Transient Analysis of *Age-MRSPN*s by the Method of Supplementary Variables ⋆

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Abstract

In order to assist the performance evaluation of complex stochastic models, automatic program tools were developed since a long time. Stochastic Petri Nets (*SPN*) are applied as an effective model description language supported by several analytical and simulation tools. The analytical description and the numerical analysis of non-Markovian stochastic Petri Net models gained attention recently. There are different theoretical approaches and numerical methods considered in recent works, such as the Markov renewal theory and the supplementary variable approach, but to find the most effective way of the analysis of such models is still an open research problem.

The supplementary variable approach was successfully applied to the transient and steady state analysis of Markov Regenerative Stochastic Petri Nets (*MSRPN*) when the preemption policy associated with the Petri Net (*PN*) transitions was *preemptive repeat different* (*prd*), but it was not applicable with other preemption mechanisms. In this paper we extend the applicability of the supplementary variable approach to a class of *MRSPN*s in which *preemptive resume* (*prs*) policy can also be assigned to the transitions of the *PN*.

Key words: Markov Regenerative Stochastic Petri Nets, Preemption Policies, Supplementary Variable Approach

1 Introduction

The semantics of *SPN* with generally distributed random firing times has been considered for a long time. To completely define the stochastic behavior of the

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marking process a firing time distribution and a memory policy are assigned to each timed transition [1]. The latter specifies how the firing of the transition depends on its past history.

Based on the concepts presented in [1], Ajmone and Chiola developed the *Deterministic and Stochastic PN (DSPN)* model [3], where in each marking, at most one transition with deterministic firing time is allowed to be enabled. In the *DSPN* model only the *prd* policy can be assigned to the transitions with deterministic firing time. In [3] the steady state analysis of *DSPN*s was provided. Choi et al. have derived the transient solution of the same model in terms of a Markov regenerative process, and have subsequently extended the *DSPN* model by accommodating at most a single transition with generally distributed firing time [8] in each marking. They have called this class of models *Markov Regenerative Stochastic PN (MRSPN)*. Independent of [8], German and Lindemann applied a different method to the steady state [13] and German to the transient analysis [11] of *DSPN*s based on the method of supplementary variables [10]. A prototype tool was presented for the steady state analysis of *MRSPN*-s with *prd* and *prs* policies based on the supplemenentary variable approach by German in [12]. Further elaborations of *SPN* models with non-exponential distributions but restricted to *prd* policy only have been presented in [9,19].

Bobbio and Telek have enlarged the class of *MRSPN* by introducing the concept of marking processes with non-overlapping memories [7]. Roughly speaking, it means that the firing processes of the timed transitions with non-exponentially distributed firing time (referred to as MEM transitions) do not overlap. In this new framework, they have accommodated into the model the *prs* [7] and *preemptive repeat identical* (*pri*) policies [4] and provided the steady state [22,6] and the transient analysis [7,4] of the considered class of *MRSPN*s.

In the past, discrete event simulation was the only way to evaluate numerically non-Markovian *SPN*s. Due to the theoretical results available for the analysis of *MRSPN*s, the research for effective numerical analysis methods has started. There are effective computational methods, based on the solution of linear systems of equations, for the steady state analysis of these models [3,13,22,6]. With respect to the transient analysis an effective solution method is proposed for the class of *MRSPN*s with only exponentially distributed and identical deterministic firing times in [20]. The general transient analysis methods published in the literature so far are based on one of the following approaches:

- analysis of an expanded Markov model
- supplementary variable approach
- Markov renewal theory

The first approach is exact only when the firing time of the MEM transitions are Phase type distributed. The latter two approaches provide the exact analytical description of the stochastic marking process, but, in general, they do not provide closed-form expressions (in time domain) for the required model parameters. In both cases numerical analysis methods have to be applied to find the required measures and the applicability of these approaches depend on the chosen numerical method. It is still an open research problem to find the most effective methods for the transient analysis of non-Markovian *SPN* models.

In this paper we provide a detailed analytical description of a subclass of *MRSPN*s, called *Age-MRSPN*s [21], by the method of supplementary variables, which was not available before. In the *Age-MRSPN* class the MEM transitions are of *prs* type. Based on the provided analytical description we introduce a numerical method of transient analysis.

The rest of the paper is organized as follows. Section 2 introduces the preemption types of Petri Net transitions, the considered class of *SPN* models, and the base concept of the supplementary variable approach. Section 3 discusses the application of the supplementary variable approach for the analysis of *MRSPN*s with *prd* type transitions, while in Section 4 the new method for the case of *prs* type transitions is explained. Application examples are evaluated in Section 5, and the paper is concluded in Section 6.

2 Markov regenerative stochastic Petri Nets

A marked Petri Net is a tuple $PN = (P, T, I, O, H, M_0)$ where: P is the set of places, T is the set of transitions, I, O and H are the input, the output and the inhibitor functions, respectively, and M_0 is the initial marking. The reachability set $\mathcal{R}(M_0)$ is the set of all the markings that can be generated from the initial marking M_0. The marking process $\mathcal{M}(t)$ denotes the marking occupied by the *PN* at time t. In order to obtain a consistent model description, the way in which the evolution of the marking process depends on its past history has to be specified at the *PN* level. The most commonly used concept is provided in [1]. To each timed transition t_g is assigned a random firing time γ_g with a general distribution $G_g(t)$ with support on $(0\,,\,\infty)$. A clock, associated to each transition, measures the time in which the transition has been enabled [5]. An *age variable* a_g associated to the timed transition t_g keeps track of the clock count. A timed transition fires as soon as the memory variable a_g reaches the value of the firing time γ_g. Naturally, at each firing of t_g the age variable a_g is reset to 0, and the firing time γ_g is re-sampled from the same distribution. The alternative approach in [8] would result in a bit different analytical treatment of the same stochastic marking process.

In order to define properly the stochastic behavior of *SPN* models, the effect of a preemption of t_g has to be defined as well. (A preemption of t_g means that after a period of time while t_g was enabled it becomes disabled before firing.)

Adopting this concept of the firing process, the following memory policies have been introduced [5]. A timed transition t_g can be:

- *Preemptive resume (prs)*:
  The preemption of t_g does not affect a_g and γ_g. The age variable a_g is reset and γ_g is re-sampled only when t_g fires.
- *Preemptive repeat different (prd)*:
  The preemption of t_g resets a_g and re-samples γ_g.
- *Preemptive repeat identical (pri)*:
  The preemption of t_g does not affect γ_g but it resets a_g. The firing time of t_g (γ_g) is re-sampled only when t_g fires.

