j \in \Omega,$$

and its LT with respect to $\bar{x}$ can be given as

$$\mathbf{q}^{(N)*}(\bar{v}) = \int_{x_1=0}^{\infty} \ldots \int_{x_N=0}^{\infty} \mathbf{q}(\bar{x}) e^{-v_1 x_1} \ldots e^{-v_N x_N} dx_N \ldots dx_1.$$

Moreover, let $\mathbf{e}_j = (0,\ldots,0,1,0,\ldots,0)$ be the $1 \times L$ vector with 1 at the j-th position. Then the $1 \times L$ indicator vector $\mathbf{1}_{(\Omega(t))}$ is defined as

$$\mathbf{1}_{(\Omega(t))} = \sum_{j=1}^{L} 1_{(\Omega(t)=j)}\mathbf{e}_j.$$

We use the following notation

$$\mathbf{f}_i^{(N-1)*}(v_1,\ldots,v_{i-1},x_i,v_{i+1},\ldots,v_N) = \int_{x_1=0}^{\infty} \ldots \int_{x_{i-1}=0}^{\infty} \int_{x_{i+1}=0}^{\infty} \ldots \int_{x_N=0}^{\infty}$$
$$\mathbf{f}_i(\bar{x})\; e^{-v_1 x_1} \ldots e^{-v_{i-1}x_{i-1}} e^{-v_{i+1}x_{i+1}} \ldots e^{-v_N x_N} dx_N \ldots dx_{i+1} dx_{i-1} \ldots dx_1.$$

Theorem 3. *In the stable fluid non-zero switchover-times polling model with gated discipline the following relation holds for the steady-state joint vector LT of the fluid levels at the stations at arbitrary epoch:*

$$\mathbf{q}^{(N)*}(\bar{v}) \left(\sum_{j=1}^{N} \mathbf{R}_j v_j - \mathbf{Q} \right) = \tag{33}$$
$$\frac{1}{c}\sum_{i=1}^{N} \left[d_i v_i \left(\mathbf{f}_i^{(N)*}(\bar{v}) - \mathbf{m}_i^{(N)*}(\bar{v}) \right) \left(\sum_{j\neq i} \mathbf{R}_j v_j + (\mathbf{R}_i - d_i\mathbf{I})\, v_i - \mathbf{Q} \right)^{-1} \right].$$

Proof. The fluid levels at the stations at arbitrary epoch can be expressed by the help of the fluid levels at the last i-polling epoch on LT level by utilizing the transient behavior of the arrived fluid (relation (6)) and taking into account that it can fall either in service or switchover period as well as its position in the actual period. Thus it is enough to average over a polling cycle for determining the behavior at arbitrary epoch.

Therefore $\mathbf{q}^{(N)*}(\bar{v})$ is given by

$$\mathbf{q}^{(N)*}(\bar{v}) = \frac{E[\int_{t=0}^{\widetilde{c}_1} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}dt]}{E[\widetilde{c}_1]} \tag{34}$$
$$= \frac{\sum_{i=1}^{N} E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}dt] + \sum_{i=1}^{N} E[\int_{t=0}^{\widetilde{\sigma}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}dt]}{c}.$$

The fluid level at time t at station i in the service time of station i is the sum of the remaining fluid level, $\xi - td_i$, and the fluid level arrived during t. The fluid level at time t at other stations, i.e., $j \neq i$ in the service time of station i is the sum of the fluid level at the begin of the service time and the fluid amount arrived during t.

Taking into account the state change of the modulating CTMC from 0 to t the LT term $E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}dt]$ can be given as

$$E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}dt] \tag{35}$$
$$= \int_{\xi=0}^{\infty} e^{-(\xi - td_i)v_i}\mathbf{f}_i^{(N-1)*}(v_1,\ldots,v_{i-1},\xi,v_{i+1},\ldots,v_N)\int_{t=0}^{\frac{\xi}{d_i}} \mathbf{A}^{(N)*}(t,\bar{v})dtd\xi$$
$$= \int_{\xi=0}^{\infty} e^{-\xi v_i}\mathbf{f}_i^{(N-1)*}(v_1,\ldots,v_{i-1},\xi,v_{i+1},\ldots,v_N)\int_{t=0}^{\frac{\xi}{d_i}} e^{td_iv_i}\mathbf{A}^{(N)*}(t,\bar{v})dtd\xi.$$

