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ANALYSIS OF MARKOV-MODULATED FLUID POLLING SYSTEMS WITH GATED DISCIPLINE

Abstract. In this paper we provide an analysis for fluid polling models with Markov modulated load and gated discipline. The fluid arrival to the stations is modulated by a common continuous-time Markov chain (the special case when the modulating Markov chains are independent is also included). The fluid is removed at the stations during the service period by a station dependent constant rate.

Using the results obtained for fluid vacation models with gated discipline in a previous work, we establish steady-state relationships for the joint distribution of the fluid levels at the stations and the state of the modulating Markov chain among different characteristic epochs including start and end of the service at each station in Laplace transform domain. We derive the steady-state vector Laplace transform of the fluid levels at the stations at arbitrary epoch and its moments. Based on the method of supplementary variables, we also provide differential equations to obtain the joint density function of the fluid levels.

Numerical examples illustrate the applicability of the analysis method.

1. Introduction.

In fluid queueing models, the work arrives and is served in a continuous manner, it is like fluid flows into a fluid container and pumped our from the container by a server. Such models can be used as the limit for the workload in the analysis of regular queueing systems with discrete customers, for example in Heavy-Traffic analysis or stability analysis [4, 5]. The Markov modulated fluid queues, which is composed by a single input flow, a single fluid container and a single server, have been analysed by several authors using matrix analytic methods, see, e.g., [9, 1].

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In polling models there are N input flows, N buffers and a single server which circulates between the buffers [13]. The time needed for the server to arrive from one buffer to an other is referred to as switchover time. Polling models with discrete customers is also exhaustively studied in the literature (see e.g. [14] for a survey), while the fluid polling models got less attention till now.

Some of the few available results focuses on polling models with Levy input processes [7, 3]. These results provide transform domain functional equations to describe the stationary system behaviour, similar to the ones of our embedded results in Section 3, but do not contain pointers to computational methods.

Our main interest is to propose analytical descriptions which allows numerical evaluation. A series of such efforts has been devoted to fluid vacation models with Markov modulated load [11, 10, 12, 8]. Vacation model is a special case of polling models, with only one buffer, but with the presence of switchover time. These papers considered fluid vacation models with the two most common service disciplines: the gated and the exhaustive disciplines with various modeling constraint on the fluid rates.

The main contribution of this work is the *numerical* analysis of the fluid gated polling model with Markov modulated load. We present two analysis methods one based on the embedded process at server arrival and departure instances, and one based on the supplementary variable approach and propose a numerical analysis method based on both of them.

The rest of the paper is organized as follows. Section 2 gives the model description and the stability criterion of the model. Section 3 and 4 provides the analysis of the steady-state fluid levels based on the method of embedded regenerative instances and supplementary variable, respectively. Numerical examples are provided in section 5.

2. Model and Notation.

2.1. Model description. We consider a fluid polling model with Markov modulated load and gated discipline. The polling system consists of N stations. Each station has an infinite fluid buffer.

A common continuous-time Markov chain (CTMC), $\Omega(t)$, with state space $\{1,\ldots,L\}$ modulates the arriving fluid flows at the station. The generator of this background CTMC is denoted by $\mathbf{Q}$ and its initial distribution by π_0. The input fluid rates at station i are specified by diagonal fluid input rate matrix $\mathbf{R_i}$, for $i\in\{1,\ldots,N\}$. If the background CTMC is in state j ($\Omega(t)=j$) then fluid flows into the buffer of station i at rate $r_i(j)$ for $j\in\{1,\ldots,L\}$ and $i\in\{1,\ldots,N\}$. The vector of the fluid rates for station i is denoted by $\mathbf{r_i}$. When the server visits station i it removes fluid from its fluid buffer at finite rate $d_i>0$ for $i\in\{1,\ldots,N\}$. Consequently, when the server visits station i and the overall Markov chain is in state j ($\Omega(t)=j$) then the fluid level of the buffer of station i changes at rate $r_i(j)-d_i$ otherwise it changes at rate $r_i(j)$ due to the lack of service. The length of the server's visit at station i in the polling model is determined by the service discipline applied at that station. In this work we consider the gated discipline. Under gated discipline only the fluid is removed during the server visit at station i, which is present at the station already upon the server arrival. The cycle time (or simple cycle) is the time between two consecutive visits of the server to the same station. In this paper, if not stated otherwise then we understand the station index i as $mod(N)$, i.e. whenever it reaches N it continues by 1. The switchover time from station i to the next station in the consecutive cycles is independent and identically distributed. The probability distribution function (pdf) of the switchover time from station i, the associated hazard rate function, the corresponding Laplace transform (LT) and its mean are denoted by $\sigma_i(t)$, $\lambda_i(t) = \frac{\sigma_i(t)}{\int_t^\infty \sigma_i(\tau)\mathrm{d}\tau}$, $\sigma_i^*(s) = \int_0^\infty e^{-st}\sigma_i(t)\mathrm{d}t$ and $\sigma_i = \int_0^\infty t\sigma_i(t)\mathrm{d}t$, respectively. We consider non-zero switchover-times model, and we use the notation $\sigma = \sum_{i=1}^N \sigma_i$. We set the following assumptions on the fluid polling model:

- **A.1** The generator matrix $\mathbf{Q}$ of the modulating CTMC is irreducible.
- **A.2** The fluid rates $r_i(j)$ are positive and finite, i.e., $r_i(j) > 0$ for $j \in \{1,\dots,L\}$ and $i \in \{1,\dots,N\}$.

Remark 1. The case of independent fluid inputs is also included by the approach with one common modulating CTMC as special case. In that case $\mathbf{Q} = \oplus_{i=1}^N \hat{\mathbf{Q}}_\mathbf{i}$ and $\mathbf{R_i} = (\otimes_{k=1}^{i-1}\mathbf{I})\otimes\hat{\mathbf{R}}_\mathbf{i}\otimes(\otimes_{k=i+1}^{N}\mathbf{I})$, where $\hat{\mathbf{Q}}_\mathbf{i}$ and $\hat{\mathbf{R}}_\mathbf{i}$ denote the independent generator and the fluid input rate matrix of station i, for $i \in \{1,\dots,N\}$, and $\otimes$ and $\oplus$ denote the Kronecker product and Kronecker sum operations, respectively.

Let π be the stationary probability vector of the modulating Markov chain. Due to assumption **A.1**, $\pi\mathbf{Q} = 0$ and $\pi\mathbb{1} = 1$ (where $\mathbb{1}$ is the column vector of ones) uniquely determine π, the row vector of the stationary probabilities. The stationary fluid flow rate and the utilization at station i, α_i and ρ_i, respectively, are given for $i \in \{1,\dots,N\}$ as

$$\alpha_i = \pi\mathbf{R_i}\mathbb{1} \text{ and } \rho_i = \frac{\alpha_i}{d_i}, \tag{1}$$

and the total utilization is

$$\rho = \sum_{i=1}^N \rho_i. \tag{2}$$

The arrival instant of the server to station i is called i-polling epoch, and the time instant when the server departs from station i is called i-departure epoch.

$\mathbf{Z}_{j,\ell}$ denotes the j,ℓ element of the matrix $\mathbf{Z}$ and $[\mathbf{z}_i]_j$ denote the j-th element of vector $\mathbf{z}_i$. When there is a set of random variables characterized by one (two) parameters, e.g., Y_n ($Y_{k,n}$), then the n (k,n) element of its vector (matrix) LT is $E(e^{-vY_n})$ ($E(e^{-vY_{k,n}})$). When $\mathbf{M}^*(v)$, $Re(v) \geq 0$ is a matrix LT, $\mathbf{M}^{(k)}$ denotes its k-th ($k \geq 1$) moment, i.e., $\mathbf{X}^{(k)} = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}v^k}\mathbf{M}^*(v)|_{v=0}$ and $\mathbf{M}$ denotes its value at $v = 0$, i.e., $\mathbf{M} = \mathbf{M}^*(0)$. Similarly, when $\mathbf{m}^*(v)$, $Re(v) \leq 0$ is a vector LT, $\mathbf{m}^{(k)}$ denotes its k-th ($k \geq 1$) moment, i.e., $\mathbf{m}^{(k)} = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}v^k}\mathbf{m}^*(v)|_{v=0}$ and $\mathbf{m}$ denotes its value at $v = 0$, i.e., $\mathbf{m} = \mathbf{m}^*(0)$.

2.2. **Stability.** We apply a workload argument to get a necessary condition of the stability. The amount of work flowing to station i during a time unit is equal to its utilization, ρ_i. The necessary condition of the stability is that the total amount of work flowing to all stations during a time unit must be less than the work-amount of that time unit, which is 1. Therefore the necessary condition of the stability is given as

$$\rho < 1. \tag{3}$$

Remark 2. If the system would limit the work which could be done on average, i.e., when less then 1 work-amount could be done during a time unit, then further restrictions were needed for the sufficiency. However, the gated discipline is "unlimited", since it does not set any load-independent limit on the work-amount, which could be performed during a service period. Therefore the above necessary condition is also a sufficient one for the stability of the system.

3. **Regenerative analysis at embedded instances.**

3.1. **The steady-state fluid levels at polling epochs.**

3.1.1. *Transient analysis of the accumulated fluid.* In this section, we consider the joint distribution of the accumulated amount of fluid entering into the individual stations during time $t \geq 0$. We derive the joint LT of the accumulated fluid levels flowed into the stations and the state of the common modulated Markov chain as a function of time.

Let $X_i(t) \in \mathbb{R}^+$ denote the accumulated amount of fluid entering into station i until time t for $i \in \{1, \ldots, N\}$. Using the notation $\mathbf{x} = (x_1, \ldots, x_N)$ let the transition density matrix $\mathbf{A}(t, \mathbf{x})$ be composed by its elements $\mathbf{A}_{j,k}(t, \mathbf{x})$ as

$$\mathbf{A}_{j,k}(t,\mathbf{x}) = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_N}$$
$$Pr(\Omega(t) = k, X_1(t) < x_1, \ldots X_N(t) < x_N | \Omega(0) = j, X_1(0) = \ldots = X_N(0) = 0).$$

The fluid level is zero at each station i at $t = 0$ ($X_i(0) = 0$) with probability 1. Hence the transition density matrix for $t = 0$ is given as

$$\mathbf{A}(0, \mathbf{x}) = \delta(x_1) \ldots \delta(x_N)\mathbf{I}, \tag{4}$$

where $\delta(x)$ denotes the unit impulse function at x=0, whose LT is 1. Furthermore the accumulated amount of fluids are greater than zero for $t > 0$ at every stations ($X_i(t) > 0$, for $i \in \{1, \ldots, N\}$) due to assumption **A.2**. It follows that

$$\mathbf{A}(t, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_N) = \mathbf{0}, \quad t > 0, i \in \{1, \ldots, N\}, \tag{5}$$

where $\mathbf{0}$ denotes the $L \times L$ zero matrix. We also use the notation $\mathbf{v} = (v_1, \ldots, v_N)$ and we define several LTs of matrix $\mathbf{A}(t, \mathbf{x})$ as

$$\begin{aligned}
\mathbf{A}^*(s, \mathbf{x}) &= \int_{t=0}^{\infty} \mathbf{A}(t, \mathbf{x}) e^{-st} \mathrm{d}t, \\
\mathbf{A}^{N*}(t, \mathbf{v}) &= \int_{x_1=0}^{\infty} \cdots \int_{x_N=0}^{\infty} \mathbf{A}(t, \mathbf{x}) e^{-\sum_{i=1}^{N} v_i x_i} \, \mathrm{d}x_N \ldots \mathrm{d}x_1, \\
\mathbf{A}^{(N+1)*}(s, \mathbf{v}) &= \int_{x_1=0}^{\infty} \cdots \int_{x_N=0}^{\infty} \mathbf{A}^*(s, \mathbf{x}) e^{-\sum_{i=1}^{N} v_i x_i} \, \mathrm{d}x_N \ldots \mathrm{d}x_1,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{A}^{(N)*}(s, v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_N) = \\
&\quad \int_{x_1=0}^{\infty} \cdots \int_{x_{i-1}=0}^{\infty} \int_{x_{i+1}=0}^{\infty} \cdots \int_{x_N=0}^{\infty} \mathbf{A}^*(s, x_1, \ldots, x_{i-1}, 0, x_{i+1} \ldots x_N) \\
&\qquad e^{-v_1 x_1} \ldots e^{-v_{i-1} x_{i-1}} e^{-v_{i+1} x_{i+1}} \ldots e^{-v_N x_N} \, \mathrm{d}x_N \ldots \mathrm{d}x_{i+1} \mathrm{d}x_{i-1} \ldots \mathrm{d}x_1,
\end{aligned}$$

where the coefficients of $*$ in the superscript of matrix $\mathbf{A}$ denotes the number of LTs.

