A SIMPLIFIED MOMENT-BASED ESTIMATION METHOD FOR EXTREME PROBABILITIES, INFINITE AND POSITIVE CASES*

ÁRPÁD TARI[1], MIKLÓS TELEK[2] and PETER BUCHHOLZ[1]

[1] *Universität Dortmund, Informatik IV*
August-Schmidt-Str. 12, Dortmund, Germany
[2] *Budapest University of Technology and Economics, Department of Telecommunications*
Magyar tudósok körútja 2., Budapest, Hungary
{*arpi,telek*}*@webspn.hit.bme.hu; peter.buchholz@cs.uni-dortmund.de*

Abstract:
The performance analysis of highly reliable and fault tolerant systems requires the investigation of events with extremely low or high probabilities. This paper presents a simplified numerical method to bound the extreme probabilities based on the moments of the distribution. This simplified method eliminates some numerically sensitive steps of the general moments based bounding procedure. Numerical examples indicate the applicability of the proposed approach.
Keywords*: reduced moment problem, moments based distribution bounding, tail distribution*

1 INTRODUCTION

Performance analysis of real-life systems usually requires the evaluation of the distribution of some random variables. The direct analysis of these distributions is often infeasible due to the high computational complexity. A possible way to overcome this difficulty is to simplify the model or to calculate only an estimate of the measure of interest. Both types of simplification result in inaccuracies in calculation, but this is the price of the solvability.

In this paper we investigate the second option, the estimation of the measure of interest based on a set of its moments. There are several classes of performance analysis problems for which the analysis of the moments of a random variable is far less complex than the analysis of the distribution. For example, for the class of Markov reward models the moments of the reward measures can be computed by the effective methods presented in [Rácz and Telek, 1999, Telek and Rácz, 1999], while the direct analysis of the distribution of these measures based on [Nabli and Sericola, 1996, de Souza e Silva and Gail, 1998] and [Donatiello and Grassi, 1991] is far more complex and practically infeasible for models with more than 10^4 states [Horváth et al, 2004]. In these cases moments based estimation of the distribution is the only feasible solution method for large models. There are two ways of moments based estimation: to fit a certain class of distribution functions to the set of moments (e.g. [van de Liefvoort, 1990] presents a method for fitting with matrix exponential distribution); and to calculate maximal and minimal values for the distribution among all possible distributions having the prescribed set of moments. The first approach results in an unknown error if the performance measure does not belong to the considered class of distributions. To bound the error of moments based distribution approximation we apply the second approach.

The performance analysis of highly reliable or safety critical fault-tolerant systems requires the analysis very unlikely events, i.e., the distribution of a random variable at very low (close to 0) or very high (close to 1) probabilities.

In the paper we focus on the analysis of these kinds

*A previous version of this paper was presented at the Eighteenth Annual UK Performance Engineering Workshop (UKPEW' 2005). This work is supported by DAAD and MÖB under the German-Hungarian R&D grant.

of extreme values and provide a simplified moments based estimation analysis algorithm with respect to the one that calculates lower and upper bounds for the distribution function based on a set of moments in the general case. The modified algorithm is numerically stable, simple and fast.

The paper is organized as follows: Section 2 introduces the previous works done on the subject. Sections 3 and 4 summarize the moments based estimation methods. The numerical procedures involved in the solution are summarized in Section 5 and some useful expressions are deduced in Section 6. An example is analyzed in Section 7. Section 8 concludes the paper.

2 RELATED WORKS

Determining a distribution function based on its moments is called the *reduced moment problem* (where reduced refers to the finite number of moments). This is a well-known problem for more than 100 years and has an extensive literature. A good overview is given in [Shohat and Tamarkin, 1946].

We denote the i^{th} moment of a distribution function $\sigma(x)$ supported in the interval $[a, b]$ by

$$\mu_i = \int_a^b x^i \, \mathrm{d}\sigma(x), \qquad i = 0, 1, 2, \ldots, m \ . \quad (1)$$

According to the value of a and b the following three main cases exist:

1. **infinite** (Hamburger moment problem): $a = -\infty, b = \infty$;
2. **positive**: a is finite, $b = \infty$ (special case: Stieltjes moment problem: $a = 0, b = \infty$);
3. **bounded**: both a and b are finite (special case: Hausdorff moment problem: $a = 0, b = 1$).

The considered task can be formalized as follows. Find the smallest and largest values, that any distribution function $\sigma(x)$ with $\mu_0, \mu_1, \ldots, \mu_m$ moments may have at a given point C, i.e:

$$L = \min \left\{ \sigma(C) : \mu_i = \int_{-\infty}^{\infty} x^i \mathrm{d}\sigma(x), i = 0, \ldots, m \right\},$$
$$U = \max \left\{ \sigma(C) : \mu_i = \int_{-\infty}^{\infty} x^i \mathrm{d}\sigma(x), i = 0, \ldots, m \right\}.$$

In 1894 Stieltjes proposed and solved completely the moment problem for distribution functions over $[0, \infty)$ (which has been named after him) in [Stieltjes, 1894] using continued fractions, however a special case of the problem was solved earlier in [Chebyshev, 1874]. After 20 years of inactivity Hamburger [Hamburger, 1920] solved Stieltjes' problem over the whole real axis in 1921 and a year later Hausdorff [Hausdorff, 1921] gave the answer for distributions over the $[0, 1]$ interval both of them achieved their results using continued fractions. The Hamburger moment problem was solved in [Nevanlinna, 1922] with the modern theory of functions in 1922 and in [Riesz, 1922] by quasi-orthogonal polynomials. [Carleman, 1922] showed the connection of the moment problem with the theories of quasi-analytic functions and of quadratic forms in infinitely many variables and gave the most general criterion the moment problem to be determined. All of these contributors considered infinitely many moments. Akhiezer and Krein generalized the work of Markov, used quadratic forms and considered finitely many given moments. They are authors of several books and papers on the subject, for example [Akhiezer, 1965, Krein and Nudelman, 1977, Akhiezer and Krein, 1968].

In 2000 in [Rácz, 2000] a notion was proposed to estimate distributions based on their moments using matrices. Founded on this [Tari et al, 2005] introduced an algorithm to bound distributions over the whole real axis and later [Tari, 2005] gave procedures for the positive and bounded cases, too.

We consider infinite and positive cases in this paper. The methods discussed here are based on [Tari et al, 2005] and [Tari, 2005] and first we briefly present the original algorithms as they are the basis of our investigation. However due to the special problem treated here, they can be considerably simplified, and we show how and under what circumstances this can be done.

3 THE INFINITE CASE

The main idea behind both algorithms is a theorem of Markov and Chebyshev [Krein and Nudelman, 1977, §IV.3], which states that the values L and U are originated from a discrete distribution (we often call it *discrete reference distribution*) that includes the point of interest C.