We define the firing process of transition t_g as the process that starts when t_g becomes enabled and ends when a_g is reset to 0 and γ_g is re-sampled. If t_g is *prd* type, the past of t_g affects the evolution of the marking process while it is enabled; if t_g is *prs* type, the past of t_g affects the evolution of the marking process while it is enabled and while it is disabled, but a_g is positive; if t_g is *pri* type, the past of t_g affects the evolution of the marking process while it is enabled and while it is disabled, but γ_g can not be re-sampled. We say that the firing process of t_g is active while t_g affects the evolution of the marking process. Transition t_g is said to be active while its firing process is active. The active period of a MEM transition can be concluded either by the firing of the transition, or by a preceding firing of an EXP transition which disables the MEM one.

The class of *SPN* in which at most one MEM transition can be active at a time satisfies the definition of the class of Markov Regenerative Stochastic Petri Nets [8]. In the rest of this paper we assume that the considered *MRSPN*s satisfies the property that at most one MEM transition can be active at a time.

In case of *prs* and *prd* type MEM transitions the age variable (a_g) sufficiently represents the effect of the MEM transition on the evolution of the marking process, so that the marking process can be analyzed by the method of supplementary variables using a single variable, while for *pri* type transitions it seems that one variable is not enough to capture the effect on the process.

Let $a(t)$ be the age of the only active MEM transition at time t, if any. Since only one MEM transition can have memory at any time, $a(t)$ is the age of the whole model at time t. Under this restriction, the marking process $\mathcal{M}(t)$ together with $a(t)$ (i.e. $(\mathcal{M}(t), a(t))$) is a *Markov process* over the state space $\mathcal{R} \times \mathbb{R}$, where $\mathcal{R}$ is the set of reachable tangible markings and $\mathbb{R}$ is the set of non-negative real numbers. The joint process can be analyzed by the method of supplementary variables [10]. This approach was followed in [13,11,16] to analyze Stochastic Petri Nets with only *prd* MEM transitions.

3 Analysis of MRSPNs with *prd* type transitions

3.1 Application of the method of supplementary variables

The method of supplementary variable has been applied to *MRSPN*s in which, in each (tangible) marking, at most one enabled transition can have non-exponential distribution with *prd* policy, having all the other enabled transitions exponential firing time distribution. Following the concept of [11] the solution approach is briefly summarized.

Let T^G be the set of MEM transitions. The tangible state space $\mathcal{R}$ is partitioned into $|T^G| + 1$ disjoint subsets. $\mathcal{R}^E$ is the set of markings in which no MEM transition is active ($a(t) = 0$ when $\mathcal{M}(t) \in \mathcal{R}^E$), and $\mathcal{R}^g,\ t_g \in T^G$ is the set of markings in which the general transition t_g can be active, i.e., a_g can be greater than 0. The superscript ^{E} refers to the states in $\mathcal{R}^E$ and the superscript ^{g} (or $^\ell$) refers to the states in $\mathcal{R}^g$ (or $\mathcal{R}^\ell$).

$\mathbf{Q} = [q_{ij}]$ denotes the $|\mathcal{R}| \times |\mathcal{R}|$ infinitesimal generator matrix of the Markov chain describing the evolution of the marking process considering only the firing of the transitions with exponentially distributed firing time (referred to as EXP transitions). According to the above partitioning of the state space $\mathbf{Q}$ can be partitioned as well. $\mathbf{Q}^{\ell,g}$ ($\mathbf{Q}^g$) contains the intensity of the state transitions from $\mathcal{R}^\ell$ to $\mathcal{R}^g$ (inside $\mathcal{R}^g$) due to the firing of an EXP transition and $\mathbf{Q}^{E,g}$, $\mathbf{Q}^{g,E}$, and $\mathbf{Q}^E$ are similarly defined. Each row of $\mathbf{Q}$ can be generated by considering the firing rate of the enabled EXP transitions in the particular marking.

The probability of being in state i at time t is $\Pi_i(t) = Prob\{\mathcal{M}(t) = i\}$. Given that the MEM transition $t_g \in T^G$, with firing time distribution $G^g(x)$, is the only active transition in state $i \in \mathcal{R}^g$ at time t, the so called, *age rate* is defined as follows:

$$h_i(t,x) = \begin{cases} \dfrac{Prob\{\mathcal{M}(t) = i,\ x < a(t) \le x + dx\}}{dx} \cdot \dfrac{1}{1 - G^g(x)}\,, & \text{if } G^g(x) < 1 \\ 0 & \text{if } G^g(x) = 1 \end{cases}$$

where $a(t)$ equals to *age* of the only active MEM transition at time t.

The probability of state transitions due to the firing of a MEM transition is given by a branching probability matrix $\mathbf{\Delta} = \{\Delta_{ij}\}$ whose generic entry has the following meaning [2]:

$$\Delta_{ij} = Prob\{\text{next marking is } j \mid t_g \text{ fires in marking } i\}$$

The following analytical description utilizes some special features of the marking processes of the considered class of *MRSPN*s. As it is mentioned above, the active period of a MEM transition can be concluded either by the firing of the transition, or by a preceding firing of an EXP transition which disables the MEM one. This way a state transition from $\mathcal{R}^\ell$ to $\mathcal{R}^g$ $(\ell \neq g)$ due to the firing of an EXP transition concludes the activity period of t_ℓ and starts a new activity period of t_g. Further, the age of the process $(a(t))$ is continuously increasing during a sojourn in $\mathcal{R}^g, t_g \in T^G$, and $a(t)$ is constant $(a(t) = 0)$ during the sojourn in $\mathcal{R}^E$.

With the above assumptions and definitions, the evolution of the *age rate* $(h_i(t,x),\ i \in \mathcal{R}^g)$ is characterized by the following partial differential equation [11]:

$$\frac{\partial}{\partial t} h_i(t,x) + \frac{\partial}{\partial x} h_i(t,x) = \sum_{k \in \mathcal{R}^g} h_k(t,x)\, q_{ki} \tag{1}$$

hence the *age rate* vector $\mathbf{h}^g(t,x) = \{h_i(t,x)\},\ i \in \mathcal{R}^g$ satisfies:

$$\frac{\partial}{\partial t} \mathbf{h}^g(t,x) + \frac{\partial}{\partial x} \mathbf{h}^g(t,x) = \mathbf{h}^g(t,x)\, \mathbf{Q}^g \tag{2}$$

Given the proper ordering of states the transient state probability vector $\mathbf{\Pi}(t) = \{\Pi_i(t)\}$, can be calculated in the partitioned form: $\mathbf{\Pi}(t) = \{\mathbf{\Pi}^E(t), \mathbf{\Pi}^g(t), \mathbf{\Pi}^\ell(t), \ldots\}$. The process evolution in $\mathcal{R}^E$ is described by the following ordinary differential equation:

$$\begin{aligned}\frac{d}{dt} \mathbf{\Pi}^E(t) &= \mathbf{\Pi}^E(t)\, \mathbf{Q}^E \\ &+ \sum_{t_g \in T^G} \int_0^\infty \mathbf{h}^g(t,x)\, dG^g(x)\, {}^{g,E} \\ &+ \sum_{t_g \in T^G} \mathbf{\Pi}^g(t)\, \mathbf{Q}^{g,E}\end{aligned} \tag{3}$$

In (3), the state probabilities inside $\mathcal{R}^E$ can change: *i)* - by the firing of an EXP transition which results in a new marking (the diagonal elements characterize the transitions also out of $\mathcal{R}^E$) (1st term); *ii)* - by the firing of a general transition when the reached state is in $\mathcal{R}^E$ (2nd term); *iii)* - by the disabling of a general transition when the reached state is in $\mathcal{R}^E$ (3rd term).