Proposition 1. *In the fluid polling model the joint matrix LT of the accumulated amount of fluid entering in interval $(0, t]$ can be expressed as*

$$\mathbf{A}^{(N)*}(t, \mathbf{v}) = e^{-t\left(\sum_{i=1}^{N} \mathbf{R_i} v_i - \mathbf{Q}\right)}. \tag{6}$$

Proof. The Markov process $\{\Omega(t), X_1(t), \ldots, X_N(t)\}$ describes a homogenous first order fluid model. As proven in [2], its transient behavior can be characterized by forward Kolmogorov equations as

$$\frac{\partial}{\partial t}\mathbf{A}(t,\mathbf{x}) + \frac{\partial}{\partial x_1}\mathbf{A}(t,\mathbf{x})\mathbf{R_1} + \ldots + \frac{\partial}{\partial x_N}\mathbf{A}(t,\mathbf{x})\mathbf{R_N} = \mathbf{A}(t,\mathbf{x})\mathbf{Q}. \tag{7}$$

and with initial conditions (4) and (5). Taking the LT of (7) with respect to t yields

$$\mathbf{A}^*(s,\mathbf{x})s - \mathbf{A}(0,\mathbf{x}) + \frac{\partial}{\partial x_1}\mathbf{A}^*(s,\mathbf{x})\mathbf{R_1} + \ldots + \frac{\partial}{\partial x_N}\mathbf{A}^*(s,\mathbf{x})\mathbf{R_N} = \mathbf{A}^*(s,\mathbf{x})\mathbf{Q}. \tag{8}$$

Now taking the LT of (8) with respect to $x_1, \ldots, x_N$ we have

$$\begin{aligned} &\mathbf{A}^{(N+1)*}(s,\mathbf{v})s - \mathbf{A}^{(N)*}(0,\mathbf{v}) \\ &\qquad + \left(\mathbf{A}^{(N+1)*}(s,\mathbf{v})v_1 - \mathbf{A}^{(N)*}(s,0,v_2,\ldots,v_N)\right)\mathbf{R_1} + \ldots \\ &\qquad + \left(\mathbf{A}^{(N+1)*}(s,\mathbf{v})v_N - \mathbf{A}^{(N)*}(s,v_1,\ldots,v_{N-1},0)\right)\mathbf{R_N} \\ &\qquad = \mathbf{A}^{(N+1)*}(s,\mathbf{v})\mathbf{Q}. \end{aligned} \tag{9}$$

Applying (4) and (5) in (9) gives

$$\begin{aligned} \mathbf{A}^{(N+1)*}(s,\mathbf{v})s - \mathbf{I} \quad &+ \quad \mathbf{A}^{(N+1)*}(s,\mathbf{v})\mathbf{R_1}v_1 + \ldots + \mathbf{A}^{(N+1)*}(s,\mathbf{v})\mathbf{R_N}v_N \\ &= \quad \mathbf{A}^{(N+1)*}(s,\mathbf{v})\mathbf{Q}. \end{aligned} \tag{10}$$

After rearranging (10) we get

$$\mathbf{A}^{(N+1)*}(s,\mathbf{v}) = \left(\mathbf{I}s + \mathbf{R_1}v_1 + \ldots + \mathbf{R_N}v_N - \mathbf{Q}\right)^{-1}. \tag{11}$$

Taking the inverse Laplace transform of (11) with respect to s results in the statement of the proposition. ☐

3.1.2. *The governing equations of the system at polling and departure epochs.* Let $X_i(t) \in \mathbb{R}^+$ denote the actual level of the fluid buffer at station i at time t for $i \in \{1, \ldots, N\}$. Let $t_i^f(\ell)$ be the time of the i-polling epoch in the ℓ-th cycle for $\ell \geq 1$ and $i = \{1, \ldots, N\}$. We define the joint densities of the fluid levels at the stations and the state of the modulating Markov chain at the i-polling epoch in the ℓ-th cycle, for $\ell \geq 1$ and $i = \{1, \ldots, N\}$, the $1 \times L$ vector $\mathbf{f_i}(\ell, \mathbf{x})$ by its elements as

$$\begin{aligned} [\mathbf{f_i}(\ell,\mathbf{x})]_j \quad = \quad & \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_N} \\ & Pr(\Omega(t_i^f(\ell)) = j, X_1(t_i^f(\ell)) < x_1, \ldots X_N(t_i^f(\ell)) < x_N). \end{aligned}$$

The steady-state counterpart of the vector $\mathbf{f_i}(\ell, \mathbf{x})$ is defined as

$$\mathbf{f_i}(\mathbf{x}) = \lim_{\ell \to \infty} \mathbf{f_i}(\ell, \mathbf{x}),$$

and its LT is given as

$$\mathbf{f_i}^{(N)*}(\mathbf{v}) = \int_{x_1=0}^{\infty} \cdots \int_{x_N=0}^{\infty} \mathbf{f_i}(\mathbf{x}) e^{-v_1 x_1} \ldots e^{-v_N x_N} \mathrm{d}x_N \ldots \mathrm{d}x_1,$$

where $\mathbf{v} = (v_1, \ldots, v_N)$.

Analogously let $t_i^m(\ell)$ be the time of the i-departure epoch in the ℓ-th cycle for $\ell \geq 1$ and $i = \{1, \ldots, N\}$. We define the joint densities of the fluid levels at the stations and the state of the modulating Markov chain at the i-departure epoch in the ℓ-th cycle, for $\ell \geq 1$ and $i = \{1, \ldots, N\}$, the $1 \times L$ vector $\mathbf{m_i}(\ell, \mathbf{x})$ by its elements as

$$[\mathbf{m_i}(\ell,\mathbf{x})]_j = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_N} Pr(\Omega(t_i^m(\ell)) = j, X_1(t_i^m(\ell)) < x_1, \ldots X_N(t_i^m(\ell)) < x_N).$$

The steady-state joint densities of the fluid levels at the stations and the state of the modulating Markov chain at the i-departure epoch are defined as

$$\mathbf{m_i}(\mathbf{x}) = \lim_{\ell \to \infty} \mathbf{m_i}(\ell, \mathbf{x}),$$

and its LT is given as

$$\mathbf{m_i}^{(N)*}(\mathbf{v}) = \int_{x_1=0}^{\infty} \cdots \int_{x_N=0}^{\infty} \mathbf{m_i}(\mathbf{x}) e^{-v_1 x_1} \ldots e^{-v_N x_N} \mathrm{d}x_N \ldots \mathrm{d}x_1.$$

We define a notation for substituting the multivariate $L \times L$ matrix function $\mathbf{H}(\mathbf{v})$ into the defining integral of the LT $\mathbf{f_i}^{(N)*}(\mathbf{v})$ as

$$\begin{aligned} &\mathbf{f_i}^{(N)*}(v_1, \ldots, v_{i-1}, \mathbf{H}(\mathbf{v}), v_{i+1}, \ldots, v_N) = \\ &\int_{x_1=0}^{\infty} \cdots \int_{x_N=0}^{\infty} \mathbf{f_i}(\mathbf{x}) e^{-v_1 x_1} \ldots e^{-v_{i-1} x_{i-1}} e^{-\mathbf{H}(\mathbf{v}) x_i} e^{-v_{i+1} x_{i+1}} \ldots e^{-v_N x_N} \mathrm{d}x_N \ldots \mathrm{d}x_1. \end{aligned} \tag{12}$$

Theorem 3.1. *The governing equations of the stable fluid polling model with gated discipline in terms of the steady-state joint vector LTs of the fluid levels at the stations at the i-polling and i-departure epochs for $i \in \{1, \ldots, N\}$ are given as*

- *for the transition $\mathbf{f_i} \to \mathbf{m_i}$*

$$\mathbf{m_i}^{(N)*}(\mathbf{v}) = \mathbf{f_i}^{(N)*}(v_1, \ldots, v_{i-1}, \frac{\sum_{i=1}^N \mathbf{R_i} v_i - \mathbf{Q}}{d_i}, v_{i+1}, \ldots, v_N), \tag{13}$$

- *and for the transition $\mathbf{m_i} \to \mathbf{f_{i+1}}$*

$$\mathbf{f_{i+1}}^{(N)*}(\mathbf{v}) = \mathbf{m_i}^{(N)*}(\mathbf{v}))\sigma_i^*(\sum_{i=1}^{N} \mathbf{R_i} v_i - \mathbf{Q}). \tag{14}$$

Proof. Due to the gated service discipline the fluid level at station i at i-departure epoch equals the level of the fluid arriving during the service duration of station i. The fluid level at stations $j \neq i$ at i-departure epoch is the sum of the fluid level at the previous i-polling epoch and the fluid arrived in between. If the fluid level at station i at i-polling epoch equals $\xi_i > 0$ then service duration is $\frac{\xi_i}{d_i}$ due to the gated discipline. Accordingly we can express vector $\mathbf{m_i}(\mathbf{x})$ as

$$\begin{aligned} \mathbf{m_i}(\mathbf{x}) = & \int_{\xi_i=0}^{\infty} \int_{y_1=0}^{x_1} \cdots \int_{y_{i-1}=0}^{x_{i-1}} \int_{y_{i+1}=0}^{x_{i+1}} \cdots \int_{y_N=0}^{x_N} \\ & \mathbf{f_i}(x_1 - y_1, \ldots, x_{i-1} - y_{i-1}, \xi_i, x_{i+1} - y_{i+1}, \ldots, x_N - y_N) \\ & \mathbf{A}(\frac{\xi_i}{d_i}, y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_N) \mathrm{d}y_N \ldots \mathrm{d}y_{i+1} \mathrm{d}y_{i-1} \ldots \mathrm{d}y_1 \mathrm{d}\xi_i. \end{aligned}$$

Using the convolution property of the LT, the LT of $\mathbf{m}_i(\mathbf{x})$ with respect to $\mathbf{x}$ can be given as

$$\mathbf{m_i}^{(N)*}(\mathbf{v}) = \int_{\xi_i=0}^{\infty} \mathbf{f}_i^{(N-1)*}(v_1,\dots,v_{i-1},\xi_i,v_{i+1},\dots,v_N)\mathbf{A}^{(N)*}(\frac{\xi_i}{d_i},\mathbf{v})\mathrm{d}\xi_i. \quad (15)$$

Applying (6) in (15) yields

$$\mathbf{m_i}^{(N)*}(\mathbf{v}) = \int_{\xi_i=0}^{\infty} \mathbf{f_i}^{(N-1)*}(v_1,\dots,v_{i-1},\xi_i,v_{i+1},\dots,v_N)e^{-\frac{\xi_i}{d_i}\left(\sum_{i=1}^{N}\mathbf{R_i}v_i-\mathbf{Q}\right)}\mathrm{d}\xi_i. \quad (16)$$

The first statement of the theorem comes by observing that the right hand side of (16) is an LT with respect to ξ_i and applying the notation (12).