Before calculating L and U, we need to check whether the series of moments $\mu_0, \mu_1, \ldots, \mu_m$ can belong to a valid distribution function. If the moments are calculated by complex computational methods the resulted moments can accumulate the numerical errors of the preceding calculations, thus it is always recommended to check if the obtained moment sequence is valid in advance of the calculation. This can be verified through the following inequalities:

$$|\boldsymbol{M}_k| \geq 0, \qquad k = 0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor, \tag{2}$$

where

$$\boldsymbol{M}_k = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_k \\ \mu_1 & \mu_2 & \cdots & \mu_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_k & \mu_{k+1} & \cdots & \mu_{2k} \end{pmatrix} . \tag{3}$$

The maximum number of moments that satisfy (2) is denoted by $2n+1$, i.e. the considered moments are $\mu_0, \mu_1, \ldots, \mu_{2n}$, and the matrix of largest order satisfying (2) is denoted by $\boldsymbol{M} := \boldsymbol{M}_n$.

We introduce the following set of orthogonal polynomials:

$$\mathcal{M}_0(x) = 1 , \tag{4}$$

$$\mathcal{M}_i(x) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_i \\ \mu_1 & \mu_2 & \cdots & \mu_{i+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{i-1} & \mu_i & \cdots & \mu_{2i-1} \\ 1 & x & \cdots & x^i \end{vmatrix} , \quad i = 1, 2, \ldots, \lfloor (m+1)/2 \rfloor , \tag{5}$$

Let the roots of $\mathcal{M}_n(x)$ be denoted by $u_1 < u_2 < \ldots < u_n$ in increasing order. These roots are also real and simple [Szegö, 1939].

If $C = u_i$ for some i then the discrete distribution consists only of n points (including C) and we have to take $\boldsymbol{M} := \boldsymbol{M}_{n-1}$ [Shohat and Tamarkin, 1946, p. 42], so this must be checked before the calculations.

Odd number of moments is needed to form the matrices $\boldsymbol{M}_k$. As a consequence if even number of moments is given, the last one (μ_{2n+1}) does not carry further information about the distribution, so this can be ignored.

The discrete reference distribution has n points (excluding C), they are $x_1 < x_2 < \ldots < x_n$ in increasing order. Their weights are $p_1, p_2, \ldots, p_n$, respectively.

The maximal probability mass that can be concentrated at C is denoted by p and calculated by [Akhiezer, 1965]:

$$p = \frac{1}{\boldsymbol{c}^T \, \boldsymbol{M}^{-1} \, \boldsymbol{c}}, \tag{6}$$

where

$$\boldsymbol{c}^T = \left(1, C, C^2, \ldots, C^n\right)^T . \tag{7}$$

Furthermore, the difference between any two distribution functions with moments $\mu_0, \mu_1, \ldots, \mu_{2n}$ is not larger than p at C [Akhiezer, 1965]. Note that the formula for p contains the inverse of $\boldsymbol{M}$, a symmetric and positive definite (due to (2)) matrix. The computation of the inverse of symmetric positive definite matrices is numerically more stable than the inversion of general matrices.

The other points x_i of the discrete distribution are the roots of the following polynomial:

$$\mathcal{K}_n^{\mathcal{M}}(x, C) = \boldsymbol{c}^T \, \boldsymbol{M}^{-1} \, \boldsymbol{x}, \tag{8}$$

where $\boldsymbol{x} = \left(1, x, x^2, \ldots, x^n\right)^T$.

This is an order n polynomial and based on the theory of orthogonal polynomials [Szegö, 1939] its roots are all real and distinct. The corresponding probability masses are

$$p_i = \frac{1}{\boldsymbol{x_i}^T \, \boldsymbol{M}^{-1} \, \boldsymbol{x_i}}, \qquad i = 1, 2, \ldots, n, \tag{9}$$

where $\boldsymbol{x_i} = \left(1, x_i, x_i^2, \ldots, x_i^n\right)^T$.

The lower limit of the distribution is obtained as the sum of the weights of the points smaller than C. The upper limit is the sum of the lower limit and the maximum mass at C:

$$L = \sum_{i:\, x_i < C} p_i, \qquad U = L + p . \tag{10}$$

This discrete distribution is extreme in that sense, that no other distribution function with moments μ_i has either a lower or higher value at C than L and U, respectively.

The algorithm can be simplified using the following interesting property of the points x_i. These points depend on C, but their locations can be characterized by a series independent of C. The $x_1 < x_2 < \ldots < x_{j-1} < C < x_j < \ldots < x_n$ and the $u_1 < u_2 < \ldots < u_n$ roots (see (5)) are mutually separated as

$$\begin{aligned} x_1 < u_1 < x_2 < u_2 < \ldots < u_{j-1} < C < \\ < u_j < x_j < u_{j+1} < \ldots < u_n < x_n . \end{aligned} \tag{11}$$

Input

- $\mu_0, \mu_1, \ldots, \mu_m$;
- a set of C values where we need to bound the distribution.

Algorithm

1. Test if the moments satisfy the

$$|\boldsymbol{M}_k| \geq 0 \qquad k = 0, 1, \ldots, \lfloor m/2 \rfloor \qquad (14)$$

inequalities. We denote the number of applicable moments (for which the (14) inequalities hold) by $2n+1$ ($\mu_0, \ldots, \mu_{2n}$).

2. Find the roots of the polynomial $\mathcal{M}_n(x)$: They are denoted by $u_1 < u_2 < \ldots < u_n$.
3. Do for each $C < u_1$ or $C > u_n$ point of interest
   (a) Calculate the largest possible p:

$$p = \frac{1}{\boldsymbol{c}^T \, \boldsymbol{M}_n^{-1} \, \boldsymbol{c}}, \qquad (15)$$

   (b) If $C < u_1$, then $L = 0, \ U = p$.
   If $C > u_n$, then $L = 1 - p, \ U = 1$.

Figure 1. Steps of the simplified algorithm if the support interval is $(-\infty, \infty)$

The number of points x_i which are smaller (greater) than C equals the number of points u_i that are smaller (greater) than C. As a consequence the roots $u_1, u_2, \ldots, u_n$ define the number of terms considered in (10). Therefore it is sufficient to calculate only the roots x_i smaller than C (or alternatively the roots x_i greater than C). If $C < u_1$ or $C > u_n$ we do not need to calculate the points of the discrete distribution, because in these cases the lower and upper limits are determined by p as follows:

$$L = 0, \qquad U = p, \qquad \text{if } C < u_1, \qquad (12)$$

$$L = 1 - p, \qquad U = 1, \qquad \text{if } C > u_n \ . \qquad (13)$$

We use these simple relations to bound the probability of extreme events and this type of estimation is called the *simplified* case. The numerical procedure is summarized in Figure 1. We estimate the distribution in a single point. Estimation in an interval is only possible with a series of applications of the algorithm in points of the interval, but this can be done effectively repeating only parts of the algorithm (we do not have to invert the matrix $\boldsymbol{M}_n$ again).