The initial conditions are $\mathbf{\Pi}^E(0)$ and $\mathbf{h}^g(0,x) = \mathbf{\Pi}^g(0)\,\hat{\delta}(x-0)$, where $\hat{\delta}(x-0)$ is the Dirac impulse at $x=0$.

The boundary condition for Equation (1) is given by:

$$\begin{aligned}\mathbf{h}^g(t,0) &= \mathbf{\Pi}^E(t)\mathbf{Q}^{E,g} \\ &+ \sum_{t_\ell \in T^G} \int_0^\infty \mathbf{h}^\ell(t,x)\, dG^\ell(x)\, \mathbf{\Delta}^{\ell,g} \\ &+ \sum_{t_\ell \in T^G, \ell \neq g} \mathbf{\Pi}^\ell(t)\, \mathbf{Q}^{\ell,g}\end{aligned} \tag{4}$$

In (4), a general transition t_g can be activated: *i)* - by the firing of an EXP transition in $\mathcal{R}^E$ leading to a state in which t_g is enabled (1st term); *ii)* - by the firing of a general transition t_ℓ when in the reached state t_g is enabled (or re-enabled if $t_g = t_\ell$) (2nd term); *iii)* - by the firing of an EXP transition which disables the active general transition t_ℓ and in the reached marking the general transition t_g is enabled (3rd term). Immediate re-enabling of the general transition is not considered; it would require to further partition $\mathbf{Q}^g$ to distinguish the state jumps that temporarily disables t_g from those that do not.

Once $\mathbf{h}^g(t,x)$ is computed from (1), the transient state probability vector in $\mathcal{R}^g$ can be calculated from:

$$\mathbf{\Pi}^g(t) = \int_0^\infty \mathbf{h}^g(t,x)\,\Big(1 - G^g(x)\Big)\, dx \ . \tag{5}$$

3.2 A numerical analysis method

An iterative algorithm for the numerical approximation of the above equations based on an equidistant (d) discretization of the continuous variables has been proposed in [14]. The algorithm is restricted to the case when there is not initial enabling of general transitions. The steps of the algorithm are the following:

(1) Initially, set $\mathbf{h}^g(nd,0) = 0$ and compute the age rates

$$\mathbf{h}^g(nd,md) = \mathbf{h}^g((n–1)d,(m–1)d)e^{\mathbf{Q}^g\, d}$$

for the next time instant.

(2) Given the age rates $\mathbf{h}^g(nd, md)$, $m = 0, 1, \ldots$, compute the state probabilities $\mathbf{\Pi}^g(nd)$ from (5).
(3) Compute the state probabilities $\mathbf{\Pi}^E(nd)$ from the ordinary differential equation (3).
(4) Recompute the activation rate of the general transitions $\mathbf{h}^g(nd, 0)$ from the boundary conditions (4).
(5) Correct the state probabilities $\Pi^g(nd)$ in order to have the proper sum (with a discretization scheme that conserves the sum of probabilities this correction is not necessary) going back to step 2 or start with the next time instant $(n+1)d$.

An improved numerical procedure, based on the same approach, but with an adaptively varying interval length (d) has been recently described in [16]. The steady-state behavior of the considered class of *MRSPN* can be easily obtained, based on the above set of equations, by setting the time derivatives to 0 [12]. Lindemann proposed an effective numerical method to evaluate the steady-state probabilities of *DSPN* based on Markov renewal theory approach [18,19].

4 Analysis of MRSPNs with *prs* type transitions

4.1 Application of the method of supplementary variables

Let T^G be the set of MEM transitions. The tangible state space $\mathcal{R}$ is partitioned into $|T^G| + 1$ disjoint subsets, as before. $\mathcal{R}^E$ is the set of states in which no general transition can be active ($a(t) = 0$ when $\mathcal{M}(t) \in \mathcal{R}^E$), and $\mathcal{R}^g, t_g \in T^G$ is the set of states in which the general transition t_g can be come active ($a(t) = a_g$ when $\mathcal{M}(t) \in \mathcal{R}^g$). The state probability ($\Pi_i(t)$) and the *age rate* ($h_i(t, x)$) are defined as before.

The stochastic behavior of an active *prs* transition depends on its enabling condition. Let r_i denote the enabling indicator of t_g in state $i \in \mathcal{R}^g$, i.e., $r_i = 1$ if t_g is enabled in state i and $r_i = 0$ if t_g is disabled in state i. Now, we further partition the $\mathcal{R}^g, t_g \in T^G$ set into two disjoint subsets. $\mathcal{R}^{g'}$ ($\mathcal{R}^{g''}$) is the subset of $\mathcal{R}^g$ in which t_g is enabled (disabled). Transition t_g can fire only when $\mathcal{M}(t) \in \mathcal{R}^{g'}$ and it cannot fire when $\mathcal{M}(t) \in \mathcal{R}^{g''}$.

The considered class of *MRSPN* with *prs* type MEM transitions have different features from the above described *prd* case. The active period of a *prs* type MEM transition can be concluded only by the firing of the transition. Since we exclude overlapping active periods of MEM transitions, there is no state transition possible from $\mathcal{R}^g$ to any $\mathcal{R}^\ell, g \neq \ell$ or to $\mathcal{R}^E$ due to the firing of an

EXP transition, i.e., $\mathbf{Q}^{g,\ell} = \mathbf{0}$, $\forall \ell, g$, $g \neq \ell$ and $\mathbf{Q}^{g,E} = \mathbf{0}, \forall g$. (If $\mathbf{Q}^{g,\ell}$, $g \neq \ell$ were different from $\mathbf{0}$, t_g and t_ℓ might become active at the same time. If there were an $i \in \mathcal{R}^g$ and a $j \in \mathcal{R}^E$ such that $\left[\mathbf{Q}^{g,E}\right]_{ij} > 0$ then t_g would be active in $j \in \mathcal{R}^E$, which is in contrast with the definition of $\mathcal{R}^E$.)

