$M/G/1$ queue with state dependent service times*

Zsolt Saffer[1], Karl Grill[1], and Miklós Telek[2,3]

[1] Institute of Statistics and Mathematical Methods in Economics,
Vienna University of Technology, Vienna, Austria
{zsolt.saffer,karl.grill}@tuwien.ac.at
[2] Department of Networked Systems and Services,
Budapest University of Technology and Economics, Budapest, Hungary
[3] ELKH-BME Informations systems reseach group, Budapest, Hungary
telek@hit.bme.hu

Abstract. In this paper we study the state dependent $M/G/1$ queueing system in which the service time can change at departure epochs. The model is a special case of an already investigated model. As a result of the narrowed scope we get numerically more effective and closed form solutions.
We provide the steady-state distribution of the number of customers in the system and the stability condition, both in terms of quantities computed by recursions.
We also study the model with finite number of state dependent service time distributions. For this model variant, closed form expressions are provided for the probability-generating function and the mean of the steady-state number of customers, which are computed from a system of linear equations.
Finally we also investigate the model with state dependent linear interpolation of two service times. For this model, we derive an explicit expression for the probability generating function of the steady-state number of customers and establish a simple, explicit stability condition. This model behaviour implements a control of number of customers in the system.
Keywords: queueing theory, state dependent service time distribution, control of queues

1 Introduction

The requirement for controlling the behaviour of queueing systems is a natural demand in the areas of their applications. One way of achieving this control is to implement a state dependent behaviour of the server.

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State dependent queueing systems appear in the literature since the 1960's. State dependency have been studied in the context of $M/M/$ systems as system with state dependent service rates. Such system has been analysed in the early work of [3], in which the service rate is specified to be proportional to the power of the number of customers. Harris [5] investigated a model, in which the dependency of service rate on the number of customers is linear. In the work [9] a two-state state dependent M/G/1 queue is investigated and the Laplace transform (LT) of the steady-state waiting time distribution is obtained. Gupta and Rao [4] studied a finite buffer queue with state dependent arrival rates and service times. They provided the distribution of the number of customers in the system. Kerner [6] considered an M/G/1 system with state dependent arrival rates and derived a closed form expression for the distribution of the customers in the system in terms of the idle probability.

M/G/1 queue with state dependent service times has been considered by many authors [2, 8]. Abouee-Mehrizi and Baron [1] investigated an M/G/1 queue with state dependent arrival rates and service times. They provided expression for the steady-state distribution of the number of customers in the system in terms of quantities depending on the LTs of the conditional state dependent residual service times, given the state of the system. These LTs are computed recursively and the solution requires $\mathcal{O}(K^2)$ operational steps, where K is the highest state to be taken into account to get the solution in required accuracy.

In this paper we study the state dependent M/G/1 queueing system in which the service time can change at departure epochs. This model is a special case of the model studied in [1], which is obtained by omitting the state dependency of arrival rates and at arrival epochs. As a result of the narrowed scope we get numerically more effective and for some cases simple closed form solutions.

We analyse the model at embedded departure epochs and use standard queueing arguments. We establish a forward recursion for computing the steady-state distribution of the number of customers in the system. This recursion requires also $\mathcal{O}(K^2)$ operational steps. We establish the stability condition of the model in terms of the quantities computed by the above mentioned forward recursion. We also study the special case of the model, in which only the first finite number of service times are state dependent. We provide closed form expressions for the probability-generating function (PGF) and the mean of the steady-state number of customers, which are computed from a system of linear equations. Additionally we also investigate the special case of the model with state dependent linear interpolation of two service times. For this model we derive an explicit expression for the PGF of the steady-state number of customers and establish a simple, closed form sufficient stability condition. The computation of the PGF and the moments of the steady-state number of customers requires $\mathcal{O}(K)$ operational steps, where K is a highest numerical index in infinite products and sums to be taken into account to get the PGF and the moments in required accuracy. This model is appropriate to implement a kind of control of number of customers in the system.

Compared to [1], there are significant differences in the applied analysis method in the current work. First of all, the state independent arrival process makes the Pasta property valid in our model, while it does not hold in [1]. The other significant simplification is that [1] is built on the computation of the steady-state residual service time distribution as a function of the customers in the system, while we compute the performance measures based on the number of Poisson arrivals during the state dependent service time.

The rest of this paper is organized as follows. In section 2 we describe the model and the notations. The steady-state analysis of the model is provided in section 3. Section 4 is devoted to the model variant with finite number of state dependent service times. Finally the investigation of the model with state dependent linear interpolation of two service time is presented in section 5.

2 Model description

We consider an infinite buffer queue. The arrival process is Poisson with rate $0 < \lambda < \infty$. The customer service time depends on the number of customers in the system and it is set at the service start epoch of the customer in the server. B_n, $b_n(t)$, $\tilde{B}_n(s)$, b_n and $b_n^{(2)}$ denote the service time random variable, its probability density function (pdf), its LT, its mean and its second moment when the number of customers is $n \geq 0$ at the service start epoch, respectively. The customer service times are independent with finite means, i.e. $0 < b_n < \infty$ for $n \geq 0$. For notational convenience we introduce $B_0 = B_1$.