The fluid level at any station j at $i+1$-polling epoch is the sum of the fluid level at the previous i-departure epoch and the fluid arrived in between. Therefore we have

$$\begin{aligned}[\mathbf{f_{i+1}}(\mathbf{x})]_k &= \sum_{j=1}^{L}\int_{t=0}^{\infty}\int_{y_1=0}^{x_1}\dots\int_{y_N=0}^{x_N}[\mathbf{m_i}(x_1-y_1,\dots,x_N-y_N)]_j \\ &\quad \mathbf{A}_{jk}(t,y_1,\dots,y_N)\sigma_i(t)\mathrm{d}y_N\dots\mathrm{d}y_1\mathrm{d}t. \quad (17)\end{aligned}$$

Changing (17) to matrix notation and using the convolution property of LT we get

$$\mathbf{f_{i+1}}^{(N)*}(\mathbf{v}) = \int_{t=0}^{\infty}\mathbf{m_i}^{(N)*}(\mathbf{v})\mathbf{A}^{(N)*}(t,\mathbf{v})\sigma_i(t)\mathrm{d}t. \quad (18)$$

Applying (6) in (18) and rearrangement leads to

$$\mathbf{f_{i+1}}^{(N)*}(\mathbf{v}) = \mathbf{m_i}^{(N)*}(\mathbf{v})\int_{t=0}^{\infty}e^{-t\left(\sum_{i=1}^{N}\mathbf{R_i}v_i-\mathbf{Q}\right)}\sigma_i(t)\mathrm{d}t. \quad (19)$$

The second statement of the theorem comes by observing that on the r.h.s. of (19) there is an LT with respect to t. □

3.1.3. *The steady-state vector moments of the fluid levels at polling epochs.*

Corollary 1. *The relation for the transition $\mathbf{f_i}\to\mathbf{f_{i+1}}$, for $i\in\{1,\dots,N\}$ in the stable fluid polling model with gated discipline are given as*

$$\mathbf{f_{i+1}}^{(N)*}(\mathbf{v}) = \quad (20)$$
$$\mathbf{f_i}^{(N)*}(v_1,\dots,v_{i-1},\frac{\sum_{m=1}^{N}\mathbf{R_m}v_m-\mathbf{Q}}{d_i},v_{i+1},\dots,v_N)\ \sigma_i^*\left(\sum_{m=1}^{N}\mathbf{R_m}v_m-\mathbf{Q}\right),$$

Proof. The corollary comes by applying (13) in (14). □

We define the joint moments of the fluid levels at the stations as

$$\mathbf{f_i}^{(j_1,\dots,j_N)} = (-1)^{\sum_{m=1}^{N}j_m}\frac{\partial^{j_1}}{\partial v_1^{j_1}}\dots\frac{\partial^{j_N}}{\partial v_N^{j_N}}\mathbf{f_i}^{(N)*}(\mathbf{v})\Big|_{v_1=\dots=v_N=0}.$$

Furthermore, we define the following quantities

$$\mathbf{H_i}^{(j_1,\dots,j_N)}(k) = (-1)^{\sum_{m=1}^{N}j_m}\frac{1}{k!}\frac{\partial^{j_1}}{\partial v_1^{j_1}}\dots\frac{\partial^{j_N}}{\partial v_N^{j_N}}\left(\frac{\mathbf{Q}-\sum_{m=1}^{N}\mathbf{R_m}v_m}{d_i}\right)^k\Bigg|_{v_1=\dots=v_N=0}$$

$$\sigma_i^{(j_1,\dots,j_N)} = (-1)^{\sum_{m=1}^{N}j_m}\frac{\partial^{j_1}}{\partial v_1^{j_1}}\dots\frac{\partial^{j_N}}{\partial v_N^{j_N}}\sigma_i^*\left(\sum_{m=1}^{N}\mathbf{R_m}v_m-\mathbf{Q}\right)\Bigg|_{v_1=\dots=v_N=0}$$

Corollary 2. *The joint moments of the fluid levels at the stations satisfies the following infinite system of linear equations*

$$\mathbf{f_{i+1}}^{(j_1,\ldots,j_N)} = \sum_{j_{1,1}+\ldots+j_{1,3}=j_1} \binom{j_1}{j_{1,1}, j_{1,2}, j_{1,3}} \ldots \sum_{j_{i,2}+j_{i,3}=j_i} \binom{j_i}{j_{i,2}, j_{i,3}} \ldots \sum_{j_{N,1}+\ldots+j_{N,3}=j_N} \binom{j_N}{j_{N,1}, j_{N,2}, j_{N,3}} \sum_{k=0}^{\infty} \mathbf{f_i}^{(j_{1,1},\ldots,j_{i-1,1},k,j_{i+1,1},\ldots,j_{N,1})} \mathbf{H}_i^{(j_{1,2},\ldots,j_{N,2})}(k) \sigma_i^{(j_{1,3},\ldots,j_{N,3})}, \tag{21}$$

where $j_1, \ldots, j_N = 0, 1, \ldots$ *and* $i \in \{1, \ldots, N\}$.

Proof. Taking $(-1)^{\sum_{m=1}^N j_m} \frac{\partial^{j_1}}{\partial v_1^{j_1}} \ldots \frac{\partial^{j_N}}{\partial v_N^{j_N}}$ on (20) and setting $v_1 = \cdots = v_N = 0$ gives

$$\mathbf{f_{i+1}}^{(j_1,\ldots,j_N)} = (-1)^{\sum_{m=1}^N j_m} \frac{\partial^{j_1}}{\partial v_1^{j_1}} \ldots \frac{\partial^{j_N}}{\partial v_N^{j_N}} \int_{y_i=0}^{\infty} \mathbf{f_i}^{(N-1)*}(v_1, \ldots, v_{i-1}, y_i, v_{i+1}, \ldots, v_N) e^{-y_i \frac{\sum_{i=1}^N \mathbf{R_i} v_i - \mathbf{Q}}{d_i}} \mathrm{d}y_i \;\; \sigma_i^* \left(\sum_{m=1}^N \mathbf{R_m} v_m - \mathbf{Q} \right) \Bigg|_{v_1=\cdots=v_N=0}. \tag{22}$$

Rearranging (22) leads to

$$\begin{aligned}
&\mathbf{f_{i+1}}^{(j_1,\ldots,j_N)} \\
&= (-1)^{\sum_{m=1}^N j_m} \frac{\partial^{j_1}}{\partial v_1^{j_1}} \ldots \frac{\partial^{j_N}}{\partial v_N^{j_N}} \int_{y_i=0}^{\infty} \mathbf{f_i}^{(N-1)*}(v_1, \ldots, v_{i-1}, y_i, v_{i+1}, \ldots, v_N) \\
&\sum_{k=0}^{\infty} \frac{y_i^k}{k!} \left(\frac{\mathbf{Q} - \sum_{m=1}^N \mathbf{R_m} v_m}{d_i} \right)^k \mathrm{d}y_i \;\; \sigma_i^* \left(\sum_{m=1}^N \mathbf{R_m} v_m - \mathbf{Q} \right) \Bigg|_{v_1=\cdots=v_N=0} \\
&= (-1)^{\sum_{m=1}^N j_m} \frac{\partial^{j_1}}{\partial v_1^{j_1}} \ldots \frac{\partial^{j_N}}{\partial v_N^{j_N}} \sum_{k=0}^{\infty} (-1)^k \frac{\partial^k}{\partial v_i^k} \mathbf{f_i}^{(N)*}(v_1, \ldots, v_N) \Big|_{v_i=0} \\
&\frac{1}{k!} \left(\frac{\mathbf{Q} - \sum_{m=1}^N \mathbf{R_m} v_m}{d_i} \right)^k \sigma_i^* \left(\sum_{m=1}^N \mathbf{R_m} v_m - \mathbf{Q} \right) \Bigg|_{v_1=\cdots=v_N=0} \\
&= \sum_{j_{1,1}+\ldots+j_{1,3}=j_1} \binom{j_1}{j_{1,1}, j_{1,2}, j_{1,3}} \ldots \sum_{j_{i,2}+j_{i,3}=j_i} \binom{j_i}{j_{i,2}, j_{i,3}} \ldots \\
&\sum_{j_{N,1}+\ldots+j_{N,3}=j_N} \binom{j_N}{j_{N,1}, j_{N,2}, j_{N,3}} \\
&\sum_{k=0}^{\infty} \mathbf{f}_i^{(j_{1,1},\ldots,j_{i-1,1},k,j_{i+1,1},\ldots,j_{N,1})} \mathbf{H}_i^{(j_{1,2},\ldots,j_{N,2})}(k) \sigma_i^{(j_{1,3},\ldots,j_{N,3})}.
\end{aligned} \tag{23}$$

$\square$

Applying a truncation of the infinite sum from $k = 0$ to ∞ at $k = K$ in (21) results in an approximate numerical procedure to compute the joint moments of the fluid levels based on system of $N(K+1)^N$ linear equations. In a proper choice of K, the effects of all the moments $\mathbf{f_i}^{(j_1,\dots,j_N)}$, in which $j_m > K$ at least for one $m = 1,\dots,N$, can be neglected.

3.2. **The steady-state fluid levels at arbitrary epoch.**

3.2.1. *Equilibrium relationships.* Let $\widetilde{s}_i(\ell)$ be the service time at station i in the ℓ-th cycle. The mean steady-state service time at station i is defined as

$$s_i = \lim_{k\to\infty} \frac{\sum_{\ell=1}^{k} \widetilde{s}_i(\ell)}{k}.$$

Similarly let $\widetilde{c}_i(\ell)$ be the cycle time between the $\ell-1$th and the ℓth visit to station i in the ℓ-th cycle. The steady state cycle time at station i is defined as

$$c_i = \lim_{k\to\infty} \frac{\sum_{\ell=1}^{k} \widetilde{c}_i(\ell)}{k}.$$

It follows from the definitions of c_i and s_i that

$$c_i = \sigma + \sum_{j=1}^{N} s_j, \text{ and } c = c_i, \quad i \in \{1,\dots,N\}. \tag{24}$$

Let $\Lambda_i(t)$ be the accumulated fluid flowed into the buffer of station i in interval $(0,t]$. The steady state mean amount of fluid, which flows into the buffer of station i during one cycle, a_i, is defined as

$$a_i = \lim_{k\to\infty} \frac{E[\sum_{\ell=1}^{k} \Lambda_i(t_i^f(\ell+1)) - \Lambda_i(t_i^f(\ell))]}{k}.$$

The right hand side of this definition can be rearranged as

$$\lim_{k\to\infty} \frac{E[\sum_{\ell=1}^{k} \Lambda_i(t_i^f(\ell+1)) - \Lambda_i(t_i^f(\ell))]}{E[\sum_{\ell=1}^{k} \widetilde{c}_i(\ell)]} \lim_{k\to\infty} \frac{E[\sum_{\ell=1}^{k} \widetilde{c}_i(\ell)]}{k}$$

and thus we get

$$a_i = \alpha_i c, \quad i \in \{1,\dots,N\}. \tag{25}$$

Corollary 3. *In the stable fluid non-zero switchover-times polling model the steady-state mean cycle time can be expressed as*

$$c = \frac{\sigma}{1-\rho}. \tag{26}$$

Proof. We apply a classical statistical equilibrium argumenting, see e.g. in [6]. The stable model is in statistical equilibrium, which implies that the mean amount of fluid flowing into the buffer of station i during a cycle equals the mean amount of fluid removed at station i during the same cycle, which equals $s_i d_i$. Putting them together yields

$$a_i = s_i d_i. \tag{27}$$

Applying (25) in (27) and expressing s_i from it leads to

$$s_i = \frac{\alpha_i}{d_i} c. \tag{28}$$

Applying (28) in (24) and changing to the notation of utilizations results in

$$c = \sigma + \sum_{j=1}^{N} \rho_j c. \tag{29}$$

Rearranging (29) gives the statement. □

Remark 3. The relations (24), (25) and (26) are valid independently of the used service discipline and hence they have more general validity scope.