The age of the process ($a(t)$) is continuously increasing at rate 1 during a sojourn in $\mathcal{R}^{g'}, t_g \in T^G$, and $a(t)$ is constant during the sojourn in $\mathcal{R}^E$ ($a(t) = 0$) and in $\mathcal{R}^{g"}, t_g \in T^G$ ($a(t) \geq 0$). Hence, there can be a probability mass at $x = 0$ when $\mathcal{M}(t) \in \mathcal{R}^{g"}$. Let denote this by $\mathbf{H}^{g"}(t) = \{H_i^{g"}(t)\}; i \in \mathcal{R}^{g"}$, where

$$H_i^{g"}(t) = Prob\,\{\mathcal{M}(t) = i \in \mathcal{R}^{g"},\, a(t) = 0\} \geq 0\ .$$

For example this probability mass exists if the initial marking is in $\mathcal{R}^E$ and there is an exponential transition leading to a marking in $\mathcal{R}^{g"}$. More generally this mass may be accumulated if a state in $\mathcal{R}^{g"}$ may be entered by the firing of a general transition or directly from $\mathcal{R}^E$.

The next theorem gives the differential equation that describes the evolution of *age rate* for *prs* type transition:

Theorem 1 *The* age rate $h_i(t,x)$ ($i \in \mathcal{R}^g$) *satisfies the partial differential equation*

$$\frac{\partial}{\partial t} h_i(t,x) + r_i \frac{\partial}{\partial x} h_i(t,x) = \sum_{j \in \mathcal{R}^g} h_j(t,x) q_{ji}, \tag{6}$$

Proof: Let us use the notation

$$k_i(t,x) = \frac{Prob\{\mathcal{M}(t) = i, x < a(t) \leq x + dx\}}{dx} \tag{7}$$

To obtain $k_i(t+\delta, x+r_i\delta)$ the following cases have to be considered in the interval $(t, t+\delta)$:

- Neither an EXP nor the active general transition fire with probability

$$\Big(1 + q_{ii}\delta + \sigma(\delta)\Big)\left(1 - \frac{\frac{G(x+\delta)-G(x)}{\delta}}{1-G(x)}\,\delta\right) = \Big(1 + q_{ii}\delta + \sigma(\delta)\Big)\frac{1-G(x+\delta)}{1-G(x)},$$

  where $\sigma(x)$ is such that $lim_{x\to 0}\sigma(x)/x = 0$.

- There is a state transition from j to i due to the firing of an EXP transition and the general transition does not fire with probability

$$\left(q_{ji}\delta + \sigma(\delta)\right) \frac{1 - G(x+\delta)}{1 - G(x)}.$$

- There are more than one state transitions due to the firing EXP transitions with probability $\sigma(\delta)$.

The probability of firing of the active general transition does not contribute to the probability $k_i(t+\delta, x + r_i\delta)$ because its firing resets the age variable. This probability appears in the boundary conditions. In the above mentioned cases the increase of the age variable during the $(t, t+\delta)$ interval is:

- if neither an EXP nor the active general transition fire:

$$a(t) = a(t+\delta) - r_i\delta \ ,$$

- if there is a state transition from j to i due to the firing of an EXP transition and the general transition does not fire :

$$a(t+\delta) - r_j\delta < a(t) < a(t+\delta) - r_i\delta \quad \text{if:} r_i < r_j,$$

$$a(t+\delta) - r_i\delta < a(t) < a(t+\delta) - r_j\delta \quad \text{if:} r_i > r_j,$$

  to simplify the description of this cases we introduce the notation $\mathcal{O}(x)$ such that $lim_{x\to 0}\mathcal{O}(x) = 0$, hence

$$a(t) = a(t+\delta) + \mathcal{O}(\delta) \ ,$$

- if there are more than one state transitions due to the firing EXP transitions:

$$a(t{+}\delta) - r_{min}\delta < a(t) < a(t{+}\delta) - r_{max}\delta \quad \Longrightarrow \quad a(t) = a(t{+}\delta) + \mathcal{O}(\delta),$$

  where $r_{min} = \min\limits_{j\in\mathcal{R}^g} r_j$ and $r_{max} = \max\limits_{j\in\mathcal{R}^g} r_j$.

Based on these considerations:

$$k_i(t+\delta, x+r_i\delta) = \frac{1 - G(x+r_i\delta)}{1 - G(x)} \ (1 + q_{ii}\delta) \ k_i(t,x) + \sum_{j\in\mathcal{R}^g, j\neq i} \frac{1 - G(x+r_i\delta)}{1 - G(x+\mathcal{O}(\delta))} \ q_{ji}\delta \ k_j(t, x+\mathcal{O}(\delta)) + \sigma(\delta).$$

Since $k_i(t,x) = h_i(t,x)\Big(1 - G(x)\Big)$ it may be rewritten as

$$h_i(t+\delta, x+r_i\delta) = (1+q_{ii}\delta)\ h_i(t,x) + \sum_{j\in\mathcal{R}^g, j\neq i} q_{ji}\delta\ h_i(t, x+\mathcal{O}(\delta)) + \sigma(\delta),$$

from which

$$h_i(t+\delta, x+r_i\delta) - h_i(t,x) =$$

$$q_{ii}\delta\ h_i(t,x) + \sum_{j\in\mathcal{R}^g, j\neq i} q_{ji}\delta\ h_i(t, x+\mathcal{O}(\delta)) + \sigma(\delta) =$$

$$\sum_{j\in\mathcal{R}^g} q_{ji}\delta\ h_j(t, x+\mathcal{O}(\delta)) + \sigma(\delta).$$

Dividing both sides by δ and using some algebra we have

$$\frac{h_i(t+\delta, x+r_i\delta) - h_i(t, x+r_i\delta)}{\delta} + \frac{h_i(t, x+r_i\delta) - h_i(t,x)}{\delta} = \sum_{j\in\mathcal{R}^g} q_{ji}\ \ h_i(t, x+\mathcal{O}(\delta)) + \sigma(\delta).$$

Taking the limit $\delta \to 0$ gives the theorem. □

Using the vector notation

$$\frac{\partial}{\partial t}\mathbf{h}^g(t,x) + \frac{\partial}{\partial x}\mathbf{h}^g(t,x)\ \mathbf{R}^g = \mathbf{h}^g(t,x)\,\mathbf{Q}^g \tag{8}$$

where the diagonal matrix $\mathbf{R}^g = diag < r_i >,\ i \in \mathcal{R}^g$.

In (6) the enabling indicator (r_i) captures the fact that the age does not increase in state $i \in \mathcal{R}^{g}$". If for $\forall i \in \mathcal{R}^G$ $r_i = 1$ we have the same differential equation as for *prd* type transitions.

The process evolution in $\mathcal{R}^E$ is described by the following theorem.