Applying (6) in (35) and rearrangement gives

$$E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}dt] = \int_{\xi=0}^{\infty} e^{-\xi v_i}\mathbf{f}_i^{(N-1)*}(v_1,\ldots,v_{i-1},\xi,v_{i+1},\ldots,v_N)$$
$$\int_{t=0}^{\frac{\xi}{d_i}} e^{-t\left(\sum_{j\neq i}\mathbf{R}_jv_j + (\mathbf{R}_i - d_i\mathbf{I})v_i - \mathbf{Q}\right)}dtd\xi. \tag{36}$$

The internal integral can be evaluated by means of a relation, which can be obtained by the help of the Taylor-expansion of $e^{\mathbf{Z}t}$, and is given by

$$\int_{t=0}^{x} e^{-\mathbf{Z}t}dt\mathbf{Z} = (\mathbf{I} - e^{-\mathbf{Z}x}). \tag{37}$$

Applying (37) in (36) and rearrangement yields

$$E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} dt] \left(\sum_{j \neq i} \mathbf{R}_j v_j + (\mathbf{R}_i - d_i \mathbf{I}) v_i - \mathbf{Q} \right) \tag{38}$$
$$= \int_{\xi=0}^{\infty} e^{-\xi v_i} \mathbf{f}_i^{(N-1)*}(v_1, \ldots, v_{i-1}, \xi, v_{i+1}, \ldots, v_N)$$
$$\left(\mathbf{I} - e^{-\frac{\xi}{d_i}\left(\sum_{j\neq i} \mathbf{R}_j v_j + (\mathbf{R}_i - d_i \mathbf{I}) v_i - \mathbf{Q}\right)} \right) \, d\xi.$$

Rearrangement and applying (13) in (38) leads to

$$E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} dt] \left(\sum_{j \neq i} \mathbf{R}_j v_j + (\mathbf{R}_i - d_i \mathbf{I}) v_i - \mathbf{Q} \right) \tag{39}$$
$$= \mathbf{f}_i^{(N)*}(\bar{v}) - \mathbf{f}_i^{(N)*}(v_1, \ldots, v_{i-1}, \frac{\sum_{i=1}^{N} \mathbf{R}_i v_i - \mathbf{Q}}{d_i}, v_{i+1}, \ldots, v_N)$$
$$= \mathbf{f}_i^{(N)*}(\bar{v}) - \mathbf{m}_i^{(N)*}(\bar{v}).$$

Further rearranging of (39) yields

$$E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} dt] \left(\sum_{j=1}^{N} \mathbf{R}_j v_j - \mathbf{Q} \right) \tag{40}$$
$$= \mathbf{f}_i^{(N)*}(\bar{v}) - \mathbf{m}_i^{(N)*}(\bar{v}) + d_i v_i E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} dt].$$

Now we consider the term $E[\int_{t=0}^{\widetilde{\sigma}_i} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} dt]$. The fluid level at time t at station j, $j \in \{1, \ldots, N\}$, in the switchover time after the service of station i is the sum of the fluid level at station j at start of the switchover time, and the fluid level arrived during t. Taking into account the state change of the modulating CTMC from 0 to t the LT term $E[\int_{t=0}^{\widetilde{\sigma}_i} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} dt]$ can be given as

$$E[\int_{t=0}^{\widetilde{\sigma}_i} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} dt] = \mathbf{m}_i^{(N)*}(\bar{v}) \int_{\tau=0}^{\infty} \int_{t=0}^{\tau} \mathbf{A}^{(N)*}(t, \bar{v}) dt \; \sigma(\tau) \; d\tau. \tag{41}$$

Applying (6) in (41) yields

$$E[\int_{t=0}^{\widetilde{\sigma}_i} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} dt]$$
$$= \mathbf{m}_i^{(N)*}(\bar{v}) \int_{\tau=0}^{\infty} \int_{t=0}^{\tau} e^{-t\left(\sum_{j=1}^{N} \mathbf{R}_j v_j - \mathbf{Q}\right)} dt \; \sigma(\tau) \; d\tau. \tag{42}$$

We apply again (37), now in (42), which gives

$$E[\int_{t=0}^{\widetilde{\sigma}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}dt]\left(\sum_{j=1}^{N}\mathbf{R}_j v_j - \mathbf{Q}\right) = \tag{43}$$

$$\mathbf{m}_i^{(N)*}(\bar{v})\int_{\tau=0}^{\infty}\left(\mathbf{I} - e^{-\tau\left(\sum_{j=1}^{N}\mathbf{R}_j v_j - \mathbf{Q}\right)}\right)\ \sigma(\tau)\ d\tau.$$