We impose the usual assumptions on the model. The arrival process and the customer service times are mutually independent. The customers are served in First-In-First-Out (FIFO) order as well as the service during the service period is work conserving and non-preemptive. We denote the above described M/G/1 queue as $M/G_n/1$ queue.

When $\widehat{x}(z)$ is a PGF, $\widehat{x}^{(k)}$ denotes its k-th derivative at $z = 1$ for $k \geq 1$, i.e., $\widehat{x}^{(k)} = \frac{d^k}{dz^k}\widehat{x}(z)|_{z=1}$. Similarly when $\tilde{y}(s)$ is a LT, then $\tilde{y}^{(k)}(s_0)$ denotes its k-th derivative at $s = s_0$ for $k \geq 1$, i.e., $\tilde{y}^{(k)}(s_0) = \frac{d^k}{ds^k}\tilde{y}(s)|_{s=s_0}$. Additionally $\tilde{y}^{(0)}(s_0)$ denotes $\tilde{y}(s_0)$.

3 General $M/G_n/1$ system

3.1 Relation for the PGF of the steady-state number of customers

Let t_k^d stand for the epoch just after the departure of the k-th customer. Let $N(t)$ denote the number of customers in the system at time t for $t \geq 0$. Then the probabilities at arbitrary epoch, p_n, and at departure epochs, p_n^d, are defined as

$$p_n = \lim_{t\to\infty} P\left\{N(t) = n\right\}, \quad n \geq 0,$$
$$p_n^d = \lim_{k\to\infty} P\left\{N(t_k^d) = n\right\}, \quad n \geq 0.$$

The corresponding steady-state PGFs are defined as

$$\widehat{P}(z) = \lim_{t\to\infty} \sum_{n=0}^{\infty} P\left\{N(t) = n\right\} z^n, \quad |z| \le 1,$$
$$\widehat{P}^d(z) = \lim_{k\to\infty} \sum_{n=0}^{\infty} P\left\{N(t_k^d) = n\right\} z^n, \quad |z| \le 1.$$

We also define the joint transform $\bar{Q}(s,z)$ as

$$\bar{Q}(s,z) = \sum_{n=0}^{\infty} p_n \tilde{B}_n(s) z^n, \quad Re(s) \ge 0 \text{ and } |z| \le 1.$$

Theorem 1. *In the stable $M/G_n/1$ system*

$$\widehat{P}(z) = \frac{\bar{Q}(\lambda - \lambda z, z) - p_0 \tilde{B}_1(\lambda - \lambda z)}{z} + p_0 \tilde{B}_1(\lambda - \lambda z). \tag{1}$$

Proof. The PGF of the number of customers arriving during a service time with pdf $b_n(t)$ is $\tilde{B}_n(\lambda - \lambda z)$. If the number of customers present at the k-th departure epoch is $n \ge 1$ then it decreases by one at the next departure epoch due to the actual customer service and increases by the number of customers arriving during the service time B_n. This can be described on PGF level as multiplication by $\frac{\tilde{B}_n(\lambda-\lambda z)}{z}$. Assuming that the system is idle at the k-th departure epoch, the customers present at the next departure epoch are the ones arriving during the service time B_1. This means a multiplication by $\tilde{B}_1(\lambda - \lambda z)$ on PGF level. Putting it together gives

$$\sum_{n=0}^{\infty} P\left\{N(t_{k+1}^d) = n\right\} z^n = \sum_{n=1}^{\infty} P\left\{N(t_k^d) = n\right\} z^n \frac{\tilde{B}_n(\lambda - \lambda z)}{z} + P\left\{N(t_k^d) = 0\right\} \tilde{B}_1(\lambda - \lambda z).$$

Taking $\lim_{k\to\infty}$ we get

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} P\left\{N(t_{k+1}^d) = n\right\} z^n = \lim_{k\to\infty} \sum_{n=1}^{\infty} P\left\{N(t_k^d) = n\right\} z^n \frac{\tilde{B}_n(\lambda - \lambda z)}{z} + \lim_{k\to\infty} P\left\{N(t_k^d) = 0\right\} \tilde{B}_1(\lambda - \lambda z).$$

Due to stability the limit and sum can be exchanged. Applying it on the first term on the r.h.s and using the notations for the steady-state quantities we get

$$\widehat{P}^d(z) = \sum_{n=1}^{\infty} p_n^d z^n \frac{\tilde{B}_n(\lambda - \lambda z)}{z} + p_0^d \tilde{B}_1(\lambda - \lambda z). \tag{2}$$

In this $M/G_n/1$ system the state of the system can be changed only in unit step and PASTA also holds due to Poisson arrivals. Thus the distribution of the number of customers at departure epochs, at arrival epochs as well as at arbitrary epochs are the same (see e.g. in [7]). Utilizing it we can rearrange (2) as

$$\widehat{P}(z) = \sum_{n=1}^{\infty} p_n z^n \frac{\tilde{B}_n(\lambda - \lambda z)}{z} + p_0 \tilde{B}_1(\lambda - \lambda z). \tag{3}$$