3.2.2. *The steady-state moments of the service time at station i.* The steady state pdf of the service time at station i, $s_i(t)$, and the corresponding LT, $s_i^*(v)$, for $t \geq 0$ are defined as

$$s_i(t) = \lim_{k\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} \frac{E[\sum_{\ell=1}^{k} 1_{(\tilde{s}_i(\ell)<t)}]}{k}, \text{ and } s_i^*(v) = \int_{t=0}^{\infty} s_i(t) e^{-st} \mathrm{d}t,$$

where $1_{(\mathrm{con})}$ denotes the indicator of condition "con".

Let $\mathbf{f_i}(x_i)$ and $\mathbf{f_i}^*(v)$ stand for steady-state vector density of the fluid level at station i at i-polling epoch and its LT, respectively. They can be obtained from $\mathbf{f_i}(\mathbf{x})$ and $\mathbf{f_i}^{(N)*}(\mathbf{v})$ as

$$\mathbf{f_i}(x_i) = \int_{x_1=0}^{\infty} \dots \int_{x_{i-1}=0}^{\infty} \int_{x_{i+1}=0}^{\infty} \dots \int_{x_N=0}^{\infty} \mathbf{f_i}(\mathbf{x})\, \mathrm{d}x_N \dots \mathrm{d}x_{i+1} \mathrm{d}x_{i-1} \dots \mathrm{d}x_1,$$
$$\mathbf{f_i}^*(v) = \mathbf{f_i}^{(N)*}(\mathbf{v})\Big|_{v_1=\ldots=v_{i-1}=v_{i+1}=\ldots=v_N=0, v_i=v}.$$

Theorem 3.2. *In the stable fluid non-zero switchover-times polling model with gated discipline the steady-state LT of the service time at station i can be expressed as*

$$s_i^*(v) = \mathbf{f_i}^*(\frac{v}{d_i})\mathbb{1}, \quad i \in \{1, \dots, N\}. \tag{30}$$

Proof. If the fluid level at station i is x_i at i-polling epoch then the service time at station i is $\frac{x_i}{d_i}$. Therefore the steady-state LT of the service time at station i can be obtained as

$$s_i^*(v) = \int_{x_i=0}^{\infty} \mathbf{f_i}(x_i) e^{-v\frac{x_i}{d_i}} \mathrm{d}x_i \mathbb{1}, \tag{31}$$

which can be rearranged as (30). □

Corollary 4. *In the stable fluid non-zero switchover-times polling model with gated discipline the steady-state moments of the service time at station i are given as*

$$s_i^{(k)} = \frac{1}{d_i^k} \mathbf{f_i}^{(k)} \mathbb{1}, \quad k \geq 1, \quad i \in \{1, \dots, N\}. \tag{32}$$

Proof. Taking the k-th derivative of (30) with respect to v at $v = 0$ and multiplying it by $(-1)^k$ results in the statement. □

3.2.3. *The steady-state joint vector LT of the fluid levels at the stations at arbitrary epoch.* The steady-state joint density of the fluid levels at the stations and the state of the modulating Markov chain at an arbitrary epoch, the $1 \times L$ row vector $\mathbf{q}(\mathbf{x})$ is defined by its j-th element as

$$[\mathbf{q}(\mathbf{x})]_j = \lim_{t\to\infty} \frac{\partial}{\partial x_1} \dots \frac{\partial}{\partial x_N} Pr(\Omega(t) = j, X_1(t) < x_1, \dots X_N(t) < x_N), \ j \in \Omega,$$

and its LT with respect to $\mathbf{x}$ can be given as

$$\mathbf{q}^{(N)*}(\mathbf{v}) = \int_{x_1=0}^{\infty} \dots \int_{x_N=0}^{\infty} \mathbf{q}(\mathbf{x}) e^{-v_1 x_1} \dots e^{-v_N x_N} dx_N \dots dx_1.$$

Moreover, let $\mathbf{e_j} = (0, \dots, 0, 1, 0, \dots, 0)$ be the $1 \times L$ vector with 1 at the j-th position. Then the $1 \times L$ indicator vector $\mathbf{1}_{(\Omega(t))}$ is defined as

$$\mathbf{1}_{(\Omega(t))} = \sum_{j=1}^{L} 1_{(\Omega(t)=j)} \mathbf{e_j}.$$

We use the following notation

$$\begin{aligned}\mathbf{f_i}^{(N-1)*}(v_1, \dots, v_{i-1}, x_i, v_{i+1}, \dots, v_N) &= \int_{x_1=0}^{\infty} \dots \int_{x_{i-1}=0}^{\infty} \int_{x_{i+1}=0}^{\infty} \dots \int_{x_N=0}^{\infty} \\ \mathbf{f_i}(\mathbf{x})\ e^{-v_1 x_1} \dots e^{-v_{i-1}x_{i-1}} e^{-v_{i+1}x_{i+1}} &\dots e^{-v_N x_N} \mathrm{d}x_N \dots \mathrm{d}x_{i+1} \mathrm{d}x_{i-1} \dots \mathrm{d}x_1.\end{aligned}$$

Theorem 3.3. *In the stable fluid non-zero switchover-times polling model with gated discipline the following relation holds for the steady-state joint vector LT of the fluid levels at the stations at arbitrary epoch:*

$$\begin{aligned}&\mathbf{q}^{(N)*}(\mathbf{v}) \left(\sum_{j=1}^{N} \mathbf{R_j} v_j - \mathbf{Q} \right) \qquad (33)\\ &= \frac{1}{c} \sum_{i=1}^{N} \left[d_i v_i \left(\mathbf{f_i}^{(N)*}(\mathbf{v}) - \mathbf{m_i}^{(N)*}(\mathbf{v}) \right) \left(\sum_{j \neq i} \mathbf{R_j} v_j + (\mathbf{R_i} - d_i \mathbf{I})\, v_i - \mathbf{Q} \right)^{-1} \right].\end{aligned}$$

Proof. The fluid levels at the stations at arbitrary epoch can be expressed by the help of the fluid levels at the last i-polling epoch on LT level by utilizing the transient behavior of the arrived fluid (relation (6)) and taking into account that it can fall either in service or switchover period as well as its position in the actual period. Thus it is enough to average over a polling cycle for determining the behavior at arbitrary epoch.

Therefore $\mathbf{q}^{(N)*}(\mathbf{v})$ is given by

$$\begin{aligned}\mathbf{q}^{(N)*}(\mathbf{v}) &= \frac{E[\int_{t=0}^{\widetilde{c}_1} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t]}{E[\widetilde{c}_1]} \qquad (34)\\ &= \frac{\sum_{i=1}^{N} E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t] + \sum_{i=1}^{N} E[\int_{t=0}^{\widetilde{\sigma}_i} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t]}{c}.\end{aligned}$$

The fluid level at time t at station i in the service time of station i is the sum of the remaining fluid level, $\xi - t d_i$, and the fluid level arrived during t. The fluid level at time t at other stations, i.e., $j \neq i$ in the service time of station i is the sum of the fluid level at the begin of the service time and the fluid amount arrived during t.

Taking into account the state change of the modulating CTMC from 0 to t the LT term $E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t]$ can be given as

$$\begin{aligned}&E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t) v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t] \qquad (35)\\ &= \int_{\xi=0}^{\infty} e^{-(\xi - t d_i) v_i} \mathbf{f_i}^{(N-1)*}(v_1, \dots, v_{i-1}, \xi, v_{i+1}, \dots, v_N) \int_{t=0}^{\frac{\xi}{d_i}} \mathbf{A}^{(N)*}(t, \mathbf{v}) \mathrm{d}t \mathrm{d}\xi \\ &= \int_{\xi=0}^{\infty} e^{-\xi v_i} \mathbf{f_i}^{(N-1)*}(v_1, \dots, v_{i-1}, \xi, v_{i+1}, \dots, v_N) \int_{t=0}^{\frac{\xi}{d_i}} e^{t d_i v_i} \mathbf{A}^{(N)*}(t, \mathbf{v}) \mathrm{d}t \mathrm{d}\xi.\end{aligned}$$

Applying (6) in (35) and rearrangement gives

$$E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t] = \int_{\xi=0}^{\infty} e^{-\xi v_i} \mathbf{f_i}^{(N-1)*}(v_1, \ldots, v_{i-1}, \xi, v_{i+1}, \ldots, v_N) \\ \times \int_{t=0}^{\frac{\xi}{d_i}} e^{-t\left(\sum_{j\neq i} \mathbf{R_j} v_j + (\mathbf{R_i} - d_i \mathbf{I}) v_i - \mathbf{Q}\right)} \mathrm{d}t \mathrm{d}\xi. \quad (36)$$

The internal integral can be evaluated by means of a relation, which can be obtained by the help of the Taylor-expansion of $e^{\mathbf{Z}t}$, and is given by

$$\int_{t=0}^{x} e^{-\mathbf{Z}t} \mathrm{d}t \mathbf{Z} = (\mathbf{I} - e^{-\mathbf{Z}x}). \quad (37)$$

Applying (37) in (36) and rearrangement yields

$$E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t] \left(\sum_{j\neq i} \mathbf{R_j} v_j + (\mathbf{R_i} - d_i \mathbf{I})\, v_i - \mathbf{Q}\right) \quad (38) \\ = \int_{\xi=0}^{\infty} e^{-\xi v_i} \mathbf{f_i}^{(N-1)*}(v_1, \ldots, v_{i-1}, \xi, v_{i+1}, \ldots, v_N) \\ \left(\mathbf{I} - e^{-\frac{\xi}{d_i}\left(\sum_{j\neq i} \mathbf{R_j} v_j + (\mathbf{R_i} - d_i \mathbf{I}) v_i - \mathbf{Q}\right)}\right) \mathrm{d}\xi.$$

Rearrangement and applying (13) in (38) leads to

$$E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t] \left(\sum_{j\neq i} \mathbf{R_j} v_j + (\mathbf{R_i} - d_i \mathbf{I})\, v_i - \mathbf{Q}\right) \quad (39) \\ = \mathbf{f_i}^{(N)*}(\mathbf{v}) - \mathbf{f_i}^{(N)*}(v_1, \ldots, v_{i-1}, \frac{\sum_{i=1}^{N} \mathbf{R_i} v_i - \mathbf{Q}}{d_i}, v_{i+1}, \ldots, v_N) \\ = \mathbf{f_i}^{(N)*}(\mathbf{v}) - \mathbf{m_i}^{(N)*}(\mathbf{v}).$$