Theorem 2 *The transient state probability* $\Pi_i(t), i \in \mathcal{R}^E$ *satisfies the follow-*

ing ordinary differential equation:

$$\frac{d}{dt}\Pi_i(t) = \sum_{j\in\mathcal{R}^E} \Pi_j(t) q_{ji} + \sum_{t_g\in T^G} \sum_{j\in\mathcal{R}^g} \int_0^\infty h_j(t,x)\ r_j\ dG^g(x)\ \Delta_{ji}^{g,E}. \qquad (9)$$

Proof: As in Theorem 1 we consider the cases that play role in the evolution of $\Pi_i(t)$ in the interval $(t, t+\delta)$:

- No EXP transition fires with probability $1 + q_{ii}\delta + \sigma(\delta)$.
- There is one state transition from j to i due to the firing of an EXP transition with probability $q_{ji}\delta + \sigma(\delta)$.
- There are multiple state transitions due to the firing of EXP transitions with probability $\sigma(\delta)$.
- An active MEM transition fires and the next tangible state is i with probability (using the notation defined in Theorem 1)

$$\sum_{t_g\in T^G} \sum_{j\in\mathcal{R}^g} \int_{x=0}^\infty k_j(t,x)\ r_j\ \frac{G^g(x+\delta) - G^g(x)}{1 - G^g(x)}\ dx\ \Delta_{ji}^{g,E} + \sigma(\delta) =$$

$$\sum_{t_g\in T^G} \sum_{j\in\mathcal{R}^g} \int_{x=0}^\infty h_j(t,x)\ r_j\ \Big(G^g(x+\delta) - G^g(x)\Big)\ dx\ \Delta_{ji}^{g,E} + \sigma(\delta).$$

Considering the above possibilities we have

$$\Pi_i(t+\delta) = (1 + q_{ii}\delta)\Pi_i(t) + \sum_{j\in\mathcal{R}^E, j\neq i} q_{ji}\delta\ \Pi_j(t)$$
$$+ \sum_{t_g\in T^G} \sum_{j\in\mathcal{R}^g} \int_{x=0}^\infty h_j(t,x)\ r_j\ (G^g(x+\delta) - G^g(x))\ dx\ \Delta_{ji}^{g,E} + \sigma(\delta),$$

which may be rewritten as (using similar steps as in Theorem 1)

$$\frac{\Pi_i(t+\delta) - \Pi_i(t)}{\delta} = \sum_{j\in\mathcal{R}^E} \Pi_j(t) q_{ji}$$
$$+ \sum_{t_g\in T^G} \sum_{j\in\mathcal{R}^g} \int_{x=0}^\infty h_j(t,x)\ r_j\ \frac{G^g(x+\delta) - G^g(x)}{\delta}\ dx\ \Delta_{ji}^{g,E} + \sigma(\delta),$$

and taking the limit $\delta \to 0$ gives the theorem. □

Using vector notations (9) may be written as

$$\begin{aligned} \frac{d}{dt}\,\mathbf{\Pi}^E(t) &= \mathbf{\Pi}^E(t)\,\mathbf{Q}^E \\ &+ \sum_{t_g \in T^G} \int_0^\infty \mathbf{h}^g(t,x)\ \mathbf{R}^g\, dG^g(x)\,\mathbf{\Delta}^{g,E} \end{aligned} \tag{10}$$

The evolution of the probability mass $\mathbf{H}^{g"}(t)$ in $\mathcal{R}^{g"}$ is described by the following ordinary differential equation (this and the following equations may be proved using similar steps as in Theorem 1 and 2):

$$\begin{aligned} \frac{d}{dt}\mathbf{H}^{g"}(t) &= \mathbf{\Pi}^E(t)\mathbf{Q}^{E,g"} + \mathbf{H}^{g"}(t)\ \mathbf{Q}^{g"} \\ &+ \sum_{t_\ell \in T^G} \int_0^\infty \mathbf{h}^\ell(t,x)\ \mathbf{R}^\ell\, dG^\ell(x)\,\mathbf{\Delta}^{\ell,g"} \end{aligned} \tag{11}$$

Equation (11) is very similar to (10), but in (11) there can be state transition due to the firing of an EXP transition not only inside $\mathcal{R}^{g"}$, but also from $\mathcal{R}^E$ to $\mathcal{R}^{g"}$.

The boundary condition for Equation (6) is given by:

$$\begin{aligned} \mathbf{h}^{g'}(t,0) &= \mathbf{\Pi}^E(t)\ \mathbf{Q}^{E,g'} + \mathbf{H}^{g"}(t)\ \mathbf{Q}^{g",g'} \\ &+ \sum_{t_\ell \in T^G} \int_0^\infty \mathbf{h}^\ell(t,x)\ \mathbf{R}^\ell\, dG^\ell(x)\,\mathbf{\Delta}^{\ell,g'} \end{aligned} \tag{12}$$

The boundary condition (12) means that the real initialization of t_g, when its age starts increasing, can happen: *i)* - by the firing of an EXP transition in $\mathcal{R}^E$ leading to a state in which t_g is enabled (1st term); *ii)* - by the firing of an EXP transition when t_g was disabled and its age was 0 (2nd term); *iii)* - by the firing of a MEM transition t_ℓ when in the reached state t_g is enabled (or re-enabled if $t_g = t_\ell$) (3rd term). For the technique to prove (12) see [17].

The transient state probability vector in $\mathcal{R}^g$ can be calculated as:

$$\begin{aligned} \mathbf{\Pi}^{g'}(t) &= \int_0^\infty \mathbf{h}^{g'}(t,x)\ (1 - G^g(x))\,dx \\ \mathbf{\Pi}^{g"}(t) &= \int_0^\infty \mathbf{h}^{g"}(t,x)\ (1 - G^g(x))\,dx + \mathbf{H}^{g"}(t) \end{aligned} \tag{13}$$

The initial conditions are $\mathbf{\Pi}^E(0)$, $\mathbf{h}^{g'}(0,x) = \mathbf{\Pi}^{g'}(0)\,\hat{\delta}(x-0)$ and $\mathbf{H}^{g"}(0) =$

$\mathbf{\Pi}^{g"}(0)$.

The steady state behavior [22] can also be obtained based on the above description of the process by eliminating the time dependence from the description measures [15] (i.e., eliminating the parameter t of the functions and setting all the derivatives of t to 0).

4.2 A numerical analysis method

In this section, we introduce a simple but asymptotically correct numerical method to approximate the stochastic behavior described by the above set of equations. We also use equidistant discretization of the time and the age, but in contrast with the previously discussed iterative method based on an implicit discretization scheme [14], we propose an explicit scheme for the calculation of the transient parameters in each time interval. This method is faster and simpler (easier to implement), but might be less accurate.

The initial parameters are given by the initial marking of the *PN*, i.e., $\mathbf{\Pi}^E(0)$ and $\mathbf{h}^g(0,0) = \mathbf{\Pi}^g(0)$.