The statement comes by applying the definition of $\bar{Q}(s,z)$ in (3). □

Corollary 1. *In the stable $M/G_n/1$ system, the steady-state probability that the system is in idle state, p_0 can be given as*

$$p_0 = 1 - \lambda \sum_{i=0}^{\infty} p_i b_i. \tag{4}$$

Proof. Expressing p_0 from (1) gives

$$p_0 = \frac{z\widehat{P}(z) - \bar{Q}(\lambda - \lambda z, z)}{(z-1)\tilde{B}_1(\lambda - \lambda z)}. \tag{5}$$

Taking $\lim_{z\to 1}$ on (5) and using the L'Hospital rule we get

$$\begin{aligned} p_0 &= \frac{1 + \widehat{P}^{(1)}(1) - \left.\frac{\partial \bar{Q}(s,z)}{\partial s}\right|_{s=\lambda-\lambda z, z=1} (-\lambda) - \left.\frac{\partial \bar{Q}(s,z)}{\partial z}\right|_{s=\lambda-\lambda z, z=1}}{1} \\ &= 1 + \widehat{P}^{(1)}(1) - \lambda \sum_{i=0}^{\infty} p_i b_i - \widehat{P}^{(1)}(1) = 1 - \lambda \sum_{i=0}^{\infty} p_i b_i. \end{aligned}$$

□

Remark 1. Probability that the queue is busy and the Little's law
Observe that the sum in (4) is exactly the mean service time, i.e. $E[B] = \sum_{i=0}^{\infty} p_i b_i$. Thus the steady-state probability of the system being busy is given as

$$P\{\text{busy}\} = 1 - p_0 = \lambda E[B],$$

which justifies the Little's law for this system due to $E[N_B] = P\{\text{busy}\}$, where N_B is the number of customers in the server.

Remark 2. Utilization and stability
The utilization, ρ of the system is defined as

$$\rho = \lambda E[B],$$

Thus the stability condition of the system can be given on two equivalent ways as

$$\rho < 1 \quad \Leftrightarrow \quad 0 < p_0.$$

3.2 The steady-state distribution of the number of customers

We define the following auxiliary quantities

$$
\begin{aligned}
c_{i,0} &= \tilde{B}_i(\lambda), \quad i \geq 1 \\
c_{i,j} &= \frac{(-\lambda)^j}{j!} \tilde{B}_i^{(j)}(\lambda), \quad i, j \geq 1.
\end{aligned}
$$

Remark 3. Interpretation of the quantities $c_{i,j}$

The quantity $c_{i,j}$, for $i \geq 1$ and $j \geq 0$ can be interpreted as the probability of arriving j customers during the service time B_i. This can be seen as

$$
\begin{aligned}
\frac{(-\lambda)^j}{j!} \tilde{B}_i^{(j)}(\lambda) &= \frac{(-\lambda)^j}{j!} \int_{x=0}^{\infty} (-x)^j e^{-\lambda x} b_i(x) dx \\
&= \int_{x=0}^{\infty} \frac{(\lambda x)^j}{j!} e^{-\lambda x} b_i(x) dx = P\left\{j \text{ arrivals during } B_i\right\}.
\end{aligned}
$$

The above integral provides a way to compute the quantities $c_{i,j}$.

Theorem 2. *In the stable $M/G_n/1$ system the steady-state probabilities of the number of customers are given by*

$$
\begin{aligned}
p_0 &= \frac{1}{\sum_{i=0}^{\infty} \alpha_i}, \\
p_i &= p_0 \alpha_i \quad i \geq 0,
\end{aligned}
\tag{6}
$$

where α_i-s can be determined recursively as

$$
\begin{aligned}
\alpha_0 &= 1, \\
\alpha_1 &= \frac{1 - c_{1,0}}{c_{1,0}}, \\
\alpha_n &= \alpha_{n-1} \frac{1 - c_{n-1,1}}{c_{n,0}} - \sum_{i=1}^{n-2} \alpha_i \frac{c_{i,n-i}}{c_{n,0}} - \frac{c_{1,n-1}}{c_{n,0}} \quad n \geq 2.
\end{aligned}
\tag{7}
$$

Proof. Let v_n be the probability that a stationary departing customer leaves n customers in the system. The $c_{i,j}$ probabilities define the following relation of the v_n probabilities

$$
v_n = \sum_{i=1}^{n+1} v_i c_{i,n-i+1} + v_0 c_{1,n}. \tag{8}
$$

Utilizing that the number of customers in the queue can change by one at a time, a stationary arriving customer finds n customers in the queue with probability v_n. Additionally utilizing the PASTA property, we have $p_n = v_n$ for $n \geq 0$.