Further rearranging of (39) yields

$$E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t] \left(\sum_{j=1}^{N} \mathbf{R_j} v_j - \mathbf{Q}\right) \quad (40) \\ = \mathbf{f_i}^{(N)*}(\mathbf{v}) - \mathbf{m_i}^{(N)*}(\mathbf{v}) + d_i v_i E[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t].$$

Now we consider the term $E[\int_{t=0}^{\widetilde{\sigma}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t]$. The fluid level at time t at station j, $j \in \{1, \ldots, N\}$, in the switchover time after the service of station i is the sum of the fluid level at station j at start of the switchover time, and the fluid level arrived during t. Taking into account the state change of the modulating CTMC from 0 to t the LT term $E[\int_{t=0}^{\widetilde{\sigma}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t]$ can be given as

$$E[\int_{t=0}^{\widetilde{\sigma}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j} \mathbf{1}_{(\Omega(t))} \mathrm{d}t] = \mathbf{m_i}^{(N)*}(\mathbf{v}) \int_{\tau=0}^{\infty} \int_{t=0}^{\tau} \mathbf{A}^{(N)*}(t, \mathbf{v}) \mathrm{d}t \; \sigma(\tau) \; \mathrm{d}\tau. \quad (41)$$

Applying (6) in (41) yields

$$E[\int_{t=0}^{\widetilde{\sigma_i}} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}\mathrm{d}t] = \mathbf{m_i}^{(N)*}(\mathbf{v})\int_{\tau=0}^{\infty}\int_{t=0}^{\tau} e^{-t\left(\sum_{j=1}^{N}\mathbf{R_j}v_j-\mathbf{Q}\right)}\mathrm{d}t\ \sigma(\tau)\ \mathrm{d}\tau. \tag{42}$$

We apply again (37), now in (42), which gives

$$\begin{aligned} &E[\int_{t=0}^{\widetilde{\sigma_i}} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}\mathrm{d}t]\left(\sum_{j=1}^{N}\mathbf{R_j}v_j-\mathbf{Q}\right) \\ &= \mathbf{m_i}^{(N)*}(\mathbf{v})\int_{\tau=0}^{\infty}\left(\mathbf{I}-e^{-\tau\left(\sum_{j=1}^{N}\mathbf{R_j}v_j-\mathbf{Q}\right)}\right)\ \sigma(\tau)\ \mathrm{d}\tau. \end{aligned} \tag{43}$$

Rearranging (42) and applying (14) in it gives the relation for $E[\int_{t=0}^{\widetilde{\sigma_i}} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}\mathrm{d}t]$ as

$$\begin{aligned} &E[\int_{t=0}^{\widetilde{\sigma_i}} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}\mathrm{d}t]\left(\sum_{j=1}^{N}\mathbf{R_j}v_j-\mathbf{Q}\right) \\ &= \mathbf{m_i}^{(N)*}(\mathbf{v})\left(\mathbf{I}-\sigma_i^*\left(\sum_{j=1}^{N}\mathbf{R_j}v_j-\mathbf{Q}\right)\right) = \mathbf{m_i}^{(N)*}(\mathbf{v})-\mathbf{f_{i+1}}^{(N)*}(\mathbf{v}). \end{aligned} \tag{44}$$

Using (40) and (44) in (34) and rearranging gives

$$\begin{aligned} &\mathbf{q}^{(N)*}(\mathbf{v})\left(\sum_{j=1}^{N}\mathbf{R_j}v_j-\mathbf{Q}\right) \\ &= \frac{1}{c}\Bigg(\sum_{i=1}^{N}\left(\mathbf{f_i}^{(N)*}(\mathbf{v})-\mathbf{m_i}^{(N)*}(\mathbf{v})+d_iv_iE[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}\mathrm{d}t]\right) \\ &\quad+\sum_{i=1}^{N}\left(\mathbf{m_i}^{(N)*}(\mathbf{v})-\mathbf{f_{i+1}}^{(N)*}(\mathbf{v})\right)\Bigg) \\ &= \frac{1}{c}\sum_{i=1}^{N} d_iv_iE[\int_{t=0}^{\widetilde{s}_i} e^{-\sum_{j=1}^{N} X_j(t)v_j}\mathbf{1}_{(\Omega(t))}\mathrm{d}t]. \end{aligned} \tag{45}$$

The statement of the theorem comes by applying (39) in (45). □

4. **Analysis with the method of supplementary variable.** We recall that $\Omega(t)$ is the state of the CTMC, and $X_i(t)$ is the fluid level at station i at time t. Let $Z(t)$ be the fluid arrived during service of the served station, and $Y(t)$ the amount of fluid to serve in the current service period at time t. That is, while station i is served $Z(t)+Y(t)=X_i(t)$ holds. During a switchover period, $V_i(t)$ denotes the time since the start of the ongoing switchover period from station i at time t. Furthermore, we introduce vector $\mathbf{h}_i(t,\mathbf{x},y)$ and $\mathbf{g_i}(t,\mathbf{x},y)$, whose jth elements are defined as

$$\begin{aligned} [\mathbf{H_i}(t,\mathbf{x},y)]_j = Pr(\Omega(t)=j, X_1(t)<x_1,\dots,Z(t)<x_i,\dots,X_N(t)<x_N,& \\ Y(t)<y, \text{station } i \text{ is served at } t)& \end{aligned}$$

$$[\mathbf{h_i}(t,\mathbf{x},y)]_j = \frac{\partial}{\partial x_1}\dots\frac{\partial}{\partial x_N}\frac{\partial}{\partial y}[\mathbf{H_i}(t,\mathbf{x},y)]_j$$

and

$$\begin{aligned}[\mathbf{G_i}(t,\mathbf{x},y)]_j = Pr(&\Omega(t)=j, X_1(t)<x_1,\dots,X_N(t)<x_N, V(t)<y,\\ &\text{switchover from } i \text{ to } i+1 \text{ at } t),\end{aligned}$$

$$[\mathbf{g_i}(t,\mathbf{x},y)]_j = \frac{\partial}{\partial x_1}\dots\frac{\partial}{\partial x_N}\frac{\partial}{\partial y}[\mathbf{G_i}(t,\mathbf{x},y)]_j\,,$$

where $\mathbf{x}=(x_1,\dots,x_N)$. Both, vector $\mathbf{h_i}(t,\mathbf{x},y)$ and $\mathbf{g_i}(t,\mathbf{x},y)$ describe the evolution of the process with a supplementary variable. During the service period, the supplementary variable, $Y(t)$, starts from a positive value (the fluid in the buffer of the served station at polling epoch) and decreases continuously at rate d_i until it gets zero and the service period ends. During the switchover period the supplementary variable, $V(t)$, starts from zero and increases continuously at rate 1, and the switchover period ends according to the value of the hazard rate function $\lambda_i(V(t))$. By definition

$$\sum_{i=1}^{N}\int_{\mathbf{x}}\int_y \mathbf{h_i}(t,\mathbf{x},y)\mathrm{d}y\mathrm{d}\mathbf{x} + \sum_{i=1}^{N}\int_{\mathbf{x}}\int_y \mathbf{g_i}(t,\mathbf{x},y)\mathrm{d}y\mathrm{d}\mathbf{x} = \pi_0 e^{\mathbf{Q}t},$$

where $\int_{\mathbf{x}} \bullet \mathrm{d}\mathbf{x} = \int_{x_1}\dots\int_{x_N}\bullet\mathrm{d}x_N\dots\mathrm{d}x_1$, since

$$\int_{\mathbf{x}}\int_y [\mathbf{h_i}(t,\mathbf{x},y)]_j\,\mathrm{d}y\mathrm{d}\mathbf{x} = Pr(\Omega(t)=j, \text{station } i \text{ is served at } t),$$

$$\int_{\mathbf{x}}\int_y [\mathbf{g_i}(t,\mathbf{x},y)]_j\,\mathrm{d}y\mathrm{d}\mathbf{x} = Pr(\Omega(t)=j, \text{switchover from } i \text{ to } i+1 \text{ at } t)$$

and the jth element of vector $\pi_0 e^{\mathbf{Q}t}$ is $Pr(\Omega(t)=j)$.

Theorem 4.1. *For $0<t,x_1,\dots,x_N,y$, $\mathbf{h_i}(t,\mathbf{x},y)$ and $\mathbf{g_i}(t,\mathbf{x},y)$ satisfy*

$$\frac{\partial}{\partial t}\mathbf{h_i}(t,\mathbf{x},y) + \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\mathbf{h_i}(t,\mathbf{x},y)\mathbf{R_i} - d_i\frac{\partial}{\partial y}\mathbf{h_i}(t,\mathbf{x},y) = \mathbf{h_i}(t,\mathbf{x},y)\mathbf{Q} \tag{46}$$

and

$$\frac{\partial}{\partial t}\mathbf{g_i}(t,\mathbf{x},y) + \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\mathbf{g_i}(t,\mathbf{x},y)\mathbf{R_i} + \frac{\partial}{\partial y}\mathbf{g_i}(t,\mathbf{x},y) = \mathbf{g_i}(t,\mathbf{x},y)(\mathbf{Q}-\lambda_i(y)\mathbf{I}). \tag{47}$$

For $0<t,x_1,\dots,x_N$, $\mathbf{h_i}(t,\mathbf{x},y)$ and $\mathbf{g_i}(t,\mathbf{x},y)$ satisfy the boundary equations

$$\mathbf{h_i}(t,\mathbf{x_i},x_i)\mathbf{R_i} = \int_0^\infty \lambda_{i-1}(y)\mathbf{g_{i-1}}(t,\mathbf{x},y)\mathrm{d}y, \tag{48}$$

$$\mathbf{g_i}(t,\mathbf{x},0) = d_i\mathbf{h_i}(t,\mathbf{x},0), \tag{49}$$

where $\mathbf{x_i}=(x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_N)$.

For $\forall i,m\in\{1,\dots,N\}$, $0<t,x_1,\dots,x_{i-1},x_{i+1},\dots,x_N$ and $y\ge 0$, $\mathbf{h_i}(t,\mathbf{x},y)$ and $\mathbf{g_i}(t,\mathbf{x},y)$ satisfy the "empty buffer" boundary equations

$$\mathbf{h_i}(t,\mathbf{x_m},0) = \mathbf{0}, \tag{50}$$

$$\mathbf{h_i}(t,\mathbf{x_m},y) = \mathbf{0},\ \textit{for } m\neq i, \tag{51}$$

$$\mathbf{g_i}(t,\mathbf{x_m},y) = \mathbf{0}. \tag{52}$$

Proof. Following a forward differential argument we can write

$$\begin{aligned}\left[\mathbf{H_i}(t+\Delta,\mathbf{x},y)\right]_j =&(1+q_{jj}\Delta)\left[\mathbf{H_i}(t,x_1-[\mathbf{r_1}]_j\Delta,\dots,x_N-[\mathbf{r_N}]_j\Delta,y+d_i\Delta)\right]_j\\ &+\sum_{k,k\neq j} q_{kj}\Delta\left[\mathbf{H_i}(t,\mathbf{x}-\mathbf{\Theta}(\Delta),y+\Delta)\right]_k+\theta(\Delta)\end{aligned}$$

and

$$\begin{aligned}\left[\mathbf{G_i}(t+\Delta,\mathbf{x},y)\right]_j =&(1+q_{jj}\Delta-\lambda_i(y)\Delta)\left[\mathbf{G_i}(t,x_1-[\mathbf{r_1}]_j\Delta,\dots,x_N-[\mathbf{r_N}]_j\Delta,y+\Delta)\right]_j\\ &+\sum_{k,k\neq j} q_{kj}\Delta\left[\mathbf{G_i}(t,\mathbf{x}-\mathbf{\Theta}(\Delta),y+\Delta)\right]_k+\theta(\Delta),\end{aligned}\tag{53}$$