The evaluation of the transient behavior at time nd is composed by the following steps:

(1) Compute the age rates in the next time instant ($i \in \mathcal{R}^g$, and $m \geq 1$):

$$h_i(nd, md) = \sum_{k \in \mathcal{R}^g} h_k((n–1)d, (m–r_k)d) \; \left[e^{\mathbf{Q}^g \, d}\right]_{ki} \; .$$

(2) Compute the state probabilities $\mathbf{\Pi}^E(nd)$ from the ordinary differential equation (10) ($i \in \mathcal{R}^E$):

$$\mathbf{\Pi}^E(nd) = \mathbf{\Pi}^E((n–1)d) \; e^{\mathbf{Q}^E \, d} + \sum_{t_g \in T^G} \sum_{j=1}^{j^g_{max}} \mathbf{h}^{g'}((n–1)d, (j–1)d) \; e^{\mathbf{Q}^{g'} \, d} \; \mathbf{\Delta}^{g',E} \; G^g_j$$

where $G^g_j = G^g(jd) - G^g((j–1)d)$ and the firing time of t_g is less than $j^g_{max}d$.

(3) Compute the probability mass $\mathbf{H}^{g"}(nd)$ from the ordinary differential

equation (11) ($i \in \mathcal{R}^{g"}$):

$$
\begin{aligned}
&\mathbf{H}^{g"}(nd) = \mathbf{H}^{g"}((n–1)d)\ e^{\mathbf{Q}^{g"}\,d} \\
&\ + \boldsymbol{\Pi}^{E}((n–1)d)\ e^{\mathbf{Q}^{E,g"}\,d} \\
&\ + \sum_{t_\ell \in T^G} \sum_{j=1}^{j^g_{max}} \mathbf{h}^{\ell'}((n–1)d,(j–1)d)\ e^{\mathbf{Q}^{\ell'}\,d}\ \boldsymbol{\Delta}^{\ell',g"}\ G^g_j
\end{aligned}
$$

(4) Compute the initialization rate of the general transitions $\mathbf{h}^{g'}(id, 0)$ from the boundary conditions (12) ($i \in \mathcal{R}^{g'}$):

$$
\begin{aligned}
&\mathbf{h}^{g'}(nd, 0) = \boldsymbol{\Pi}^{E}((n–1)d)\ e^{\mathbf{Q}^{E,g'}\,d} \\
&\ + \mathbf{H}^{g"}((n–1)d)\ e^{\mathbf{Q}^{g",g'}\,d} \\
&\ + \sum_{t_\ell \in T^G} \sum_{j=1}^{j^\ell_{max}} \mathbf{h}^{\ell'}((n–1)d,(j–1)d)\ e^{\mathbf{Q}^{\ell'}\,d}\ \boldsymbol{\Delta}^{\ell',g'}\ G^g_j
\end{aligned}
$$

(5) Given the age rates $\mathbf{h}^{g}(nd, md),\ m = 0, 1, \ldots$, compute the state probabilities $\boldsymbol{\Pi}^{g}(nd)$ from (13). For $\mathcal{R}^{g'}$:

$$
\boldsymbol{\Pi}^{g'}(nd) = \sum_{j=0}^{j^g_{max}} \mathbf{h}^{g'}(nd, jd)\ \Big(1 - G^g(jd)\Big)
$$

and for $\mathcal{R}^{g"}$:

$$
\boldsymbol{\Pi}^{g"}(nd) = \mathbf{H}^{g"}(nd) + \sum_{j=1}^{j^g_{max}} \mathbf{h}^{g"}(nd, jd)\ \Big(1 - G^g(jd)\Big)
$$

(6) Repeat the same steps at the next time instant $(n + 1)d$.

The algorithm is able to handle the initial enabling of general transitions, this case is captured when one of the age rates $\mathbf{h}^{g'}(0, 0) = \boldsymbol{\Pi}^{g'}(0)$ is positive. Due to the described equidistant explicit discretization scheme the discrete quantities in $\mathbf{h}^{g'}(t, x)$ are carried precisely and in a natural way.

Let us denote the number of markings in which a general transition t_g is active by n_g and the largest considered firing time of t_g by u_g. Assuming that the number of markings in which no general transition is active is n_E the size of the discretized state space is

$$
n_E + \sum_{t_g \in T^G} n_g \lceil u_g/d \rceil,
$$

where $\lceil x \rceil$ denotes the smallest natural number larger than x.

partial differential equation
$\frac{\partial}{\partial t}\mathbf{h}^g(t,x) + \frac{\partial}{\partial x}\mathbf{h}^g(t,x)\ \mathbf{R}^g = \mathbf{h}^g(t,x)\,\mathbf{Q}^g$
ordinary differential equations
$\frac{d}{dt}\mathbf{\Pi}^E(t) = \mathbf{\Pi}^E(t)\,\mathbf{Q}^E + \sum_{t_g \in T^G} \int_0^\infty \mathbf{h}^g(t,x)\ \mathbf{R}^\ell\, dG^g(x)\,\mathbf{\Delta}^{g,E} + \sum_{t_g \in T^G_{prd}} \mathbf{\Pi}^g(t)\,\mathbf{Q}^{g,E}$
$\begin{aligned}\frac{d}{dt}\mathbf{H}^{g''}(t) &= \mathbf{\Pi}^E(t)\mathbf{Q}^{E,g''} + \mathbf{H}^{g''}(t)\ \mathbf{Q}^{g''} \\ &+ \sum_{t_\ell \in T^G} \int_0^\infty \mathbf{h}^\ell(t,x)\ \mathbf{R}^\ell\, dG^\ell(x)\,\mathbf{\Delta}^{\ell,g''} + \sum_{t_\ell \in T^G_{prd}, \ell \neq g} \mathbf{\Pi}^\ell(t)\,\mathbf{Q}^{\ell,g''}\end{aligned}$
boundary condition
$\begin{aligned}\mathbf{h}^{g'}(t,0) &= \mathbf{\Pi}^E(t)\ \mathbf{Q}^{E,g'} + \mathbf{H}^{g''}(t)\ \mathbf{Q}^{g'',g'} \\ &+ \sum_{t_\ell \in T^G} \int_0^\infty \mathbf{h}^\ell(t,x)\ \mathbf{R}^\ell\, dG^\ell(x)\,\mathbf{\Delta}^{\ell,g'} + \sum_{t_\ell \in T^G_{prd}, \ell \neq g} \mathbf{\Pi}^\ell(t)\,\mathbf{Q}^{\ell,g'}\end{aligned}$
state probabilities
$\mathbf{\Pi}^{g'}(t) = \int_0^\infty \mathbf{h}^{g'}(t,x)\ (1 - G^g(x))\,dx$
$\mathbf{\Pi}^{g''}(t) = \int_0^\infty \mathbf{h}^{g''}(t,x)\ (1 - G^g(x))\,dx + \mathbf{H}^{g''}(t)$

Table 1
Set of equations for *SPN*s with *prd* and *prs* general transitions

4.3 Complex models

The *SPN* model of real systems may contain MEM transitions with associated *prd* and *prs* policy as well. The set of equations provided for the analytical description of the marking process with only *prd* type MEM transitions (Section 3) and the set of equation describes the *SPN* models with only *prs* type MEM transitions (Section 4) can be combined for the analysis of these *SPN* models.