Using $p_n = v_n$ and expressing $p_{n+1}c_{n+1,0}$ from (8) gives

$$p_1 c_{1,0} = p_0(1 - c_{1,0})$$
$$p_{n+1}c_{n+1,0} = p_n(1 - c_{n,1}) - \sum_{i=1}^{n-1} p_i c_{i,n-i+1} - p_0 c_{1,n} \quad n \geq 1.$$

Changing the index $n+1 \to n$ leads to the expression of p_n as

$$p_1 = p_0 \frac{1 - c_{1,0}}{c_{1,0}}$$
$$p_n = p_{n-1} \frac{1 - c_{n-1,1}}{c_{n,0}} - \sum_{i=1}^{n-2} p_i \frac{c_{i,n-i}}{c_{n,0}} - p_0 \frac{c_{1,n-1}}{c_{n,0}} \quad n \geq 2. \tag{9}$$

The recursive forms for determining α_i-s are coming from applying $p_i = p_0\alpha_i$ for $i \geq 0$, from (6), in (9). Finally p_0 can be determined from the normalization condition $1 = \sum_{i=0}^{\infty} p_i = p_0 \sum_{i=0}^{\infty} \alpha_i$. □

Corollary 2. *The necessary and sufficient condition of the stability of the $M/G_n/1$ system is given by*

$$0 < \sum_{i=0}^{\infty} \alpha_i < \infty. \tag{10}$$

Proof. It follows directly from the expression of p_0 in (6). □

4 $M/G_n/1$ system with a finite number of different service time distributions

Let $K \geq 0$, such that

$$B_i = B_\infty, \quad i \geq K. \tag{11}$$

and $b_\infty(t)$, $\tilde{B}_\infty(s)$, b_∞ and $b_\infty^{(2)}$ stands for the related pdf, LT, first and second moment, respectively. That is, when the number of customers in the system is above K the service time pdf is always $b_\infty(t)$. This modelling restrictions allows to derive closed form expressions for the PGF of the steady-state number of customers and the mean steady-state number of customers.

4.1 The PGF of the steady-state number of customers

Proposition 1. *In the stable $M/G_n/1$ system with finite number of state dependent service times, the steady-state probability that the system is idle is*

$$p_0 = 1 - \lambda \left(b_\infty + \sum_{i=0}^{K-1} p_i (b_i - b_\infty) \right). \tag{12}$$

Proof. For the system with finite number of state dependent service times, the sum $\sum_{i=0}^{\infty} p_i b_i$ can be rewritten as

$$\sum_{i=0}^{\infty} p_i b_i = \sum_{i=0}^{K-1} p_i b_i + \sum_{i=K}^{\infty} p_i b_\infty = \sum_{i=0}^{K-1} p_i b_i + (1 - \sum_{i=0}^{K-1} p_i) b_\infty$$
$$= b_\infty + \sum_{i=K}^{\infty} p_i (b_i - b_\infty). \tag{13}$$

Applying (13) in (4) gives the statement. □

Let $d_0 = 1 + \lambda(b_1 - b_\infty)$ and for $i = 1, \ldots, K-1$ let $d_i = \lambda(b_i - b_\infty)$.

Theorem 3. *In the stable $M/G_n/1$ system with finite number of state dependent service times, the PGF of the steady-state number of customers in the system is*

$$\widehat{P}(z) = p_0 \frac{(1-z)\tilde{B}_1(\lambda - \lambda z)}{\tilde{B}_\infty(\lambda - \lambda z) - z} + \frac{\sum_{i=0}^{K-1} p_i (\tilde{B}_\infty(\lambda - \lambda z) - \tilde{B}_i(\lambda - \lambda z)) z^i}{\tilde{B}_\infty(\lambda - \lambda z) - z}, \tag{14}$$

and the steady-state probabilities can be expressed as

$$(p_0, \ldots, p_{K-2}, p_{K-1}) = (0, \ldots, 0, 1 - \lambda b_\infty)\mathbf{M}^{-1}, \tag{15}$$

where the coefficient matrix $\mathbf{M}$ *is given by*

$$\mathbf{M} = \begin{pmatrix} c_{1,0} - 1 & c_{1,1} & c_{1,2} & \ldots & \ldots & c_{1,K-2} & d_0 \\ c_{1,0} & c_{1,1} - 1 & c_{1,2} & \ldots & \ldots & c_{1,K-2} & d_1 \\ & c_{2,0} & c_{2,1} - 1 & \ldots & \ldots & c_{2,K-3} & d_2 \\ & & c_{3,0} & \ddots & \ldots & c_{3,K-4} & d_3 \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & c_{K-2,0} & c_{K-2,1} - 1 & d_{K-2} \\ & & & & & c_{K-1,0} & d_{K-1} \end{pmatrix}.$$

Proof. For the system with finite number of state dependent service times, the joint transform $\bar{Q}(s,z)$ can be rearranged as

$$\bar{Q}(s,z) = \sum_{i=0}^{\infty} p_i \tilde{B}_i(s) z^i = \sum_{i=0}^{K-1} p_i \tilde{B}_i(s) z^i + \sum_{i=K}^{\infty} p_i \tilde{B}_\infty(s) z^i$$
$$= \sum_{i=0}^{K-1} p_i \tilde{B}_i(s) z^i + \tilde{B}_\infty(s) \left(\widehat{P}(z) - \sum_{i=0}^{K-1} p_i z^i \right)$$
$$= \tilde{B}_\infty(s) \widehat{P}(z) + \sum_{i=0}^{K-1} p_i \left(\tilde{B}_i(s) - \tilde{B}_\infty(s) \right) z^i. \tag{16}$$

Applying (16) in (1) and rearranging it gives

$$\left(z - \tilde{B}_\infty(\lambda - \lambda z) \right) \widehat{P}(z) = p_0 (z-1) \tilde{B}_1(\lambda - \lambda z)$$
$$+ \sum_{i=0}^{K-1} p_i \left(\tilde{B}_n(\lambda - \lambda z) - \tilde{B}_\infty(\lambda - \lambda z) \right) z^i. \tag{17}$$

Further rearranging of (17) results in the expression (14).