Rearranging (42) and applying (14) in it gives the relation for $E[\int_{t=0}^{\widetilde{\sigma}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}dt]$ as

$$E[\int_{t=0}^{\widetilde{\sigma}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}dt]\left(\sum_{j=1}^{N}\mathbf{R}_j v_j - \mathbf{Q}\right) \tag{44}$$

$$= \mathbf{m}_i^{(N)*}(\bar{v})\left(\mathbf{I} - \sigma_i^*\left(\sum_{j=1}^{N}\mathbf{R}_j v_j - \mathbf{Q}\right)\right) = \mathbf{m}_i^{(N)*}(\bar{v}) - \mathbf{f}_{i+1}^{(N)*}(\bar{v}).$$

Using (40) and (44) in (34) and rearranging gives

$$\begin{aligned}
&\mathbf{q}^{(N)*}(\bar{v})\left(\sum_{j=1}^{N}\mathbf{R}_j v_j - \mathbf{Q}\right)\\
&= \frac{1}{c}\Bigg(\sum_{i=1}^{N}\left(\mathbf{f}_i^{(N)*}(\bar{v}) - \mathbf{m}_i^{(N)*}(\bar{v}) + d_i v_i E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}dt]\right)\\
&\quad + \sum_{i=1}^{N}\left(\mathbf{m}_i^{(N)*}(\bar{v}) - \mathbf{f}_{i+1}^{(N)*}(\bar{v})\right)\Bigg)\\
&= \frac{1}{c}\sum_{i=1}^{N} d_i v_i E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}dt].
\end{aligned}\tag{45}$$

The statement of the theorem comes by applying (39) in (45). □

Let $\mathbf{q}_i^*(v)$ denote the steady-state vector LT of the fluid level at station i at arbitrary epoch. $\mathbf{q}_i^*(v)$ can be obtained as

$$\mathbf{q}_i^*(v) = \mathbf{q}^{(N)*}(\bar{v})\Big|_{v_1=\ldots=v_{i-1}=v_{i+1}=\ldots=v_N=0, v_i=v}.$$

Let $\mathbf{m}_i(x_i)$ and $\mathbf{m}_i^*(v)$ stand for steady-state vector density of the fluid level at station i at i-departure epoch and its LT, respectively. They can be obtained from $\mathbf{m}_i(\bar{x})$ and $\mathbf{m}_i^{(N)*}(\bar{v})$ as

$$\mathbf{m}_i(x_i) = \int_{x_1=0}^{\infty}\ldots\int_{x_{i-1}=0}^{\infty}\int_{x_{i+1}=0}^{\infty}\ldots\int_{x_N=0}^{\infty}\mathbf{m}_i(\bar{x})\ dx_N\ldots dx_{i+1}dx_{i-1}\ldots dx_1,$$

$$\mathbf{m}_i^*(v) = \mathbf{m}_i^{(N)*}(\bar{v})\Big|_{v_1=\ldots=v_{i-1}=v_{i+1}=\ldots=v_N=0, v_i=v}.$$

Corollary 5. *In the stable fluid non-zero switchover-times polling model with gated discipline the following relation holds for the steady-state vector LT of the fluid level at station i at arbitrary epoch:*

$$\mathbf{q}_i^*(v)\left(\mathbf{R}_i v - \mathbf{Q}\right)\left(\left(\mathbf{R}_i - d_i\mathbf{I}\right)v - \mathbf{Q}\right) = \frac{1}{c}d_i v\left(\mathbf{f}_i^*(v) - \mathbf{m}_i^*(v)\right). \tag{46}$$

Proof. The statement comes by setting $v_1 = \ldots = v_{i-1} = v_{i+1} = \ldots = v_N = 0, v_i = v$ in (33). □

Remark 4. The relation (46) holds also for fluid vacation model with gated discipline (see (61) in [6]).

Corollary 6. *In the stable fluid non-zero switchover-times polling model with gated discipline the steady-state vector mean of the fluid level at station i at arbitrary epoch can be determined as*