where $\theta(\Delta)$ and $\Theta(\Delta)$ are such that $\lim_{\Delta\to 0}\theta(\Delta)/\Delta=0$ and $\lim_{\Delta\to 0}\Theta(\Delta)=0$ and $\mathbf{x}-\mathbf{\Theta}(\Delta)=(x_1-\Theta(\Delta),\dots,x_N-\Theta(\Delta))$. In these expressions, apart of a $\theta(\Delta)$ error term, $1+q_{jj}\Delta$ is the probability that the Markov chain stays in state j in $(t,t+\Delta)$, $1+q_{jj}\Delta-\lambda_i(y)\Delta$ is the probability that the Markov chain stays in state j and the switchover period does not complete in $(t,t+\Delta)$, $q_{kj}\Delta$ is the probability that the Markov chain moves from k to j in $(t,t+\Delta)$ and $\lambda_i(y)\Delta$ is the probability that the switchover period completes in $(t,t+\Delta)$. For completeness, we demonstrate the steps of the forward differential argument for obtaining $\left[\mathbf{h_i}(t,\mathbf{x},y)\right]_j$. First we write

$$\begin{aligned}&\frac{\left[\mathbf{H_i}(t+\Delta,\mathbf{x},y)\right]_j-\left[\mathbf{H_i}(t,x_1-[\mathbf{r_1}]_j\Delta,\dots,x_N-[\mathbf{r_N}]_j\Delta,y+d_i\Delta)\right]_j}{\Delta}\\ &=\sum_k q_{kj}\left[\mathbf{H_i}(t,\mathbf{x}-\mathbf{\Theta}(\Delta),y+\Delta)\right]_k+\frac{\theta(\Delta)}{\Delta},\end{aligned}$$

from which the limit at $\Delta\to 0$ is

$$\frac{\partial}{\partial t}\left[\mathbf{H_i}(t,\mathbf{x},y)\right]_j+[\mathbf{r_i}]_j\frac{\partial}{\partial\mathbf{x}}\left[\mathbf{H_i}(t,\mathbf{x},y)\right]_j-d_i\frac{\partial}{\partial y}\left[\mathbf{H_i}(t,\mathbf{x},y)\right]_j=\sum_k q_{kj}\left[\mathbf{H_i}(t,\mathbf{x},y)\right]_k,$$

and differentiating with respect to $\mathbf{x}$ and y gives

$$\frac{\partial}{\partial t}\left[\mathbf{h_i}(t,\mathbf{x},y)\right]_j+[\mathbf{r_i}]_j\frac{\partial}{\partial\mathbf{x}}\left[\mathbf{h_i}(t,\mathbf{x},y)\right]_j-d_i\frac{\partial}{\partial y}\left[\mathbf{h_i}(t,\mathbf{x},y)\right]_j=\sum_k q_{kj}\left[\mathbf{h_i}(t,\mathbf{x},y)\right]_k,$$

1 whose vector from is (46). (47) is obtained by the same steps from (53).

We introduce $\mathbf{x_i}+[\mathbf{r_i}]_j\Delta\mathbf{e_i}=(x_1,\dots,x_{i-1},[\mathbf{r_i}]_j\Delta,x_{i+1},\dots,x_N)$, where $\mathbf{e_i}$ is the ith unit vector and for the boundary equations we write

$$\begin{aligned}&\left[\mathbf{H_i}(t+\Delta,\mathbf{x_i}+[\mathbf{r_i}]_j\Delta\mathbf{e_i},x_i)\right]_j\\ &=\sum_{n=0}^{\infty}\lambda_i(n\Delta)\Delta\Big(\left[\mathbf{G_{i-1}}(t,\mathbf{x}-\mathbf{\Theta}(\Delta),(n+1)\Delta)\right]_j-\left[\mathbf{G_{i-1}}(t,\mathbf{x}-\mathbf{\Theta}(\Delta),n\Delta)\right]_j\Big)+\theta(\Delta)\end{aligned}$$

and

$$\left[\mathbf{G_i}(t+\Delta,\mathbf{x},\Delta)\right]_j=\left[\mathbf{H_i}(t,\mathbf{x}-\mathbf{\Theta}(\Delta),d_i\Delta)\right]_j+\theta(\Delta).\tag{54}$$

$\left[\mathbf{H_i}(t+\Delta,\mathbf{x_i}+[\mathbf{r_i}]_j\Delta\mathbf{e_i},x_i)\right]_j$ means that during a service period of station i at time $t+\Delta$ the accumulated fluid is less than $[\mathbf{r_i}]_j\Delta$. It implies that the switchover period ended in $(t,t+\Delta)$ and the fluid level was less than x_i, apart of a $\theta(\Delta)$ error term, at time t. When the length of the switchover period is between $n\Delta$ and $(n+1)\Delta$, the probability that the switchover period ends in $(t,t+\Delta)$ is $\lambda_i(n\Delta)\Delta$, apart of a $\theta(\Delta)$ error term again. The probability that the switchover period ended and the Markov chain had a state transition in $(t, t+\Delta)$ is as small as $\theta(\Delta)$. Now we write the Taylor series of $\left[\mathbf{H_i}(t+\Delta, \mathbf{x_i}+[\mathbf{r_i}]_j\Delta\mathbf{e_i}, x_i)\right]_j$ as

$$\begin{aligned}\left[\mathbf{H_i}(t+\Delta, \mathbf{x_i}+[\mathbf{r_i}]_j\Delta\mathbf{e_i}, x_i)\right]_j &= \left[\mathbf{H_i}(t+\Delta, \mathbf{x_i}, x_i)\right]_j \\ &\quad + [\mathbf{r_i}]_j\Delta\left[\mathbf{H_i}^{(0,\mathbf{e_i},0)}(t+\Delta, \mathbf{x_i}, x_i)\right]_j + \theta(\Delta),\end{aligned}$$

where the superscripts in brackets refer to the derivatives, that is

$$f^{(j,\mathbf{v},\ell)}(t, \mathbf{x}, y) = \frac{\partial^j}{\partial t^j}\frac{\partial^{v_1}}{\partial x_1^{v_1}}\cdots\frac{\partial^{v_N}}{\partial x_N^{v_N}}\frac{\partial^\ell}{\partial y^\ell}f(t, \mathbf{x}, y).$$

By this notation $\left[\mathbf{H_i}^{(0,\mathbf{1},1)}(t, \mathbf{x}, y)\right]_j = [\mathbf{h_i}(t, \mathbf{x}, y)]_j$, where $\mathbf{1}$ denotes the vector composed of ones. Substituting the results of the expansion gives

$$\underbrace{\left[\mathbf{H_i}(t+\Delta, \mathbf{x_i}, x_i)\right]_j}_{0} + [\mathbf{r_i}]_j\Delta\left[\mathbf{H_i}^{(0,\mathbf{e_i},0)}(t+\Delta, \mathbf{x_i}, x_i)\right]_j + \theta(\Delta) =$$

$$\sum_{n=0}^{\infty}\lambda_{i-1}(n\Delta)\Delta\Big(\left[\mathbf{G_{i-1}}(t, \mathbf{x}-\boldsymbol{\Theta}(\Delta), (n+1)\Delta)\right]_j - \left[\mathbf{G_{i-1}}(t, \mathbf{x}-\boldsymbol{\Theta}(\Delta), n\Delta)\right]_j\Big) + \theta(\Delta).$$

Dividing both sides by Δ and letting $\Delta \to 0$ results

$$[\mathbf{r_i}]_j\left[\mathbf{H_i}^{(0,\mathbf{e_i},0)}(t, \mathbf{x_i}, x_i)\right]_j = \int_0^\infty \lambda_{i-1}(y)\left[\mathbf{G_{i-1}}^{(0,\mathbf{0},1)}(t, \mathbf{x}, y)\right]_j \mathrm{d}y.$$

Finally, a derivative with respect to $x_1, \ldots, x_N$ gives

$$[\mathbf{r_i}]_j\left[\mathbf{h_i}(t, \mathbf{x_i}, x_i)\right]_j = \int_0^\infty \lambda_{i-1}(y)\left[\mathbf{g_{i-1}}(t, \mathbf{x}, y)\right]_j \mathrm{d}y,$$

whose vector form is (48).

The derivation of (49) based on (54) follows the same pattern and is omitted.

For the empty buffer boundary equations, (50) and (52), we note that for $y > \Delta\min_j[\mathbf{r_m}]_j$

$$\left[\mathbf{G_i}(t, \mathbf{x_m}+[\mathbf{r_m}]_j\Delta\mathbf{e_m}, y)\right]_j = 0, \tag{55}$$

that is, if the switchover period is longer than $\Delta\min_j[\mathbf{r_m}]_j$ the amount of fluid in buffer m accumulated during the switchover period is larger than $[\mathbf{r_m}]_j\Delta$. When y is small (smaller than $\Delta\min_j[\mathbf{r_m}]_j$) we need to backtrack the process evolution:

$$\left[\mathbf{G_i}(t, \mathbf{x_m}+3[\mathbf{r_m}]_j\Delta\mathbf{e_m}, \Delta)\right]_j = \left[\mathbf{H_i}(t-\Delta, \mathbf{x_m}+2[\mathbf{r_m}]_j\Delta\mathbf{e_m}, d\Delta)\right]_j + \theta(\Delta),$$

where the $\theta(\Delta)$ error term also contains the state transition of the Markov chain.

$$\begin{aligned}&\left[\mathbf{H_i}(t-\Delta, \mathbf{x_m}+2[\mathbf{r_m}]_j\Delta\mathbf{e_m}, d\Delta)\right]_j = \\ &\sum_{n=0}^{\infty}\lambda_{i-1}(n\Delta)\Delta\Big(\left[\mathbf{G_{i-1}}(t-2\Delta, \mathbf{x_m}+[\mathbf{r_m}]_j\Delta\mathbf{e_m}, (n+1)\Delta)\right]_j \\ &\qquad - \left[\mathbf{G_{i-1}}(t-2\Delta, \mathbf{x_m}+[\mathbf{r_m}]_j\Delta\mathbf{e_m}, n\Delta)\right]_j\Big) + \theta(\Delta),\end{aligned}$$

where $\left[\mathbf{G_{i-1}}(t-2\Delta, \mathbf{x_m}+[\mathbf{r_m}]_j\Delta\mathbf{e_m}, n\Delta)\right]_j = 0$ for large n values according to (55). That is both $\left[\mathbf{G_i}(t, \mathbf{x_m}+\Theta(\Delta)\mathbf{e_m}, \Theta(\Delta))\right]_j$ and $\left[\mathbf{H_i}(t, \mathbf{x_m}+\Theta(\Delta)\mathbf{e_m}, \Theta(\Delta))\right]_j$

can be non-negligible only if the previous switchover periods are shorter than $\Theta(\Delta)$ and the probability of 2 such short switchover periods is $\theta(\Delta)$. $\square$