The relation (12) can be rearranged as

$$p_0 + \lambda \left(\sum_{i=0}^{K-1} p_i (b_i - b_\infty) \right) = 1 - \lambda b_\infty. \tag{18}$$

This provides a linear equation for $p_0, \ldots, p_{K-1}$, which is represented by the last column of matrix $\mathbf{M}$. The linear relations in Equation (8) for $n = 0, \ldots, K-2$ are represented by the first $K-1$ columns of matrix $\mathbf{M}$. □

Equation (15) provides an explicit expression for $(p_0, \ldots, p_{K-2}, p_{K-1})$, which can be evaluated efficiently utilizing the quasi-triangular structure of matrix $\mathbf{M}$.

4.2 The mean steady-state number of customers

Corollary 3. *In the stable $M/G_n/1$ system with finite number of state dependent service times, the mean steady-state number of customers in the system, $p^{(1)}$, is given as*

$$\begin{aligned} p^{(1)} = \frac{\lambda^2 b_\infty^{(2)}}{2(1-\lambda b_\infty)} + \frac{(2\lambda b_1 + \lambda^2 (b_1^{(2)} - b_\infty^{(2)})) p_0}{2(1-\lambda b_\infty)} \\ + \frac{\sum_{i=1}^{K-1} p_i (\lambda^2 b_i^{(2)} - \lambda^2 b_\infty^{(2)}) + 2 \sum_{i=1}^{K-1} i \; p_i \lambda (b_i - b_\infty)}{2(1-\lambda b_\infty)}. \end{aligned} \tag{19}$$

Proof. The expression (14) can be rearranged as

$$\begin{aligned} \left(\tilde{B}_\infty(\lambda - \lambda z) - z \right) \widehat{P}(z) = p_0 (1-z) \tilde{B}_1(\lambda - \lambda z) \\ + \sum_{i=0}^{K-1} p_i (\tilde{B}_\infty(\lambda - \lambda z) - \tilde{B}_i(\lambda - \lambda z)) z^i. \end{aligned} \tag{20}$$

Taking the second derivative of (20) with respect to z and setting $z = 1$ gives

$$\begin{aligned} \lambda^2 b_\infty^{(2)} + 2(\lambda b_\infty - 1) p^{(1)} = -p_0 2 \lambda b_1 \\ + \sum_{i=0}^{K-1} p_i \left((\lambda^2 b_\infty^{(2)} - \lambda^2 b_i^{(2)}) + 2\lambda (b_\infty - b_i) i \right). \end{aligned}$$

This can be rearranged as

$$\begin{aligned} \lambda^2 b_\infty^{(2)} + 2(\lambda b_\infty - 1) p^{(1)} = -(2\lambda b_1 + \lambda^2 (b_1^{(2)} - b_\infty^{(2)})) p_0 \\ + \sum_{i=1}^{K-1} p_i \left((\lambda^2 b_\infty^{(2)} - \lambda^2 b_i^{(2)}) + 2\lambda (b_\infty - b_i) i \right), \end{aligned}$$

from which the statement comes by expressing $p^{(1)}$. □

5 $M/G_n/1$ system with state dependent linear interpolation of two service times

In this section, we consider the special case when the state dependent service time is

$$B_n = (1 - Y_n)B_f + Y_n B_s, \tag{21}$$

and Y_n is a Bernoulli distributed random variable with $P\{Y_n = 1\} = \eta(1-\delta^{n-1})$, with $0 \leq \eta, \delta \leq 1$. That is, the state dependent service time is characterized by two service times B_f and B_s. Parameter η determines the portion of the first service time, B_f, in B_n, while parameter δ controls the dependence on the number of customers in the system. Qualitatively, for small n values the service time is B_f with high probability and for large n values it is B_s with high probability.

In Laplace transform domain

$$\begin{aligned}\tilde{B}_n(s) &= \left((1-\eta) + \eta\delta^{n-1}\right)\tilde{B}_f(s) + \eta(1-\delta^{n-1})\tilde{B}_s(s) \\ &= (1-\eta)\tilde{B}_f(s) + \eta\left(\delta^{n-1}\tilde{B}_f(s) + (1-\delta^{n-1})\tilde{B}_s(s)\right),\end{aligned} \tag{22}$$

where $\tilde{B}_f(s)$ and $\tilde{B}_s(s)$ are the LT of B_f and B_s, respectively, and $\tilde{B}_0(s) = \tilde{B}_1(s) = \tilde{B}_f(s)$ by definition.

This model enables also that the second service time is higher than the first one until keeping the queue stable. However in the usual application scenarios of this model, where $B_s < B_f$ (meaning that $P\{B_s < t\} \geq P\{B_f < t\}$ for $\forall t > 0$), the second service time is implemented in order to realize a kind of control of number of customers in the system. This control mechanism makes the number of customers in the system and the waiting time lower.