$$\begin{aligned}
\mathbf{q}_i^{(1)} &= \frac{1}{6\lambda_i(\lambda_i - d_i)}\mathbf{r}^{(3)}\mathbf{e}\boldsymbol{\pi} \qquad (47)\\
&-\frac{1}{2(\lambda_i - d_i)}\mathbf{r}^{(2)}\frac{1}{\lambda_i}\left(\mathbf{I} - \frac{1}{(\lambda_i - d_i)}\mathbf{e}\boldsymbol{\pi}(\mathbf{R}_i - d_i\mathbf{I})\right)\\
&\times\left(\mathbf{Q} + \mathbf{e}\boldsymbol{\pi}\right)^{-1}(\mathbf{R}_i - d_i\mathbf{I})\mathbf{e}\boldsymbol{\pi}\\
&-\frac{1}{2(\lambda_i - d_i)}\mathbf{r}^{(2)}\mathbf{e}\boldsymbol{\pi}\left(\mathbf{Q} + \mathbf{e}\boldsymbol{\pi}\right)^{-1}\left(\frac{\mathbf{R}_i\mathbf{e}\boldsymbol{\pi}}{\lambda_i} - \mathbf{I}\right)\\
&+\mathbf{r}^{(1)}\left(\mathbf{Q} + \mathbf{e}\boldsymbol{\pi}\right)^{-1}\left(\frac{1}{(\lambda_i - d_i)}(\mathbf{R}_i - d_i\mathbf{I})\mathbf{e}\boldsymbol{\pi} - \mathbf{I}\right)\\
&\times\left(\frac{-1}{\lambda_i(\lambda_i - d_i)}(\mathbf{R}_i - d_i\mathbf{I})\left(\mathbf{Q} + \mathbf{e}\boldsymbol{\pi}\right)^{-1}(\mathbf{R}_i - d_i\mathbf{I})\mathbf{e}\boldsymbol{\pi}\right.\\
&\left.+\left(\mathbf{Q} + \mathbf{e}\boldsymbol{\pi}\right)^{-1}\left(\frac{\mathbf{R}_i\mathbf{e}\boldsymbol{\pi}}{\lambda_i} - \mathbf{I}\right)\right)\\
&+\boldsymbol{\pi}\mathbf{R}_i\left(\mathbf{Q} + \mathbf{e}\boldsymbol{\pi}\right)^{-1}\left(\frac{\mathbf{R}_i\mathbf{e}\boldsymbol{\pi}}{\lambda_i} - \mathbf{I}\right).
\end{aligned}$$

where c is given by (26) and $\mathbf{r}^{(1)}$, $\mathbf{r}^{(2)}$ and $\mathbf{r}^{(3)}$ are given by

$$\begin{aligned}
\mathbf{r}^{(1)} &= -\frac{d_i}{c}(\mathbf{f} - \mathbf{m}),\\
\mathbf{r}^{(2)} &= -\frac{2d_i}{c}(\mathbf{f}^{(1)} - \mathbf{m}^{(1)}),\\
\mathbf{r}^{(3)} &= -\frac{3d_i}{c}(\mathbf{f}^{(2)} - \mathbf{m}^{(2)}).
\end{aligned}$$

Proof. The proof of the statement can be found in [6] (proof of Corollary 6). □

Acknowledgment

This work is supported by the OTKA K-123914 project and by the ÚNKP-17-4-III New National Excellence Program of the Ministry of Human Capacities.

References

1. Dai, J.G.: On the positive Harris recurrence for multiclass queueing networks: A unified approach via fluid limit models. Ann. Appl. Prob., vol. 5, pp. 49-77. (1995)
2. Dai, J.G, Meyn, S.P.: Stability and convergence of moments for multiclass queueing networks via fluid limit models. IEEE Trans. Automat. Control, vol. 40, pp. 1-16. (1995)
3. Kulkarni, V.G.: Fluid models for single buffer systems. In: Dshalalow, J. (eds.) Frontiers in Queueing. CRC, Boca Raton, FL, pp. 321-338. (1997)
4. Ahn, S., Ramaswami, V.: Efficient algorithms for transient analysis of stochastic fluid flow models. Journal of Applied Probability, vol. 42(2), pp. 531-549. (2005)
5. Czerniak, O., Yechiali, U.: Fluid polling systems. Queueing System, vol. 63, pp. 401-435. (2009)
6. Saffer, Z., Telek, M.: Fluid vacation model with Markov modulated load and gated discipline. In: 9th International Conference on Queueing Theory and Network Applications (QTNA2014), pp. 184-197. (2014)
7. Saffer, Z., Telek, M.: Fluid vacation model with Markov modulated load and exhaustive discipline. In: Horváth, A., Wolter, K., (eds.) 11th European Workshop on Performance Engineering, EPEW 2014. LNCS, vol. 8721, pp. 59-73. Springer International Publishing (2014)
8. Saffer, Z., Telek, M.: Exhaustive fluid vacation model with Markov modulated load. Performance Evaluation (PEVA), vol. 98, pp. 19–35. (2016)
9. Horváth, G., Telek, M.: Exhaustive fluid vacation model with positive fluid rate during service. Performance Evaluation, vol. 91, pp. 286 - 302. (2015)
10. Eisenberg, M.: Queues with Periodic Service and Changeover Time. Operations Research, vol. 20, pp. 440–451. (1972)