4 THE POSITIVE CASE

The solution of the moment problem in the positive case is detailed in [Tari, 2005, §3], we discuss here the main results only. The task is to find a discrete distribution with certain number of points and in many ways the algorithm is similar to the procedure of the infinite case, though it is more complex.

The necessary and sufficient conditions that the series of numbers $\mu_0, \mu_1, \ldots, \mu_m$ define a distribution supported over the $[a, \infty)$ semi-axis are

$$|\boldsymbol{M}_k| \geq 0, \ k = 0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor, \qquad (16)$$

$$|\boldsymbol{N}_k - a\boldsymbol{M}_k| \geq 0, \ k = 0, 1, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor, \qquad (17)$$

where

$$\boldsymbol{N}_k = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_{k+1} \\ \mu_2 & \mu_3 & \cdots & \mu_{k+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{k+1} & \mu_{k+2} & \cdots & \mu_{2k+1} \end{pmatrix} . \qquad (18)$$

In the positive case both even and odd number of input moments can be considered. The solution is a bit different according to the parity of the maximum number os usable moments.

For further discussions we need the following set of orthogonal polynomials:

$$\mathcal{N}_0(x) = 1 \ , \qquad (19)$$

$$\mathcal{N}_i(x) = \begin{vmatrix} \mu_1 - a\mu_0 & \cdots & \mu_{i+1} - a\mu_i \\ \mu_2 - a\mu_1 & \cdots & \mu_{i+2} - a\mu_{i+1} \\ \vdots & \ddots & \vdots \\ \mu_i - a\mu_{i-1} & \cdots & \mu_{2i} - \mu_{2i-1} \\ 1 & \cdots & x^i \end{vmatrix} , \qquad (20)$$

$$i = 1, 2, \ldots, \lfloor m/2 \rfloor \ .$$

We will also refer to the orthogonal polynomials $\mathcal{M}_i(x)$ (see (5)).

4.1 Even number of moments

First we us assume that the have even number of moments that satisfy (16) and (17): $\mu_0, \mu_1, \ldots, \mu_{2n+1}$.

The highest degree orthogonal polynomials and their roots are denoted by

$$\mathcal{M}_{n+1}(x) \ , \qquad u_1 < u_2 < \ldots < u_{n+1} \ , \qquad (21a)$$

$$\mathcal{N}_n(x) \ , \qquad v_1 < v_2 < \ldots < v_n \ . \qquad (21b)$$

These roots are mutually separated as

$$u_1 < v_1 < u_2 < \ldots < u_n < v_n < u_{n+1} \ . \quad (22)$$

In the infinite case the mass p at C is originated from the biggest possible matrix in the set of conditional equations (2). In the positive case we have two different sets of matrices: $\boldsymbol{M}_i$ (see (16)) and $\boldsymbol{N}_i - a\boldsymbol{M}_i$ (see (17)), and thus we have two candidates for the value of p:

$$p_{\text{inf}} = \frac{1}{\boldsymbol{c}^T \ \boldsymbol{M}_n^{-1} \ \boldsymbol{c}} \ , \quad (23)$$

$$p_{\text{pos}} = \frac{1}{(C-a) \ \boldsymbol{c}^T (\boldsymbol{N}_n - a\boldsymbol{M}_n)^{-1} \boldsymbol{c}} \ . \quad (24)$$

(We denote the value in (23) by p_{inf} since it is the same mass that we get using infinite estimation; the other p is denoted by p_{pos} because it is derived from the set of matrices (17) which is characteristic for the positive estimation.) The actual mass is the minimum of them:

$$p = \min \{p_{\text{inf}} \ ; \ p_{\text{pos}}\} \ . \quad (25)$$

1. If p is obtained by p_{inf} then all the other parameters of the discrete reference distribution are the same as in the infinite case, i.e. the points x_i are the roots of

$$\mathcal{K}_n^{\mathcal{M}}(x, C) = \boldsymbol{c}^T \ \boldsymbol{M}_n^{-1} \ \boldsymbol{x} \ , \quad (26)$$

and the weights are

$$p_i = \frac{1}{\boldsymbol{x_i}^T \ \boldsymbol{M}_n^{-1} \ \boldsymbol{x_i}} \ , \quad i = 1, 2, \ldots, n \ . \quad (27)$$

2. If $p = p_{\text{pos}}$ then we have a nonzero mass at the left end point of the support interval a:

$$p_a = \frac{1}{(a-C) \ \boldsymbol{a}^T (\boldsymbol{N}_n - C\boldsymbol{M}_n)^{-1} \ \boldsymbol{a}} \ . \quad (28)$$

The points x_i are the roots of

$$\mathcal{K}_n^{\mathcal{N}}(x, C) = \boldsymbol{x}^T (\boldsymbol{N}_n - a\boldsymbol{M}_n)^{-1} \boldsymbol{c} \ , \quad (29)$$

the weights are

$$p_i = \frac{1}{(x_i - a) \ \boldsymbol{x_i}^T (\boldsymbol{N}_n - a\boldsymbol{M}_n)^{-1} \ \boldsymbol{x_i}} \ , \quad i = 1, 2, \ldots, n \ . \quad (30)$$

A special case: if $C = v_i$ for some i, then we have to use the matrices $\boldsymbol{M}_{n-1}$ and $\boldsymbol{N}_{n-1}$ in the above formulas and the discrete reference distribution contains only $n-1$ point (excluding C). We call it *pure positive* case, as the calculations are based on the matrix $\boldsymbol{N}_n - a\boldsymbol{M}_n$ which bears the additional information about distributions supported in $[a, \infty)$ compared to the infinite case.

We can determine which one of the two sets of formulas is valid regarding the interval where the point of interest C is located:

$[a, u_1]$	infinite ($p_a = 0$),
$(u_i, v_i], \ i = 1, 2, \ldots, n$	pure positive ($p_a \neq 0$),
$(v_i, u_{i+1}], \ i = 1, 2, \ldots, n$	infinite ($p_a = 0$),
(u_{n+1}, ∞)	pure positive ($p_a \neq 0$).

Summing the weights of the points smaller than C we get the lower limit L, the upper limit is the sum of L and p, hence they are calculated as in (10).

The positions of the points of the discrete distribution can be characterized by C and v_i: the number of points x_i left (right) from C is equal to the number of v_i left (right) from C. As a consequence if $C < v_1$ or $C > v_n$ then no discrete distribution is needed and

$$L = p_a \ , \qquad U = p + p_a \ , \quad \text{if } C < v_1 \ , \quad (31)$$

$$L = 1 - p \ , \quad U = 1 \ , \qquad \text{if } C > v_n \ , \quad (32)$$

If $a \leq C \leq u_1$ then the infinite estimation is valid, hence $p_a = 0$:

$$L = 0 \ , \qquad U = p \ , \qquad \text{if } a \leq C \leq u_1 \ . \quad (33)$$

This is the simplified bounding in the positive case when even number of moments is applied for estimation.