4.1. **Stationary behavior.** To analyze the stationary behavior we introduce $[\mathbf{h_i}(\mathbf{x},y)]_j = \lim_{t\to\infty}[\mathbf{h_i}(t,\mathbf{x},y)]_j$ and $[\mathbf{g_i}(\mathbf{x},y)]_j = \lim_{t\to\infty}[\mathbf{g_i}(t,\mathbf{x},y)]_j$, for which based on (26) and the definition of π, ρ, σ, $\mathbf{h_i}(t,\mathbf{x},y)$ and $\mathbf{g_i}(t,\mathbf{x},y)$ we have

$$\sum_{i=1}^{N}\int_{\mathbf{x}}\int_{y}\mathbf{h_i}(\mathbf{x},y)\mathrm{d}y\mathrm{d}\mathbf{x}+\int_{\mathbf{x}}\int_{y}\mathbf{g_i}(\mathbf{x},y)\mathrm{d}y\mathrm{d}\mathbf{x}=\pi,$$

$$\int_{\mathbf{x}}\int_{y}\mathbf{h_i}(\mathbf{x},y)\mathbb{1}\mathrm{d}y\mathrm{d}\mathbf{x}=\lim_{t\to\infty}Pr(\text{station } i \text{ is served at } t)=\rho_i,$$

and

$$\int_{\mathbf{x}}\int_{y}\mathbf{g_i}(\mathbf{x},y)\mathbb{1}\mathrm{d}y\mathrm{d}\mathbf{x}=\lim_{t\to\infty}Pr(\text{switchover from } i \text{ to } i+1 \text{ at } t)=\frac{(1-\rho)\sigma_i}{\sigma}.$$

Corollary 5. *At the stationary limit, for $0<x_1,\dots,x_N,y$, $\mathbf{h_i}(\mathbf{x},y)$ and $\mathbf{g_i}(\mathbf{x},y)$ satisfy*

$$\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\mathbf{h_i}(\mathbf{x},y)\mathbf{R_j}-d_i\frac{\partial}{\partial y}\mathbf{h_i}(\mathbf{x},y)=\mathbf{h_i}(\mathbf{x},y)\mathbf{Q} \tag{56}$$

and

$$\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\mathbf{g_i}(\mathbf{x},y)\mathbf{R_j}+\frac{\partial}{\partial y}\mathbf{g_i}(\mathbf{x},y)=\mathbf{g_i}(\mathbf{x},y)(\mathbf{Q}-\lambda_i(y)\mathbf{I}). \tag{57}$$

For $0<x_1,\dots,x_N$, $\mathbf{h_i}(\mathbf{x},y)$ and $\mathbf{g_i}(\mathbf{x},y)$ satisfy the boundary equations

$$\mathbf{h_i}(\mathbf{x_i},x_i)\mathbf{R_i}=\int_0^\infty\lambda_{i-1}(y)\mathbf{g_{i-1}}(\mathbf{x},y)\mathrm{d}y, \tag{58}$$

$$\mathbf{g_i}(\mathbf{x},0)=d_i\mathbf{h_i}(\mathbf{x},0). \tag{59}$$

For $\forall i,m\in\{1,\dots,N\}$, $0<x_1,\dots,x_{i-1},x_{i+1},\dots,x_N$ and $y\ge 0$, $\mathbf{h_i}(\mathbf{x},y)$ and $\mathbf{g_i}(\mathbf{x},y)$ satisfy the "empty buffer" boundary equations

$$\mathbf{h_i}(\mathbf{x_i},0)=\mathbf{0}, \tag{60}$$

$$\mathbf{h_i}(\mathbf{x_m},y)=\mathbf{0},\ \textit{for } m\ne i, \tag{61}$$

$$\mathbf{g_i}(\mathbf{x_m},y)=\mathbf{0}. \tag{62}$$

Proof. The corollary comes by making the $t\to\infty$ limit at Theorem 4.1. $\square$

4.2. **Stationary polling and departure rates.**

Theorem 4.2.

$$\int_{\mathbf{x}}\int_{y}\mathbf{g_i}(\mathbf{x},y)\mathbb{1}\lambda_i(y)\mathrm{d}y\mathrm{d}\mathbf{x}=\frac{1}{c}$$

and

$$d_i\int_{\mathbf{x}}\mathbf{h_i}(\mathbf{x},0)\mathbb{1}\mathrm{d}\mathbf{x}=\frac{1}{c}$$

Proof. On the one hand, i to $i+1$ switchover ($i+i$ polling) and service i completion (i departure) occurs once in every cycle, whose mean length is c, from which

$$\lim_{t\to\infty} Pr(i \text{ to } i+1 \text{ switchover ends in } (t,t+\Delta)) = \frac{\Delta}{c} + \theta(\Delta),$$

$$\lim_{t\to\infty} Pr(\text{service } i \text{ completion in } (t,t+\Delta)) = \frac{\Delta}{c} + \theta(\Delta).$$

On the other hand

$$\lim_{t\to\infty} Pr(i \text{ to } i+1 \text{ switchover ends in } (t,t+\Delta)) = \int_{\mathbf{x}}\int_{y} \mathbf{g_i}(\mathbf{x},y)\mathbb{1}\lambda_i(y)\mathrm{d}y\mathrm{d}\mathbf{x}\Delta + \theta(\Delta),$$

$$\lim_{t\to\infty} Pr(\text{service } i \text{ completion in } (t,t+\Delta)) = \mathbf{H_i}(\infty,d_i\Delta)\mathbb{1} = d_i \int_{\mathbf{x}} \mathbf{h_i}(\mathbf{x},0)\mathrm{d}\mathbf{x}\mathbb{1}\Delta + \theta(\Delta).$$

Dividing the equations by Δ and making the $\Delta \to 0$ limit gives the theorem. □

Theorem 4.3.

$$\mathbf{f_{i+1}}(\mathbf{x}) = c\int_0^\infty \mathbf{g_i}(\mathbf{x},y)\lambda_i(y)\mathrm{d}y \tag{63}$$

$$\mathbf{m_i}(\mathbf{x}) = cd_i\mathbf{h_i}(\mathbf{x},0) \tag{64}$$

Proof.

$$\begin{aligned}
[\mathbf{f_{i+1}}(\mathbf{x})]_j &= \lim_{\Delta\to0}\lim_{t\to\infty}\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_N} \\
&\frac{Pr(\Omega(t)=j, X_1(t)<x_1,\ldots,X_N(t)<x_N, i \text{ to } i{+}1 \text{ switchover ends in } (t,t{+}\Delta))}{Pr(i \text{ to } i{+}1 \text{ switchover ends in } (t,t{+}\Delta))} \\
&= \lim_{\Delta\to0}\frac{\int_0^\infty [\mathbf{g_i}(\mathbf{x},y)]_j\,\lambda_i(y)\mathrm{d}y\Delta + \theta(\Delta)}{\frac{\Delta}{c}+\theta(\Delta)} = c\int_0^\infty [\mathbf{g_i}(\mathbf{x},y)]_j\,\lambda_i(y)\mathrm{d}y
\end{aligned}$$

$$\begin{aligned}
[\mathbf{m_i}(\mathbf{x})]_j &= \lim_{\Delta\to0}\lim_{t\to\infty}\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_N} \\
&\frac{Pr(\Omega(t)=j, X_1(t)<x_1,\ldots,X_N(t)<x_N, \text{service } i \text{ completion in } (t,t{+}\Delta))}{Pr(\text{service } i \text{ completion in } (t,t{+}\Delta))} \\
&= \lim_{\Delta\to0}\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_N}\frac{[\mathbf{H_i}(\mathbf{x},d_i\Delta)]_j}{\frac{\Delta}{c}+\theta(\Delta)} = \lim_{\Delta\to0}\frac{d_i\,[\mathbf{h_i}(\mathbf{x},0)]_j\,\Delta+\theta(\Delta)}{\frac{\Delta}{c}+\theta(\Delta)} = cd_i\,[\mathbf{h_i}(\mathbf{x},0)]_j
\end{aligned}$$

□

4.3. **Relation of the analysis approaches.** The N-fold and $N+1$-fold Laplace transform of $\mathbf{h_i}(\mathbf{x},y)$ and $\mathbf{g_i}(\mathbf{x},y)$ are denoted by ${\mathbf{h_i}}^{(N)*}(\mathbf{v},y)$, ${\mathbf{g_i}}^{(N)*}(\mathbf{v},y)$, ${\mathbf{h_i}}^{(N+1)*}(\mathbf{v},u)$ and ${\mathbf{g_i}}^{(N+1)*}(\mathbf{v},u)$, respectively.

Theorem 4.4. *The relation* $\mathbf{m_i}(\mathbf{x}) \to \mathbf{f_{i+1}}(\mathbf{x})$ *reads as*

$${\mathbf{f_{i+1}}}^{(N)*}(\mathbf{v}) = {\mathbf{m_i}}^{(N)*}(\mathbf{v})\int_{y=0}^\infty e^{y(\mathbf{Q}-\sum_{j=1}^N v_j\mathbf{R_j})}\sigma_i(y)\mathrm{d}y.$$

Proof. The N-fold Laplace transform of (57) is

$$\sum_{j=1}^{N}\left(v_j \mathbf{g_i}^{(N)*}(\mathbf{v},y) - \underbrace{\mathbf{g_i}^{(N-1)*}(\mathbf{v_j},y)}_{\mathbf{0}\ \text{ due to (62)}}\right)\mathbf{R_j} + \frac{\partial}{\partial y}\mathbf{g_i}^{(N)*}(\mathbf{v},y)$$
$$= \mathbf{g_i}^{(N)*}(\mathbf{v},y)(\mathbf{Q} - \lambda_i(y)\mathbf{I}),$$

which can be written as

$$\frac{\partial}{\partial y}\mathbf{g_i}^{(N)*}(\mathbf{v},y) = \mathbf{g_i}^{(N)*}(\mathbf{v},y)\left(\mathbf{Q} - \sum_{j=1}^{N} v_j \mathbf{R_j} - \lambda_i(y)\mathbf{I}\right). \tag{65}$$

The solution of (65) is

$$\begin{aligned}
\mathbf{g_i}^{(N)*}(\mathbf{v},y) &= \mathbf{g_i}^{(N)*}(\mathbf{v},0)e^{y(\mathbf{Q}-\sum_{j=1}^{N} v_j\mathbf{R_j}-\lambda_i(y)\mathbf{I})} \\
&= d_i\mathbf{h_i}^{(N)*}(\mathbf{v},0)e^{y(\mathbf{Q}-\sum_{j=1}^{N} v_j\mathbf{R_j})}e^{-y\lambda_i(y)} \\
&= \frac{1}{c}\mathbf{m_i}^{(N)*}(\mathbf{v})e^{y(\mathbf{Q}-\sum_{j=1}^{N} v_j\mathbf{R_j})}e^{-y\lambda_i(y)},
\end{aligned}$$

where we used (59) and (64). Multiplying both sides with $\lambda_i(y)$ and integrating from 0 to ∞ we get

$$\int_{y=0}^{\infty}\mathbf{g_i}^{(N)*}(\mathbf{v},y)\lambda_i(y)\mathrm{d}y = \frac{1}{c}\mathbf{m_i}^{(N)*}(\mathbf{v})\int_{y=0}^{\infty} e^{y(\mathbf{Q}-\sum_{j=1}^{N} v_j\mathbf{R_j})}\underbrace{e^{-y\lambda_i(y)}\lambda_i(y)}_{\sigma(y)}\mathrm{d}y.$$

1 Substituting $\mathbf{f_{i+1}}^{(N)*}(\mathbf{v})$ from (63) to the right hand side gives the theorem. □

Theorem 4.5. *The relation* $\mathbf{f_i}(\mathbf{x}) \to \mathbf{m_i}(\mathbf{x})$ *reads as*

$$\begin{aligned}
\mathbf{m_i}^{(N)*}(\mathbf{v}) &= \mathbf{f_i}^{(N)*}\Big(v_1,\dots,v_{i-1},\frac{1}{d_i}\sum_{j=1}^{N} v_j\mathbf{R_j}-\mathbf{Q},v_{i+1},\dots,v_N\Big) \\
&= \int_{z=0}^{\infty}\mathbf{f_i}^{(N-1)*}(\mathbf{v_i}+z\mathbf{e_i})e^{-z\frac{1}{d_i}(\sum_{j=1}^{N} v_j\mathbf{R_j}-\mathbf{Q})}\mathrm{d}z
\end{aligned}$$