5.1 The PGF of the steady-state number of customers in the system

Theorem 4. *In the stable $M/G_n/1$ system with state dependent linear interpolation of two service times, $\widehat{P}(z)$ satisfies*

$$\widehat{P}(z) = \frac{\beta(\lambda-\lambda z)}{\alpha(\lambda-\lambda z)-z}\widehat{P}(\delta z) + \frac{\alpha(\lambda-\lambda z)-\beta(\lambda-\lambda z)-z\tilde{B}_f(\lambda-\lambda z)}{\alpha(\lambda-\lambda z)-z}p_0, \tag{23}$$

where $\alpha(s) = (1-\eta)\tilde{B}_f(s) + \eta\tilde{B}_s(s)$ and $\beta(s) = \frac{\eta}{\delta}\left(\tilde{B}_s(s) - \tilde{B}_f(s)\right)$.

Proof. For this system the joint transform $\bar{Q}(s,z)$ can be written as

$$\begin{aligned}
\bar{Q}(s,z) &= \sum_{n=0}^{\infty} p_n \tilde{B}_n(s) z^n = p_0 \tilde{B}_0(s) + \sum_{n=1}^{\infty} p_n \tilde{B}_n(s) z^n \\
&= p_0 \tilde{B}_f(s) + \sum_{n=1}^{\infty} p_n \Big((1-\eta)\tilde{B}_f(s) + \eta \left(\delta^{n-1}\tilde{B}_f(s) + (1-\delta^{n-1})\tilde{B}_s(s) \right) \Big) z^n \\
&= \tilde{B}_f(s) p_0 + \left((1-\eta)\tilde{B}_f(s) + \eta \tilde{B}_s(s) \right) \sum_{n=1}^{\infty} p_n z^n \\
&+ \eta \left(\tilde{B}_f(s) - \tilde{B}_s(s) \right) \sum_{n=1}^{\infty} p_n \delta^{n-1} z^n \\
&= \tilde{B}_f(s) p_0 + \left((1-\eta)\tilde{B}_f(s) + \eta \tilde{B}_s(s) \right) \left(\widehat{P}(z) - p_0 \right) \\
&+ \frac{\eta}{\delta} \left(\tilde{B}_f(s) - \tilde{B}_s(s) \right) \left(\widehat{P}(\delta z) - p_0 \right).
\end{aligned}$$

Further rearrangement gives

$$\begin{aligned}
\bar{Q}(s,z) = &\left((1-\eta)\tilde{B}_f(s) + \eta \tilde{B}_s(s) \right) \widehat{P}(z) - \frac{\eta}{\delta} \left(\tilde{B}_s(s) - \tilde{B}_f(s) \right) \widehat{P}(\delta z) \\
&- \Big(\left((1-\eta)\tilde{B}_f(s) + \eta \tilde{B}_s(s) \right) - \frac{\eta}{\delta} \left(\tilde{B}_s(s) - \tilde{B}_f(s) \right) - \tilde{B}_f(s) \Big) p_0.
\end{aligned}$$

This can be written by means of the functions $\alpha(s)$ and $\beta(s)$ as

$$\bar{Q}(s,z) = \alpha(s)\widehat{P}(z) - \beta(s)\widehat{P}(\delta z) - \left(\alpha(s) - \beta(s) - \tilde{B}_f(s) \right) p_0. \tag{24}$$

Applying (24) in (1) and rearranging it gives

$$\begin{aligned}
\left(z - \alpha(\lambda - \lambda z) \right) \widehat{P}(z) = &-\beta(\lambda - \lambda z)\widehat{P}(\delta z) \\
&- \left(\alpha(\lambda - \lambda z) - \beta(\lambda - \lambda z) - z\tilde{B}_f(\lambda - \lambda z) \right) p_0.
\end{aligned} \tag{25}$$

The statement comes by further rearranging of (25). □

Theorem 5. *In the stable $M/G_n/1$ system with state dependent linear interpolation of two service times, the PGF of the steady-state number of customers in the system, $\widehat{P}(z)$, is given as*

$$\begin{aligned}
\widehat{P}(z) = &\prod_{k=0}^{\infty} \frac{\beta(\lambda - \lambda\delta^k z)}{\alpha(\lambda - \lambda\delta^k z) - \delta^k z} p_0 \\
&+ \sum_{k=0}^{\infty} \frac{\alpha(\lambda - \lambda\delta^k z) - \beta(\lambda - \lambda\delta^k z) - \delta^k z \tilde{B}_f(\lambda - \lambda\delta^k z)}{\alpha(\lambda - \lambda\delta^k z) - \delta^k z} \\
&\times \prod_{i=0}^{k-1} \frac{\beta(\lambda - \lambda\delta^i z)}{\alpha(\lambda - \lambda\delta^i z) - \delta^i z} p_0,
\end{aligned} \tag{26}$$

where p_0 is given by

$$p_0 = \left(\prod_{k=0}^{\infty} \frac{\beta(\lambda - \lambda\delta^k)}{\alpha(\lambda - \lambda\delta^k) - \delta^k} + \sum_{k=0}^{\infty} \frac{\alpha(\lambda - \lambda\delta^k) - \beta(\lambda - \lambda\delta^k) - \delta^k \tilde{B}_f(\lambda - \lambda\delta^k)}{\alpha(\lambda - \lambda\delta^k) - \delta^k} \prod_{i=0}^{k-1} \frac{\beta(\lambda - \lambda\delta^i)}{\alpha(\lambda - \lambda\delta^i) - \delta^i} \right)^{-1}. \tag{27}$$