4.2 Odd number of moments

In this section we consider odd number of moments satisfying (16) and (17): $\mu_0, \mu_1, \ldots, \mu_{2n}$.

The orthogonal polynomials of highest degree and their roots are denoted by

$$\mathcal{M}_n(x) \ , \qquad u_1 < u_2 < \ldots < u_n \ , \quad (34a)$$

$$\mathcal{N}_n(x) \ , \qquad v_1 < v_2 < \ldots < v_n \ . \quad (34b)$$

The roots u_i and v_i interlace as

$$u_1 < v_1 < u_2 < v_2 < \ldots < u_n < v_n \ . \quad (35)$$

The mass at C is the minimum of the following two values:

$$p_{\rm inf} = \frac{1}{\boldsymbol{c}^T \boldsymbol{M}_n^{-1}\,\boldsymbol{c}} \,, \tag{36}$$

$$p_{\rm pos} = \frac{1}{(C-a)\,\boldsymbol{c}^T(\boldsymbol{N}_{n-1} - a\boldsymbol{M}_{n-1})^{-1}\,\boldsymbol{c}} \,. \tag{37}$$

1. (*infinite*) If $p = p_{\rm inf}$ then the parameters of the discrete reference distribution are

$$\boldsymbol{c}^T\,\boldsymbol{M}_n^{-1}\,\boldsymbol{x} = 0 \implies x_i \,,$$
$$p_i = \frac{1}{\boldsymbol{x_i}^T\,\boldsymbol{M}_n^{-1}\,\boldsymbol{x_i}} \,, \quad i = 1,2,\ldots,n \,.$$

   If $C = u_i$ for some i then we have to apply the matrix $\boldsymbol{M}_{n-1}$ instead of $\boldsymbol{M}_n$.

2. (*pure positive*) If $p = p_{\rm pos}$ then there is a nonzero mass at a:

$$p_a = \frac{1}{(a-C)\,\boldsymbol{a}^T(\boldsymbol{N}_{n-1} - C\boldsymbol{M}_{n-1})^{-1}\boldsymbol{a}} \,,$$
$$\boldsymbol{x}^T(\boldsymbol{N}_{n-1} - a\boldsymbol{M}_{n-1})^{-1}\boldsymbol{c} = 0 \implies x_i \,,$$
$$p_i = \frac{1}{(x_i - a)\,\boldsymbol{x_i}^T(\boldsymbol{N}_{n-1} - a\boldsymbol{M}_{n-1})^{-1}\boldsymbol{x_i}} \,.$$

The applicability of the above two sets of formulas according to the interval which contains C:

$[a, u_1]$	infinite ($p_a = 0$),
$(u_i, v_i],\ i = 1,2,\ldots,n$	pure positive ($p_a \neq 0$),
$(v_i, u_{i+1}],\ i = 1,2,\ldots,n-1$	infinite ($p_a = 0$),
(v_n, ∞)	infinite ($p_a = 0$).

The number of x_i smaller than C is the number of v_i smaller than C, however the number of points of the reference discrete distribution right from C is the number of u_i greater than C. As a result if $C < v_1$ or $C > u_n$ then no discrete distribution is necessary:

$$L = p_a \,, \qquad U = p + p_a \,, \quad \text{if } C < v_1 \,, \tag{38}$$
$$L = 1 - p \,, \quad U = 1 \,, \qquad \text{if } C > u_n \,. \tag{39}$$

Furthermore if $C \in [a, u_1]$ then

$$L = 0 \,, \qquad U = p \,, \qquad \text{if } a \le C \le u_1 \,. \tag{40}$$

These are the formulas and the conditions of applicability of the simplified bounding in the positive case (when the number of usable moments is odd).

Note that if $C > v_n$ or $C \le u_1$ then the lower and upper bounds are exactly the same as in the infinite case using the moments $\mu_0, \mu_1, \ldots, \mu_{2n}$. The only differences are in the intervals $(u_1, v_1]$ and $(u_n, v_n]$.

5 COMPUTATIONAL COMPLEXITY

Some tasks in the proposed algorithm may involve numerical difficulties in the general case (e.g. evaluating determinants, inverting matrices, finding roots of polynomials), however the matrices and the polynomial considered here have special properties that make it possible to use numerically more stable methods to calculate them.

To calculate the determinants of symmetric matrices we use the *LU decomposition* [Press et al, 1993, p. 43 – 50]. Testing with known distributions the maximum dimension of the matrix whose determinant could be computed correctly is 15×15, bigger matrices resulted negative determinants showing numerical instabilities in the method. Therefore the limit of the applicability is 29 moments using standard floating point arithmetic, but the maximum number of moments that satisfy (2) or (16)–(17) largely depends on the original distribution: our experiences show that in general the number of usable moments is around 20, but in some cases it is below 15.

We use *Cholesky decomposition* with backsubstitution to invert the positive definite matrices $\boldsymbol{M}_n$ and $\boldsymbol{N}_n - a\boldsymbol{M}_n$ [Press et al, 1993, p. 96–98]. This method is known to be extremely stable numerically and approximately two times faster than the alternative methods for solving linear equations. It fails only if the matrix is not positive definite.

It is hard to find the roots of a polynomial if we do not know anything about the location of the roots. But all the roots of $\mathcal{M}_n(x)$ and $\mathcal{N}_n(x)$ are real, and in this case *Laguerre's method* [Press et al, 1993, p. 371 – 374] works well as it is theoretically guaranteed that this algorithm converges to a root from any starting point.

Figure 1 shows that at different values of C only p has to be recalculated, hence the overall algorithm is neither CPU, nor memory intensive. Table 1 shows the required operations in order to estimate a distribution at N different values of C using $m+1 = 2n+1$ moments ($\mu_0, \mu_1, \ldots, \mu_{2n}$). Similar numbers of operations are shown in Table 2 in case of the positive estimation.

6 CLOSED-FORM EXPRESSIONS

The applicability of the simplified estimation depends on the smallest and the largest roots of $\mathcal{M}_n(x)$ and $\mathcal{N}_n(x)$. If the degree of these polynomials is less

Input

- a series of numbers $\mu_0, \mu_1, \ldots, \mu_m$;
- the left end point of the support interval;
- a set of C values where we need to bound the distribution..