In order to have the set of equations for the combined case T^G is partitioned into two disjoint sets: T^G_{prd} (T^G_{prs}) contains the general transitions with associated *prd* (*prs*) policy. Since overlapping active periods are excluded we have the following restrictions on $\mathbf{Q}$: $\mathbf{Q}^{g,\ell} = \mathbf{0}, \forall g \in T^G_{prs}, \forall \ell \in T^G$, and $g \neq \ell$, $\mathbf{Q}^{g,E} = \mathbf{0}, \forall g \in T^G_{prs}$. As defined in Section 4, r_i is the enabling indicator of t_g in state $i \in \mathcal{R}^g$, i.e., $r_i = 1$ if t_g is enabled in state i and $r_i = 0$ if t_g is disabled in state i (note that $r_i = 1, \forall i \in \mathcal{R}^g, \forall g \in T^G_{prd}$). The differential equation that describes the evolution of the *age rate* is the same for the combined case as (8). The process evolution in $\mathcal{R}^E$ is described by the same ordinary differential equation as for *prd* type transitions (3). The boundary condition for the *age rate* associated with a *prd* type transition $t_g, g \in T^G_{prd}$ is (4), while it is (12) for a *prs* type transition $t_g, g \in T^G_{prs}$. The evolution of the probability mass $\mathbf{H}^{g"}(t)$ in $\mathcal{R}^{g"}$ ($g \in T^G_{prs}$) is described by (11). The transient state probabilities may be calculated using (5) and (13). The set of equations describes the behavior of *SPN*s with *prd* and *prs* general transitions is provided in Table 1.

There is an *SPN* model with combined firing policies evaluated among the subsequent numerical examples.

5 Numerical examples

The above introduced numerical method (with equidistant intervals, explicit evaluation at each interval) has been used to evaluate the following examples.

5.1 Terminal system

The *SPN* of Figure 1a models a system of 2 terminals. The jobs submitted by terminal 2 have higher priority and preempt the jobs submitted by terminal 1. The server adopts a *prs* service discipline, i.e. after a preemption of the lower priority job the service of the same preempted job resumes from the point it was preempted, once the server becomes available again. Place p_1 (p_3) signifies that terminal 1 (2) is in the thinking phase, while place p_2 (p_4) indicates job from terminal 1 (2) under service. Transitions t_1 and t_3 are EXP and model the submission of a job of type 1 or 2, respectively. t_2 is a MEM transition and represents the completion of service of the lower priority job (coming from terminal 1). The firing time of transition t_2 is assumed to be uniformly distributed with a *prs* service discipline. Transition t_4 models the service time of a higher priority job. Its firing time is exponentially distributed. The inhibitor arc from p_4 to t_2 models the described preemption mechanism: as soon as a job from terminal 2 is submitted for processing, the job from terminal 1 under service (if any) is interrupted. After the higher priority is processed, the service of the lower priority job is continued. The associated reachability graph is shown in Figure 1b.

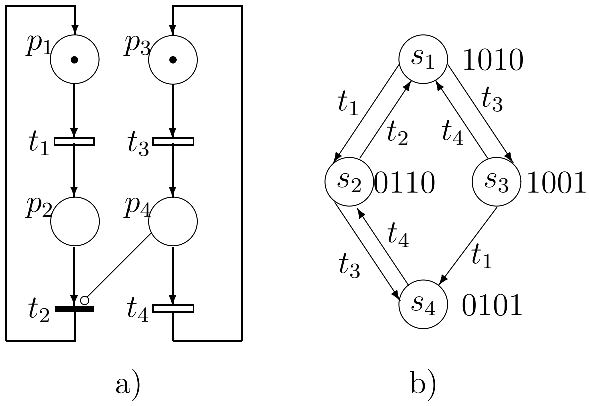

Fig. 1. *PN* model of the terminal system

The transient and the steady state probabilities of this model was solved assuming the following values:

- firing rate of EXP transitions t_1,t_3 and t_4: $\lambda_1 = \lambda_3 = 0.5$, $\lambda_4 = 1$;
- service time of lower priority job (transition t_2): uniformly distributed between 0.5 and 1.5;
- step size: $d = 0.01$.

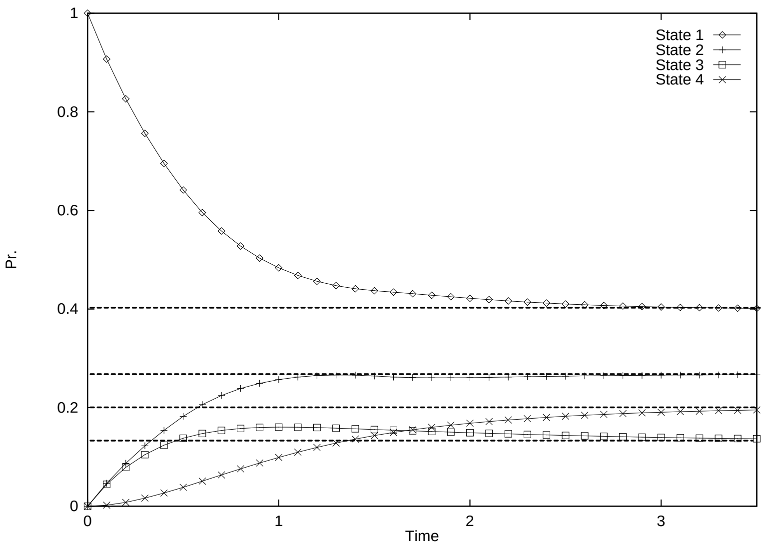

Fig. 2. Transient probabilities of the marking process

The numerical results obtained by the method based on the Markov renewal theory [5,7]. and the supplementary variable approach, proposed in this paper, were compared. The exact steady state probabilities are $p_{s1} = 6/15, p_{s2} =$

$4/15, p_{s3} = 2/15, p_{s4} = 3/15$. The numerical results obtained by the two methods shows a proper coincidence (Figure 2).

Note, that in the job completion example both $\mathbf{H}^{t_3"}(t)$ and $\mathbf{H}^{t_4"}(t)$ were always 0, i.e., t_3 (t_4) was always enabled before reaching $\mathcal{R}^{t_3"}$ ($\mathcal{R}^{t_4"}$), and there was not probability mass at $x = 0$. In this second example there is a probability mass at $x = 0$ in $\mathcal{R}^{t_2"} = s_4$ to consider.