Proof. Replacing z by $\delta^k z$ in (23) for $k \geq 0$ yields

$$\widehat{P}(\delta^k z) = \frac{\beta(\lambda - \lambda\delta^k z)}{\alpha(\lambda - \lambda\delta^k z) - \delta^k z} \widehat{P}(\delta^{k+1} z) + \frac{\alpha(\lambda - \lambda\delta^k z) - \beta(\lambda - \lambda\delta^k z) - \delta^k z \tilde{B}_f(\lambda - \lambda\delta^k z)}{\alpha(\lambda - \lambda\delta^k z) - \delta^k z} p_0.$$

Solving the above equation by recursive substitution for $k \geq 0$ leads to

$$\widehat{P}(z) = \prod_{k=0}^{\infty} \frac{\beta(\lambda - \lambda\delta^k z)}{\alpha(\lambda - \lambda\delta^k z) - \delta^k z} \lim_{k\to\infty} \widehat{P}(\delta^k z) + \sum_{k=0}^{\infty} \frac{\alpha(\lambda - \lambda\delta^k z) - \beta(\lambda - \lambda\delta^k z) - \delta^k z \tilde{B}_f(\lambda - \lambda\delta^k z)}{\alpha(\lambda - \lambda\delta^k z) - \delta^k z} \times \prod_{i=0}^{k-1} \frac{\beta(\lambda - \lambda\delta^i z)}{\alpha(\lambda - \lambda\delta^i z) - \delta^i z} p_0. \tag{28}$$

Due to $\delta < 1$

$$\lim_{k\to\infty} \widehat{P}(\delta^k z) = \widehat{P}(0) = p_0.$$

Applying this limit in (28) gives the first part of the statement, the relation (26). The second relation, (27) comes by setting $z = 1$ in (26) and expressing p_0 from it. □

Remark 4. Numerical complexity
The computation of $\widehat{P}(z)$ by means of (26) and the steady-state moments of the number of customers in the system requires $\mathcal{O}(K)$ operational steps, where K is the highest index in the infinite products and sums to be taken into account to get the PGF and the moments in required accuracy. This is because these computations require the computation of K points of the LTs $\tilde{B}_f(s)$, $\tilde{B}_s(s)$ and their derivatives.

5.2 Stability

Proposition 2. *The necessary and sufficient condition of the stability of $M/G_n/1$ system with state dependent linear interpolation of two service times is*

$$\prod_{k=0}^{\infty} \frac{\beta(\lambda-\lambda\delta^k)}{\alpha(\lambda-\lambda\delta^k)-\delta^k} + \sum_{k=0}^{\infty} \frac{\alpha(\lambda-\lambda\delta^k)-\beta(\lambda-\lambda\delta^k)-\delta^k\tilde{B}_f(\lambda-\lambda\delta^k)}{\alpha(\lambda-\lambda\delta^k)-\delta^k} \prod_{i=0}^{k-1} \frac{\beta(\lambda-\lambda\delta^i)}{\alpha(\lambda-\lambda\delta^i)-\delta^i} < \infty. \tag{29}$$

Proof. The necessary and sufficient condition of the stability is $p_0 > 0$, which is equivalent with the denominator of (27) being convergent. Thus this statement is a direct consequence of the expression (27). □

Corollary 4. *A sufficient condition of the stability of $M/G_n/1$ system with state dependent linear interpolation of two service times is*

$$\begin{aligned} \eta\left(\frac{1}{\delta}-1\right)\left(\tilde{B}_s(\lambda)-\tilde{B}_f(\lambda)\right) &< \tilde{B}_f(\lambda), \quad \textit{if} \;\; \tilde{B}_s(\lambda) \geq \tilde{B}_f(\lambda), \\ \eta\left(\frac{1}{\delta}+1\right)\left(\tilde{B}_f(\lambda)-\tilde{B}_s(\lambda)\right) &< \tilde{B}_f(\lambda), \quad \textit{if} \;\; \tilde{B}_s(\lambda) < \tilde{B}_f(\lambda). \end{aligned} \tag{30}$$

Proof. We evaluate the convergence of the denominator of (27) under the condition

$$\left|\frac{\beta(\lambda)}{\alpha(\lambda)}\right| < 1. \tag{31}$$

Under this condition there exists an enough large K_1 for which $|\frac{\beta(\lambda-\lambda\delta^k)}{(\alpha(\lambda-\lambda\delta^k)-\delta^k)}| = r < 1$ for every $k \geq K_1$. Hence the first product term in (27) must vanish, in other words

$$\prod_{k=0}^{\infty} \frac{\beta(\lambda-\lambda\delta^k)}{\alpha(\lambda-\lambda\delta^k)-\delta^k} = 0. \tag{32}$$