Algorithm

1. Determine maximum how many moments satisfy consecutively (i.e. for all k in a range $0, 1, \ldots, k_{\text{max}}$)

$$|\boldsymbol{M}_k| \geq 0,\ k = 0, \ldots, \lfloor m/2 \rfloor, \qquad |\boldsymbol{N}_k - a\boldsymbol{M}_k| \geq 0,\ k = 0, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor, \tag{41}$$

   We denote the number of applicable moments (for which both sets of inequalities (41) hold) by $m_{\text{max}} + 1$ ($\mu_0, \ldots, \mu_{m_{\text{max}}}$).
2. If the number of applicable moments $m_{\text{max}} + 1$ is even ($\mu_0, \ldots, \mu_{2n+1}$ form a valid moment sequence):
   (a) Find the roots of the polynomials $\mathcal{M}_{n+1}(x)$ and $\mathcal{N}_n(x)$:

$$\mathcal{M}_{n+1}(x) = 0 \implies u_1 < \ldots < u_n < u_{n+1}\ , \qquad \mathcal{N}_n(x) = 0 \implies v_1 < \ldots < v_n\ .$$

   (b) Do for each $C < v_1$ or $C > v_n$ point of interest
      i. If $a \leq C \leq u_1$ or $v_n \leq C \leq u_{n+1}$ then

$$p = \frac{1}{\boldsymbol{c}^T\ \boldsymbol{M}_n^{-1}\ \boldsymbol{c}}\ , \qquad p_a = 0\ .$$

      If $u_1 < C < v_1$ or $u_{n+1} < C$ then

$$p = \frac{1}{(C-a)\,\boldsymbol{c}^T(\boldsymbol{N} - a\boldsymbol{M})^{-1}\,\boldsymbol{c}}\ , \qquad p_a = \frac{1}{(a-C)\,\boldsymbol{a}^T(\boldsymbol{N} - C\boldsymbol{M})^{-1}\,\boldsymbol{a}}\ .$$

      ii. If $C < v_1$, then $L = p_a,\ U = p + p_a$.
      If $C > v_n$, then $L = 1 - p,\ U = 1$.
3. If the number of applicable moments $m_{\text{max}} + 1$ is odd ($\mu_0, \ldots, \mu_{2n}$ form a valid moment sequence):
   (a) Find the roots of the polynomials $\mathcal{M}_n(x)$ and $\mathcal{N}_n(x)$:

$$\mathcal{M}_n(x) = 0 \implies u_1 < \ldots < u_n\ , \qquad \mathcal{N}_n(x) = 0 \implies v_1 < \ldots < v_n\ .$$

   (b) Do for each $C < v_1$ or $C > u_n$ point of interest
      i. If $a \leq C \leq u_1$ or $v_n \leq C$ then

$$p = \frac{1}{\boldsymbol{c}^T\ \boldsymbol{M}_n^{-1}\ \boldsymbol{c}}\ , \qquad p_a = 0\ .$$

      If $u_1 < C < v_1$ or $u_n < C < v_n$ then

$$p = \frac{1}{(C-a)\,\boldsymbol{c}^T(\boldsymbol{N} - a\boldsymbol{M})^{-1}\,\boldsymbol{c}}\ , \qquad p_a = \frac{1}{(a-C)\,\boldsymbol{a}^T(\boldsymbol{N} - C\boldsymbol{M})^{-1}\,\boldsymbol{a}}\ .$$

      ii. If $C < v_1$, then $L = p_a,\ U = p + p_a$.
      If $C > u_n$, then $L = 1 - p,\ U = 1$.

Figure 2. Steps of the simplified algorithm if the support interval is $[a, \infty)$

than 5, then closed form expressions can be deduced for u_i and v_i, though for degrees 3 and 4 these expressions are much too complicated and would fill several pages.

Infinite case If the degree of $\mathcal{M}_n(x)$ is equal to 2 (hence we have 5 moments as input: $\mu_0, \mu_1, \mu_2, \mu_3$ and μ_4) the formulas for u_1, u_2 and even for p are quite simple. Discrete construction is needed only in

Task	Nr. of executions
calculation of determinants of max. $(\lfloor\frac{m+1}{2}\rfloor+1)\times(\lfloor\frac{m+1}{2}\rfloor+1)$	$\lfloor\frac{m+1}{2}\rfloor+1$
inversion of an $(n+1)\times(n+1)$ matrix	1
finding $\lfloor\frac{m+1}{2}\rfloor$ roots of $\mathcal{M}(x)$	1
vector-matrix multiplications of size $(n+1)\times(n+1)$	$2N$
scalar product of vectors of size $(n+1)$	$2N$
reciprocal	$2N$

Table 1. Computational cost of the simplified estimation (infinite case)

Task	Nr. of executions
calculation of determinants of max. $(\lfloor\frac{m+1}{2}\rfloor+1)\times(\lfloor\frac{m+1}{2}\rfloor+1)$	$m+3$
inversion of an $(\lfloor\frac{m+1}{2}\rfloor+1)\times(\lfloor\frac{m+1}{2}\rfloor+1)$ matrix	$N+1$
finding n roots of $\mathcal{M}_n(x)$	1
finding $\lfloor\frac{m}{2}\rfloor$ roots of $\mathcal{N}(x)$	1
vector-matrix multiplications of size $(\lfloor\frac{m+1}{2}\rfloor+1)\times(\lfloor\frac{m+1}{2}\rfloor+1)$	$3N$
scalar product of vectors of size $(\lfloor\frac{m+1}{2}\rfloor+1)$	$3N$
reciprocal	$3N$

Table 2. Computational cost of the simplified estimation (positive case)

the interval $[u_1, u_2]$.

$$u_{1,2} = \frac{1}{2\mu_1^2 - 2\mu_0\mu_2}\Big(\mu_1\mu_2 - \mu_0\mu_3 \pm \sqrt{4\mu_0\mu_2^3 + 4\mu_1^3\mu_3 - 6\mu_0\mu_1\mu_2\mu_3 + \mu_0^2\mu_3^2 - 3\mu_1^2\mu_2^2}\Big), \tag{42}$$

$$\frac{1}{p} = \frac{1}{\mu_2^3 + \mu_0\mu_3^2 + \mu_1^2\mu_4 - \mu_2(2\mu_1\mu_3 + \mu_0\mu_4)} \Big(C^4(\mu_1^2 - \mu_0\mu_2) + C^3(-2\mu_1\mu_2 + 2\mu_0\mu_3) + C^2(3\mu_2^2 - 2\mu_1\mu_3 - \mu_0\mu_4) + C(-2\mu_2\mu_3 + 2\mu_1\mu_4) + (\mu_3^2 - \mu_2\mu_4)\Big) . \tag{43}$$

Having 3 input moments ($\mu_0 = 1, \mu_1$ and μ_2) the discrete reference distribution contains only 1 point: C . The only root of $\mathcal{M}_n(x)$ and the maximal concentrated mass at C are the following:

$$u_1 = \mu_1, \qquad p = \frac{\mu_2 - \mu_1^2}{C^2 - 2C\mu_1 + \mu_2} \ . \tag{44}$$

The lower and upper bounding functions can be expressed by simple formulas along the whole real axis.

$$L = \begin{cases} 0 & \text{if } C < \mu_1 \ , \\ \dfrac{(C-\mu_1)^2}{C^2 - 2C\mu_1 + \mu_2} & \text{if } C \geq \mu_1 \ , \end{cases} \tag{45}$$

$$U = \begin{cases} \dfrac{\mu_2 - \mu_1^2}{C^2 - 2C\mu_1 + \mu_2} & \text{if } C < \mu_1 \ , \\ 1 & \text{if } C \geq \mu_1 \ . \end{cases} \tag{46}$$

It is easy to see that L and U are continuous functions of C .