Proof. The N-fold Laplace transform of (56) using $y = w$ is

$$\sum_{j=1}^{N}\left(v_j\mathbf{h_i}^{(N)*}(\mathbf{v},w) - \mathbf{h_i}^{(N-1)*}(\mathbf{v_j},w)\right)\mathbf{R_j} - d_i\frac{\partial}{\partial w}\mathbf{h_i}^{(N)*}(\mathbf{v},w) \tag{66}$$
$$= \mathbf{h_i}^{(N)*}(\mathbf{v},w)\mathbf{Q}, \tag{67}$$

where according to (58) and (63), $\mathbf{h_i}^{(N-1)*}(\mathbf{v_j},w) = 0$ for $i \neq j$ and

$$\begin{aligned}
\mathbf{h_i}^{(N-1)*}(\mathbf{v_i},w)\mathbf{R_i} &= \int_{0}^{\infty}\lambda_{i-1}(y)\mathbf{g_{i-1}}^{(N-1)*}(\mathbf{v_i}+w\mathbf{e_i},y)\mathrm{d}y \\
&= \frac{1}{c}\mathbf{f_i}^{(N-1)*}(\mathbf{v_i}+w\mathbf{e_i}).
\end{aligned}$$

Using this, (67) can be written as

$$\frac{\partial}{\partial w}\mathbf{h_i}^{(N)*}(\mathbf{v},w) = \mathbf{h_i}^{(N)*}(\mathbf{v},w)\underbrace{\frac{1}{d_i}\left(\sum_{j=1}^{N} v_j\mathbf{R_j}-\mathbf{Q}\right)}_{\mathbf{A}} + \underbrace{\frac{-1}{cd_i}\mathbf{f_i}^{(N-1)*}(\mathbf{v_i}+w\mathbf{e_i})}_{\mathbf{w}(w)},$$

whose proper solution is

$$\mathbf{h_i}^{(N)*}(\mathbf{v}, w) = -\int_{z=w}^{\infty} \mathbf{w}(z) e^{(w-z)\mathbf{A}} \mathrm{d}z.$$

At $w = 0$, the solution is $\mathbf{h_i}^{(N)*}(\mathbf{v}, 0) = -\int_{z=0}^{\infty} \mathbf{w}(z) e^{-z\mathbf{A}} \mathrm{d}z$. Substituting $\mathbf{A}$, $\mathbf{w}(z)$ and (64) at $w = 0$, we get

$$\begin{aligned}\mathbf{h_i}^{(N)*}(\mathbf{v}, 0) &= \frac{1}{cd_i} \mathbf{m_i}^{(N)*}(\mathbf{v}) \\ &= \int_{z=0}^{\infty} \frac{1}{cd_i} \mathbf{f_i}^{(N-1)*}(\mathbf{v_i} + z\mathbf{e_i}) e^{-z\frac{1}{d_i}\left(\sum_{j=1}^{N} v_j \mathbf{R_i} - \mathbf{Q}\right)} \mathrm{d}z,\end{aligned}$$

which verifies the theorem. □

5. Numerical examples.

5.1. Method of embedded regenerative instances. The numerical example illustrates the computation of the steady-state vector moments of the fluid levels at polling epochs by using the approximate system of linear equations (21). We consider a system with $N = 2$ stations. The input parameters are given as

$$\mathbf{Q} = \begin{bmatrix} -0.4 & 0.4 \\ 0.8 & -0.8 \end{bmatrix}, \tag{68}$$

and

$$\mathbf{R}_1 = \begin{bmatrix} 0.7 & 0 \\ 0 & 1.4 \end{bmatrix}, \ \mathbf{R}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}. \tag{69}$$

The service rates are $d_1 = 3$ and $d_2 = 5$. The utilization of the stations are $\rho_1 = 0.3111$ and $\rho_2 = 0.3$ and hence the total utilization of the system is $\rho = 0.6111$. The vacation times are exponentially distributed, with parameters $\nu_1 = 2$ and $\nu_2 = 4$. The numeric computation is performed by the help a Matlab/Simulink implementation using symbolic (exact) arithmetic.

The first two moments, $\mathbf{f_1}^{(1)}$, $\mathbf{f_2}^{(1)}$ as well as $\mathbf{f_1}^{(2)}$ and $\mathbf{f_1}^{(2)}$ are provided in Table 1.

	1st moment element 0	**1st moment element 1**	**2nd moment element 0**	**2nd moment element 1**
Station 1:	1.0614	0.7386	2.1640	1.7821
Station 2:	2.1759	0.7170	8.3775	2.2387

TABLE 1. Steady-state vector moments of the fluid levels at polling epochs

5.2. Method of supplementary variables. In this numerical example we consider a system with $N = 2$ stations. The generator of the background process is characterized by

$$\mathbf{Q} = \begin{bmatrix} -8 & 1 & 7 \\ 0 & -1 & 1 \\ 5 & 20 & -25 \end{bmatrix}, \tag{70}$$

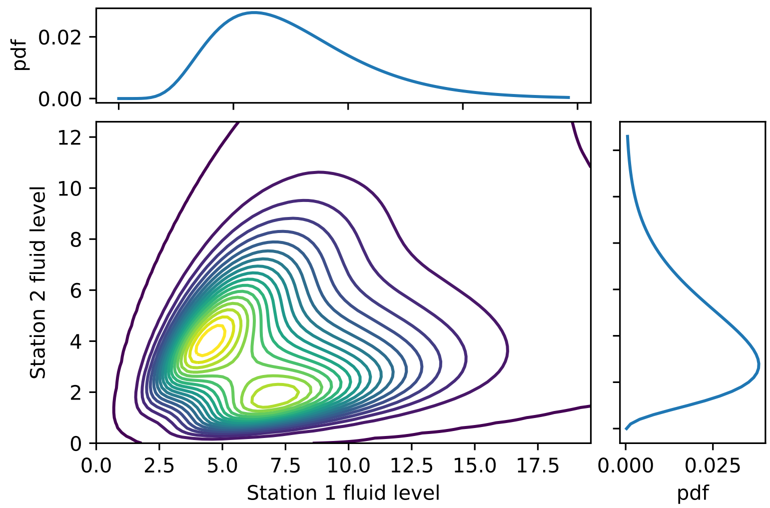

FIGURE 1. The joint distribution of the fluid level and the one-dimensional marginals

and the fluid input rate matrices associated with the two stations are given by

$$\mathbf{R}_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \ \mathbf{R}_2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{71}$$

The service rate is $d_1 = 5.7$ for station 1, and it is $d_2 = 4.9$ for station 2. With these parameters the utilization of the stations are $\rho_1 = 0.225$ and $\rho_2 = 0.416$, thus the total utilization of the system is $\rho = 0.641$.

Both vacation times are exponentially distributed, with rate parameter being $\nu_1 = 1.5$ for the first, and $\nu_2 = 1.1$ for the second station.

Our implementation is based on the temporal and spatial discretization of differential equations (46) and (47). We start with the empty system at $t = 0$ and the evolution of the fluid buffers and the background process are calculated for every Δ long time step. The length of the time step was $\Delta = 0.08$, and the discretization step for the fluid levels was $\delta = 0.2$. We found that around at $t = 25$ the steady state was reached, the results obtained are reported below. Due to the many dimensions (x_1, x_2 and the supplementary variable), we decided to prepare the implementation in the Julia programming language[1], since it has efficient memory management and almost native execution times, while maintaining a Matlab-like high level syntax.

The two dimensional density function of the fluid levels and the associated one-dimensional marginals as depicted in Figure 1. The mean fluid level is 4.164 at station 1, and it is 7.559 at station 2.

The mean fluid levels in the different phases of the service process are shown by Table 2. In line with the intuition, the fluid level of station 1 is the highest when the server is in a type-2 vacation, since in this phase a long time has passed since station 1 received service. The fluid level is the shortest when the server leaves

[1] https://julialang.org/

	St. 1. busy	**St. 1. vacation**	**St. 2. busy**	**St. 2. vacation**
Station 1:	7.559	5.827	7.861	9.418
Station 2:	3.915	5.932	4.362	2.194

TABLE 2. Mean fluid levels of the queue in different phases of the server

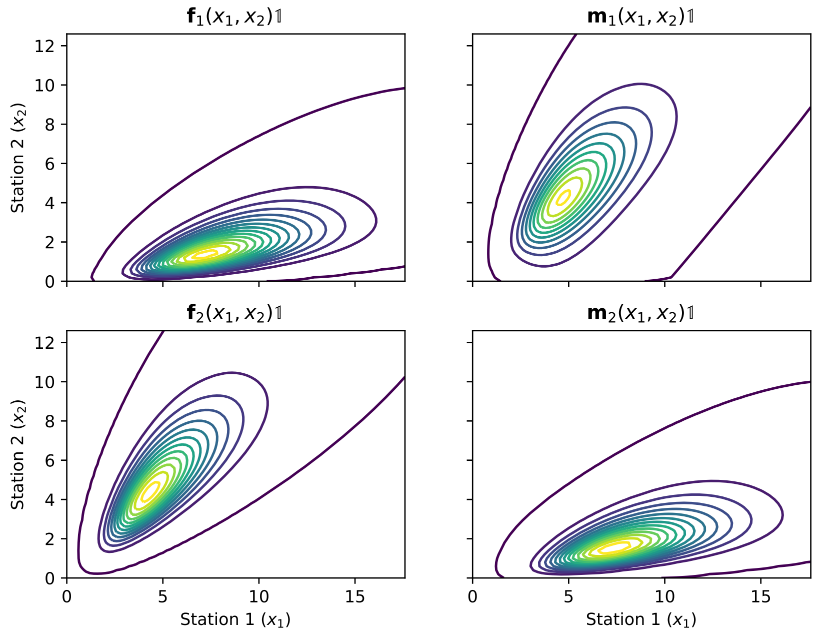

FIGURE 2. The joint distribution of the fluid level at polling and at departure epochs

station 1 and is on a type-1 vacation. The behavior of the station 2 fluid levels follows the same pattern.

The two-dimensional joint densities of the fluid levels are depicted by Figure 2 at 1-polling epoch ($\mathbf{f_1}(\mathbf{x})$), at 1- departure epoch ($\mathbf{m_1}(\mathbf{x})$), at 2-polling epoch ($\mathbf{f_2}(\mathbf{x})$), and at 2-departure epoch ($\mathbf{m_2}(\mathbf{x})$). The plots reflect the intuitive behavior of the system: at the $1-$departure epoch there is less type-1 but more type-2 fluid in the system then in the $1-$polling epoch, and similarly, at the $2-$departure epoch there is less type-2 but more type-1 fluid in the system then in the $2-$polling epoch.

The joint pdf of the fluid levels is uni-modal at the polling- and departure epochs. The two modes of the density function of the stationary fluid levels (Figure 1) is the consequence of mixing these uni-modal density functions.

6. **Conclusion.** In order to obtain computable analytical description of fluid polling models we presented two different analytical descriptions of the stationary model behaviour. The first one is based on the embedded process at server arrival and departure instances, and the second one is based on the supplementary variable approach. In the first case we provided a linear relation of the stationary

moments which can be solved if a feasible truncation limit is available and in the second case the numerical solution of a partial differential equation provides the stationary measures of interest.

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