The expression after the sum in the second term in the denominator of (27) can be upper limited for enough large k as follows. According to the above argument the fraction $-\frac{\beta(\lambda-\lambda\delta^k)}{\alpha(\lambda-\lambda\delta^k)-\delta^k} < 1$ for any $k \geq K_1$. The functions $\tilde{B}_f(\lambda-\lambda z)$ and $\alpha(\lambda-\lambda z)$ are PGFs, since both $\tilde{B}_f(s)$ and $\alpha(s)$ are LTs of continuous random variables representing a durations and hence the above functions can be interpreted as the PGF of the number of arriving customers during these random durations. PGFs have positiv values for $0 \leq z \leq 1$ and their value at z close to 0 are greater than z as far as the number of zero arrival in their above interpretations has positive probability. Thus there exists a K_2 for which for any $k \geq K_2$ $z = \delta^k$ is close enough to 0 to have $\frac{\tilde{B}_f(\lambda-\lambda\delta^k)}{\alpha(\lambda-\lambda\delta^k)-\delta^k} > 0$. The fraction $\frac{\alpha(\lambda-\lambda\delta^k)}{\alpha(\lambda-\lambda\delta^k)-\delta^k}$ is upper limited by $\frac{\alpha(\lambda-\lambda\delta^K)}{\alpha(\lambda-\lambda\delta^K)-\delta^K}$ for any $k \geq K$, where $K = max(K_1, K_2)$. Putting all these together

$$\frac{\alpha(\lambda-\lambda\delta^k) - \beta(\lambda-\lambda\delta^k) - \delta^k\tilde{B}_f(\lambda-\lambda\delta^k)}{\alpha(\lambda-\lambda\delta^k)-\delta^k} \leq \frac{\alpha(\lambda-\lambda\delta^K)}{\alpha(\lambda-\lambda\delta^K)-\delta^K} + 1 = U.$$

for any $k \geq K$.

This upper limit ensures that the infinite tail of the second term in the denominator of (27) is convergent, which can be shown as

$$\begin{aligned}
&\sum_{k=K}^{\infty} \frac{\alpha(\lambda-\lambda\delta^k) - \beta(\lambda-\lambda\delta^k) - \delta^k\tilde{B}_f(\lambda-\lambda\delta^k)}{\alpha(\lambda-\lambda\delta^k)-\delta^k} \prod_{i=0}^{k-1} \frac{\beta(\lambda-\lambda\delta^i)}{\alpha(\lambda-\lambda\delta^i)-\delta^i} \\
&\leq \sum_{k=K}^{\infty} U \prod_{i=0}^{k-1} \frac{\beta(\lambda-\lambda\delta^i)}{\alpha(\lambda-\lambda\delta^i)-\delta^i} \\
&= U \prod_{j=0}^{K_1-1} \frac{\beta(\lambda-\lambda\delta^j)}{\alpha(\lambda-\lambda\delta^j)-\delta^j} \sum_{k=K}^{\infty} \prod_{i=K_1}^{k-1} \frac{\beta(\lambda-\lambda\delta^i)}{\alpha(\lambda-\lambda\delta^i)-\delta^i} \\
&\leq U \prod_{j=0}^{K_1-1} \frac{\beta(\lambda-\lambda\delta^j)}{\alpha(\lambda-\lambda\delta^j)-\delta^j} \sum_{k-K_1=K-K_1}^{\infty} (r)^{k-K_1} < \infty. \qquad (33)
\end{aligned}$$

It follows from (32) and (33) that the condition (31) ensures the convergence of the denominator of (27), and hence it is sufficient for the stability.

Applying the expressions of the functions $\alpha(s)$ and $\beta(s)$ in (31) gives

$$\begin{aligned}
&\frac{\frac{\eta}{\delta}\left(\tilde{B}_s(\lambda) - \tilde{B}_f(\lambda)\right)}{(1-\eta)\tilde{B}_f(\lambda) + \eta\tilde{B}_s(\lambda)} < 1, \quad \text{if } \tilde{B}_s(\lambda) \geq \tilde{B}_f(\lambda), \\
&\frac{\frac{\eta}{\delta}\left(\tilde{B}_f(\lambda) - \tilde{B}_s(\lambda)\right)}{(1-\eta)\tilde{B}_f(\lambda) + \eta\tilde{B}_s(\lambda)} < 1, \quad \text{if } \tilde{B}_s(\lambda) < \tilde{B}_f(\lambda). \qquad (34)
\end{aligned}$$

The relations (34) can be rearranged as

$$\begin{aligned}
&\eta(\frac{1}{\delta} - 1)\tilde{B}_s(\lambda) < \tilde{B}_f(\lambda) + \eta(\frac{1}{\delta} - 1)\tilde{B}_f(\lambda), \quad \text{if } \tilde{B}_s(\lambda) \geq \tilde{B}_f(\lambda), \qquad (35) \\
&\eta(\frac{1}{\delta} + 1)\tilde{B}_f(\lambda) < \tilde{B}_f(\lambda) + \eta(\frac{1}{\delta} + 1)\tilde{B}_s(\lambda), \quad \text{if } \tilde{B}_s(\lambda) < \tilde{B}_f(\lambda).
\end{aligned}$$

The final form of the condition comes by rearranging (35). □

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