Positive case First assume that two moments are given: $\mu_0 = 1$ and μ_1. In this special case there is no v_i because of the too small number of moments. The only u_1 is equal to μ_1. In the interval $[a, \mu_1]$ the positive estimation is applied: the discrete reference distribution has only one point: C. Its weight is

$$p = \frac{1}{1 \cdot (\mu_0)^{-1} \cdot 1} = \mu_0 = 1 \ , \tag{47}$$

hence the lower and upper limits are 0 and 1, respectively, if we choose C from $[a, \mu_1]$. In (μ_1, ∞) the pure positive estimation is applied: the discrete distribution also has only one point C. The other parameters:

$$p_a = \frac{1}{(a-C)(\mu_1 - C\mu_0)^{-1}} = \frac{\mu_1 - C}{a - C} \ , \tag{48}$$

$$p = \frac{1}{(C-a)(\mu_1 - a\mu_0)^{-1}} = \frac{\mu_1 - a}{C - a} \ , \tag{49}$$

therefore the lower limit is p_a and the upper one is 1. Summarizing the results we get

$$L = \begin{cases} 0 & \text{if } C \le \mu_1 , \\ \dfrac{\mu_1 - C}{a - C} & \text{if } C > \mu_1 , \end{cases} \qquad U = 1 . \qquad (50)$$

(Note that considering $a = 0$ the above formulas yield the *Markov-inequality*: $\Pr(X \ge C) \le \mu_1/C$.)

Having 3 moments ($\mu_0 = 1, \mu_1, \mu_2$) the roots of the orthogonal polynomials are the following:

$$u_1 = \mu_1 \qquad < \qquad v_1 = \frac{\mu_2 - a\mu_1}{\mu_1 - a} . \qquad (51)$$

In $[a, u_1]$ and (v_1, ∞) the infinite estimation is applied with weight

$$p = \frac{\mu_2 - \mu_1^2}{C^2 - 2C\mu_1 + \mu_2} . \qquad (52)$$

In $(u_1, v_1]$ the pure positive estimation is valid with parameters

$$p_a = \frac{\mu_1 - C}{a - C} , \qquad p = \frac{\mu_1 - a}{C - a} . \qquad (53)$$

Putting these together we get the bounds along the $[a, \infty)$ semi-axis:

$$L = \begin{cases} 0 & \text{if } a \le C < \mu_1 , \\ \dfrac{\mu_1 - C}{a - C} & \text{if } \mu_1 \le C < \dfrac{\mu_2 - a\mu_1}{\mu_1 - a} , \\ \dfrac{(C - \mu_1)^2}{C^2 - 2C\mu_1 + \mu_2} & \text{if } \dfrac{\mu_2 - a\mu_1}{\mu_1 - a} \le C , \end{cases} \qquad (54)$$

$$U = \begin{cases} \dfrac{\mu_2 - \mu_1^2}{C^2 - 2C\mu_1 + \mu_2} & \text{if } a \le C < \mu_1 , \\ 1 & \text{if } \mu_1 \le C . \end{cases} \qquad (55)$$

These formulas are simple but they make only rough estimations possible. The next section shows how the increasing number of moments affects accuracy.

7 EXAMPLE OF APPLICATION

This section demonstrates the properties of the proposed approach through an example, pointing out its strengths and weaknesses.

[Fodor et al, 2002] introduced a strategy to share a telecommunication link between different traffic classes to satisfy certain pre-defined Quality of Service (QoS) constraints. Three traffic classes are defined:

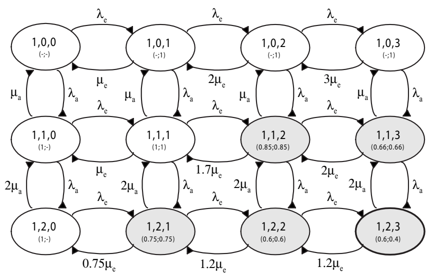


Figure 3. State space of the sample model

- **rigid**: require constant bandwidth (b_r) allocation;
- **adaptive**: characterized by peak (b_a) and minimum bandwidth (b_a^{min}) requirements, the actual bandwidth usage depends on the link utilization (for example a video stream with adaptive compression level, where quality degradation is allowed to a certain degree, but high delay variance in not);
- **elastic**: similar to the adaptive class regarding their bandwidth requirements (b_e and b_e^{min}), but they stay in the system until a given amount of data has been transmitted (for example an ftp-session, where transfer rate changes are allowed, but data loss is not).

A Markov reward model (MRM) is used to describe system behavior. The states of the system are represented by a triple (n_r, n_a, n_e) which are the number of active flows in the system belonging to the rigid, adaptive and elastic flows, respectively. The arrival rates are $\lambda_r, \lambda_a, \lambda_e$, and the departure rates are μ_r, μ_a, μ_e. μ_e is called the *maximal* departure rate of an elastic flow experienced when maximal bandwidth is available, the *actual* departure rate is proportional to the available bandwidth, which is a function of n_r, n_a and n_e. The transition rates of the MRM are calculated from these rates, and the reward rates associated with each state are the actual bandwidth of the elastic class.

Figure 3 shows a portion of the state space in case of $n_r = 1$. The states where the elastic flows do not get the maximal bandwidth are printed in grey. The numbers below the state identifiers indicate the actual bandwidth of the adaptive and elastic flows as a fraction of their peak bandwidth.

The performance measure of our interest is the distribution of the amount of time, $T(\xi)$, required to transmit ξ amount of data by an elastic traffic flow. We would like to ensure that the transmission completes before time t with a very high probability:

$$\Pr\left(T(\xi) < t\right) > \varepsilon, \tag{56}$$

where ε is a prescribed constant close to 1 $(0.99, \ldots, 0.99999)$. The amount of data is given and we are interested in the minimum value of t which means that the transfer of ξ amount of data will be finished during the interval $[0, t_{\min})$ with probability e.g. 0.9999 but this is not true for any $t < t_{\min}$.

This investigation requires evaluation of the MRM. We compare two different analysis approaches:

1. the moment-based method in [Telek and Rácz, 1999] with estimation based on the moments;
2. direct analysis of the distribution of the completion time: methods of Nabli and Sericola [Nabli and Sericola, 1996], De Souza e Silva and Gail [de Souza e Silva and Gail, 1986], Donatiello and Grassi [Donatiello and Grassi, 1991].