5.2 Job completion

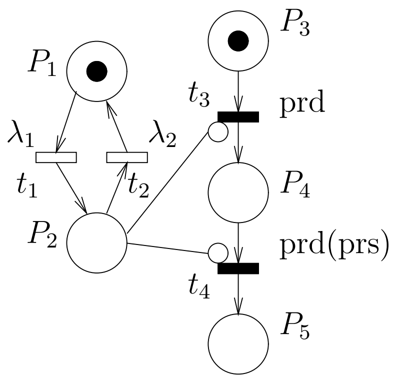

Fig. 3. *PN* model for completion time of two phase job

The *SPN* on Figure 3 models a system that executes a job with two phases. A token in P_3 (P_4) signifies that the job is in the first (second) phase. The work is done when the token moves to place P_5. Either phase of the job is preempted when there is a token at place P_2. The token between P_1 and P_2 changes its place according to two EXP transitions t_1 and t_2 with parameter λ_1 and λ_2. Transition t_3 and t_4 are generally distributed. The memory policy associated to t_3 is *prd*, so if this phase is preempted the job starts from the beginning. For the second phase we consider both *prd, prs* policies, in case of *prs* policy the job restarts from where it was preempted. Figure 5 shows the probabilities of the phases versus time when the second phase adopts *prd* policy. The same can be seen on Figure 6 in case of *prs* policy for t_4. The numerical parameters are:

- the parameters for the EXP transitions are: $\lambda_1 = 0.5, \lambda_2 = 2$,
- t_3 has deterministic distribution with firing time 1.5,
- t_4 has deterministic distribution with firing time 1.0,
- step size: $d = 0.01$.

Figure 4 shows the *SPN* of our second example. It is similar to the previous one. The difference is that if the second phase is done the work starts again. The probabilities are calculated using the same parameters (Figure 7).

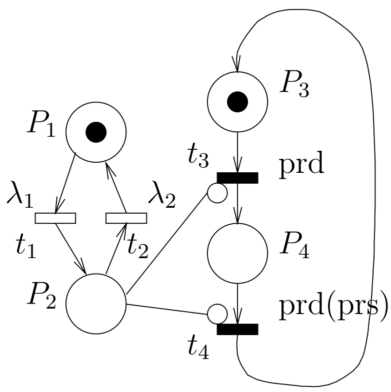

Fig. 4. *PN* model for continuous service of two phase jobs

Note that MEM transitions with associated *prd* and *prs* policy can be found in the same *SPN* models in these examples.

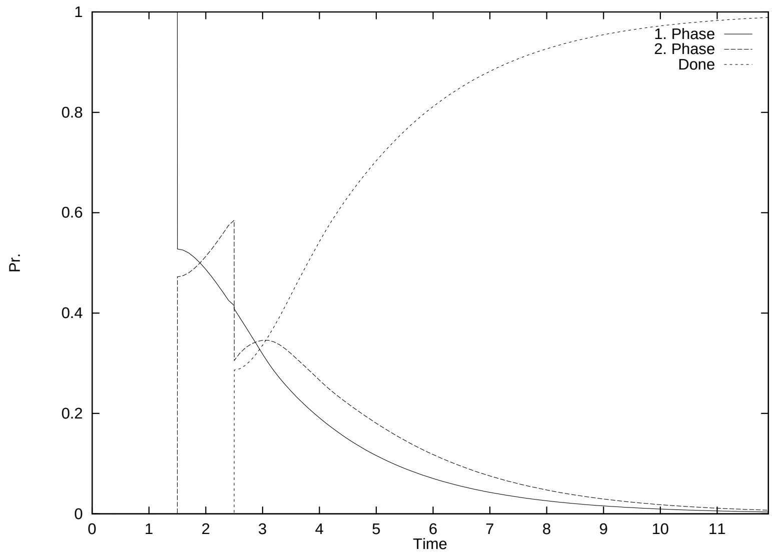

Fig. 5. Probabilities of the phases in case t_4 is *prd* type

6 Conclusion

This paper discusses the extension of the supplementary variable approach to the analysis of *MRSPN*s with *prs* type MEM transitions. It defines the set of differential and integral equations that describe the stochastic marking process.

A numerical procedure based on the exact analytical description is proposed for the (numerical) transient analysis of the considered class of models. Regarding the size of the analyzable models the applicability of this method is similar to the one proposed for *MRSPN*s with *prd* type transitions [11] and

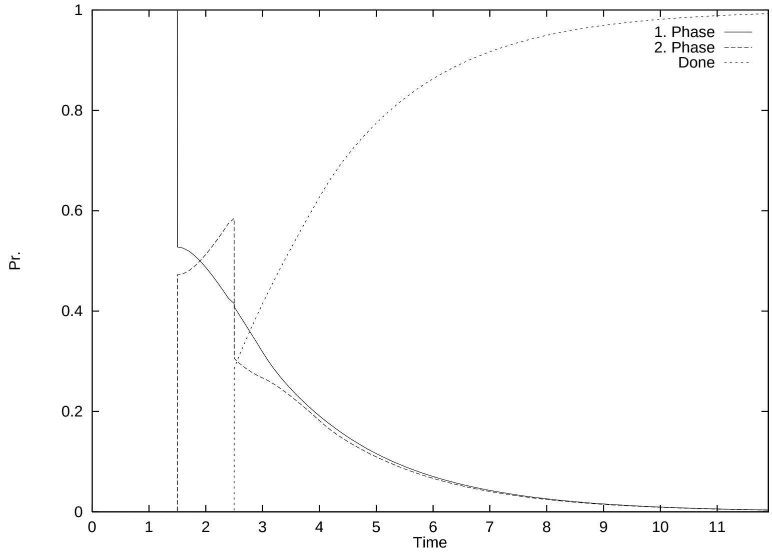

Fig. 6. Probabilities of the phases in case t_4 is *prs* type

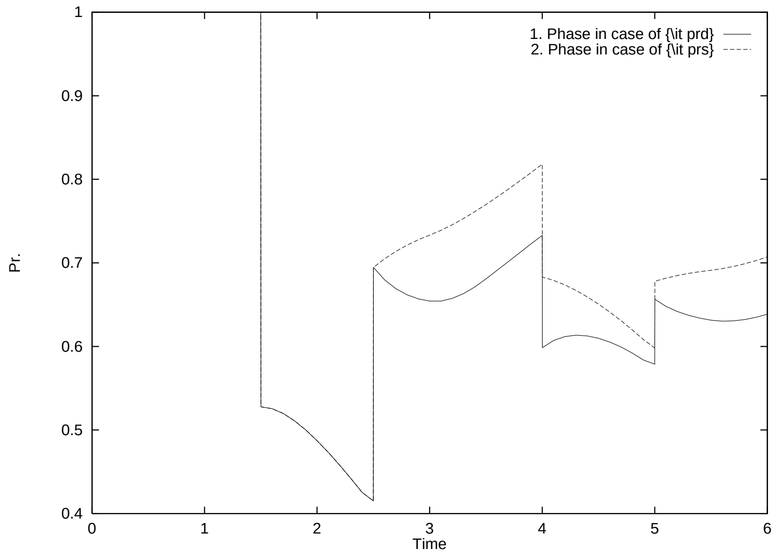

Fig. 7. Probabilities of the phases in case t_4 is *prd* or *prs* type with continuous service

usually is better than the Laplace transform domain method [7], which was the only available analysis method before.

Two application examples are evaluated. The first numerical experiences about the proposed numerical method are promising regarding both the accuracy and the computational complexity.

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