The algorithms were implemented by their original paper. We use a dual AMD Opteron 248 (2.2 GHz) system with 6 GB of RAM running Linux for computations.

7.1 Correctness

To verify the procedures we evaluate a sample system with 105 states and calculate the whole distribution of the amount of transmitted data. The three direct methods result in the same values and the moments-based method gives real bounds as it is depicted in Figure 4. The more moments are given the tighter the bounds are. It is also observable that convergence slows down with the increasing number of moments. The bounds are the widest around the mean of the distribution. We are able to do the estimations with maximum 17 moments, because using more moments results in negative determinant while testing the necessary and sufficient condition of existence (16). This is due to numerical instabilities in the procedure that calculates the determinant of a matrix.

7.2 Numerical results

We evaluate and estimate $t_{\min}$, i.e. the minimum of t that satisfies (56). Three values of ε are considered: 0.9999, 0.99999 and 0.999999, ξ is set to 100.

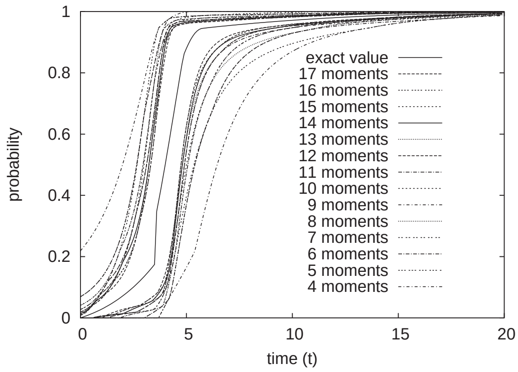


Figure 4. Distribution of the transmission time of ξ amount of elastic data

Fig. 5 shows the exact distribution and the bounds we get using different number of moments in case of $\varepsilon = 0.9999$. Thick black line represents this value. All the three direct analysis methods result the same values, the corresponding curve is labeled “exact” and $t_{\min}$ is the point where it reaches 0.9999 . When estimating a distribution based on its moments we get a lower and an upper bounding function. In these special cases that we investigate the upper bounding function is always equal to 1 and that’s why it is omitted in the figure. The lower estimation is always smaller than the real value in any point of interest C, hence all the lower bounding functions corresponding to different number of moments are below the exact distribution function. As a consequence these functions intersect the line 0.9999 at greater values of t than $t_{\min}$.

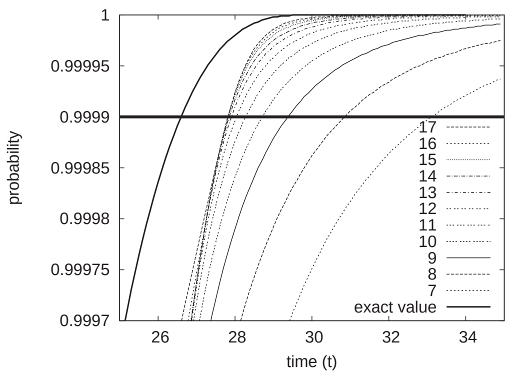


Figure 5. Lower estimation reaches 0.9999

Table 3 presents the experiences. The 3^{rd}, 4^{th} and

Moment-based method		ε		
Moments	Simple bound from	0.9999	0.99999	0.999999
4	5.402	87.601	181.983	344.845
5	13.668	48.542	78.919	122.966
6	17.428	37.704	53.769	74.907
9	23.188	29.364	34.462	44.324
12	25.734	28.032	30.671	33.499
15	27.129	27.818	29.486	31.144
17	27.698	27.814	29.169	30.405
Exact		26.590	28.373	29.145

Table 3. Moments based bounding of the tail distribution

5^{th} columns contain the results at different values of accuracy ε . The last row contains the "Exact" values which result from the direct distribution analysis. The other rows show the points where the moment-based estimation reaches the predefined level of accuracy. The "Simple bound from" column indicates the point from which the presented simple bounding methods are applicable (see (12), (31)–(33) and (38)–(40)) and no reference discrete distribution is needed.

The table clearly shows that more moments contain more information about the tail distribution, and the estimated value of $t_{\min}$ is closer to the real one in these cases. However convergence slows down as the number of used moments increases.

In this example the moment based estimation gives surpassingly accurate results, since the differences from the exact values are 4.6%, 2.8% and 4.3% (using 17 moments) according to the three different values of ε. This precision makes the moment based bounding a practically well utilizable tool for complex models or models with huge state spaces, that also facilitates transient analysis, which is very important in examining real systems.

7.3 Size of the state space

We evaluated a series of runs to determine the maximum number of states which the different types of solvers are still capable to calculate. We considered a method unusable if it resulted in clearly invalid values (e.g., negative possibilities) or the running time was more than 20× of the previous configuration.

Using the moments based method we could calculate the model with 370,000 states, while direct methods calculated the model with maximum 12,000 states. On the other hand the moments based approach yields less information about the distribution. The evaluation time of the estimation from the moments is 0.01s, its contribution to the overall calculation time in all considered cases is negligible.

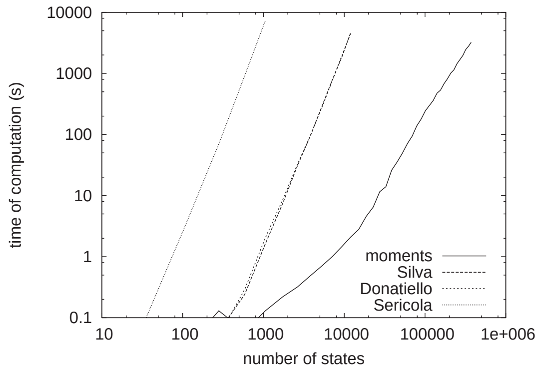


Figure 6. Evaluation time vs. state space size in logarithmic scale

8 CONCLUSION

In this paper we focus on a special use of our previously developed moments based distribution bounding method. For the computation of the distribution of extreme events the moment based analysis simplifies, because the probability mass at the point of interest (and at the left end point of the support interval a in case of positive estimation) defines the bounds of the distribution.

We present an example where the simple bounding method is efficient and accurate compared to the results of other methods that calculate directly the values of the distribution function.

We plan to increase the accuracy of our algorithm by using extended precision arithmetic and to improve our method using additional information about the distribution functions such as finite support intervals.

References

[Akhiezer, 1965] Akhiezer N. I. 1965, "The classical moment problem and some related questions in analysis". Hafner publishing company, New York

[Akhiezer and Krein, 1968] Akhiezer N. I. and Krein M. G. 1968, "Some questions in the theory of moments", volume 2 of *Translations of mathematical monographs*. American Mathematical Society, Providence, RI,

[Carleman, 1922] Carleman T. 1922, "Sur le problème des moments". In *Comptes rendus*, 174 pp 1680–1682.

[Chebyshev, 1874] Chebyshev P. 1874, "Sur les valeurs limites des intégrales". In *Journal des Mathématiques pures et appliquées*, 19 pp 157–160.

[de Souza e Silva and Gail, 1986] de Souza e Silva E. and Gail H. R. 1986, "Calculating cumulative operational time distributions of repairable computer systems". In *IEEE Transactions on Computers*, C-35, pp 322–332.

[de Souza e Silva and Gail, 1998] de Souza e Silva E. and Gail H. R. 1998, "An algorithm to calculate transient distributions of cummulative rate and impulse based reward". In *Commun. in Statist. – Stochastic Models*, 14(3) pp 509–536.

[Donatiello and Grassi, 1991] Donatiello L. and Grassi V. 1991, "On evaluating the cumulative performance distribution of fault-tolerant computer systems". In *IEEE Transactions on Computers*.

[Fodor et al, 2002] Fodor G., Rácz S. and Telek M 2002. "On providing blocking probability- and throughput guarantees in a multi-service environment". *International Journal of Communication Systems*, 15:4 pp 257–285.

[Hamburger, 1920] Hamburger H. 1920, "Über eine Erweiterung des Stieltjes'schen Momentproblems". In *Mathematische Annalen*, 81 pp 235–319.

[Hausdorff, 1921] Hausdorff F. 1921, "Momentprobleme für ein endliches Intervall". In *Mathematische Zeitschrift*, 16 pp 220–248.

[Horváth et al, 2004] Horváth G., Rácz S., Tari Á. and Telek M. 2004, "Evaluation of reward analysis methods with MRMSolve 2.0". In *1st International Conference on Quantitative Evaluation of Systems (QEST) 2004*, pp 165–174, Enschede, The Netherlands IEEE CS Press.

[Krein and Nudelman, 1977] Krein M. G. and Nudelman A. A. 1977, "The Markov moment problem and extremal problems", volume 50. Americal Mathematical Society, Providence, Rhode Island Translation of mathematical monographs.

[Nabli and Sericola, 1996] Nabli H. and Sericola B. 1996, "Performability analysis: a new algorithm". In *IEEE Transactions on Computers*, 45 pp 491–494.

[Nevanlinna, 1922] Nevanlinna. R. 1922, "Asymptotische Entwickelungen beschränkter Funktionen und das Stieltjessche Momentenproblem". In *Ann. Acad. Sci. Fenn. Ser. A*, 18, no. 5.

[Press et al, 1993] Press W. H., Flannery B. P., Teukolsky S. A. and Vetterling W. T. 1993, "Numerical Recipes in C: The Art of Scientific Computing". Cambridge University Press `http://lib-www.lanl.gov/numerical/bookcpdf.html`.

[Rácz, 2000] Rácz S. 2000, "Numerical analysis of communication systems through Markov reward models". PhD thesis, Technical University of Budapest, Hungary.

[Rácz et al, 2005] Rácz S., Tari Á. and Telek M. 2005, "A moments based distribution bounding method". In Ervin Y. Rodin, editor, *Mathematical and Computer Modelling* (USA).

[Rácz and Telek, 1999] Rácz S. and Telek M. 1999, "Performability analysis of Markov reward models with rate and impulse reward". In M. Silva B. Plateau, W. Stewart, editor, *Int. Conf. on Numerical solution of Markov chains*, pp 169–187, Zaragoza, Spain.

[Riesz, 1922] Riesz M. 1922, "Sur le probleme des moments et le theoreme de parseval correspondant". In *Acta Litt. Sci.*, 1 pp 209–225.

[Shohat and Tamarkin, 1946] Shohat J. A. and Tamarkin D. J.. "The problem of moments". Americal Mathematical Society, Providence, Rhode Island, Mathematical surveys.

[Stieltjes, 1894] Stieltjes T. J. 1894, "Recherches sur les fractions continues". In *Annales de la Faculté des Sciences de Toulouse*, 2 pp 1–122.

[Szegö, 1939] Szegö G. 1939, "Orthogonal polynomials". American Mathematical Society, Providence, Rhode Island.

[Tari, 2005] Tari Á. 2005, "Moments based bounds in stochastic models". PhD thesis, Budapest University of Technology and Economics, Hungary. `http://webspn.hit.bme.hu/~arpi/dl/dissertation.pdf`

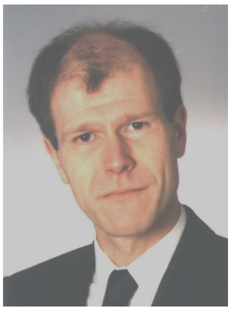
[Tari et al, 2005] Tari Á., Telek M. and Buchholz P. 2005, "A unified approach to the moments based distribution estimation – Unbounded support". In *2^{nd} European Performance Evaluation Workshop*, pp 79–93, Versailles, France, Springer LNCS 3670.

[Telek and Rácz, 1999] Telek M. and Rácz S. 1999, "Numerical analysis of large Markovian reward models". In *Performance Evaluation*, 36&37 pp 95–114.

[van de Liefvoort, 1990] van de Liefvoort A. 1990, "The moment problem for continuous distributions". Technical report, University of Missouri, WP-CM-1990-02, Kansas City.

Árpád Tari received the M. Sc. degree in Technical Informatics from the Technical University of Budapest in 2001. After that, he started Ph.D. studies at the Department of Telecommunications of the same university. He joined the University of Dortmund in 2004. His current research interest includes stochastic modeling and analysis of computer and communication systems especially with moments based analysis methods.

Miklós Telek received the M. Sc. degree in Electrical Engineering from the Technical University of Budapest in 1987. After graduation he joined the Hungarian Post Research Institute where he studied the modeling, analysis and planning aspects of communication networks. Since 1990 he has been with the Department of Telecommunications of the Technical University of Budapest, where he is a full professor now. He received the candidate of science degree from the Hungarian Academy of Science in 1995. His current research interest includes stochastic performance modeling and analysis of computer and communication systems.



Peter Buchholz holds a Diploma degree in computer science (Dipl. -Inform., 1987), a Doctoral degree (Dr.rer.nat., 1991) and a Habilitation degree (1996), all from the University of Dortmund, where he is currently a professor for modeling and simulation. Previously, he has been an associative professor for modeling and simulation at Dresden University of Technology. His research interests include techniques for performance and functional analysis of discrete event dynamic systems, especially he worked on the development of numerical analysis techniques for large Markov chains. Furthermore, he developed software tools for the qualitative and quantitative analysis of complex systems and applied the analysis techniques and tools to applications from various areas including communication systems and logistic networks. In the mentioned areas, he has published more than 70 papers in refereed journal or conference proceedings and has served on various program committees of international conferences.