Moment Bounds for Acyclic Discrete and Continuous Phase Type Distributions of Second Order

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Abstract

The problem of matching moments to phase type (PH) distributions occurs in many applications. Often, low dimensions of the selected distributions are desired. On the other hand, it is obvious that the three parameters of acyclic PH distributions of second order - be they continuous (ACPH(2)) or discrete (ADPH(2)) - can be fitted to three given moments provided that these are feasible. For both types of PH distributions, this paper provides the precise permissible ranges by giving the immanent lower and upper (if existing) bounds for the first three moments. For moments which obey these bounds an exact and minimal (with respect to the dimension of the representation) analytic mapping of three moments into ACPH(2) or ADPH(2) is presented.

1 Introduction

Continuous [13] and discrete [15] phase type (PH) distributions were formalized already in the 1970s. Since then, many research activities and application-oriented work have been devoted to this field of stochastic modeling with emphasis on the continuous PH distributions (e.g., [1, 5, 16, 14, 10]). In recent years, discrete PH distributions have attracted increasing attention, because their relation to physical observations and their usefulness in the numerical solution of non-Markovian processes have been observed.

Approximating general distributions - possibly given only partially in form of measured data or some moments - with PH distributions is widely covered in the literature. Corresponding techniques can be grouped into sophisticated numerical methods (which make use of nonlinear programming [10] or statistical procedures like maximum likelihood [3] or minimum distance [17] estimation) and straightforward analytic methods (also called the method of moments, e.g., [20, 9]). Our focus in this paper is on the method of moments. In general, the superior efficiency of such techniques renders them ideally apt for many applications. The procedure proposed in this paper was developed in the context of traffic-based decomposition of queueing networks, where moments of (continuous) service/idle times and of the number of customers served during a queue's busy period need to be matched [8]. As orders of these representations affect the involved dimensions (e.g., of matrices, state spaces) multiplicatively, it is a crucial issue to keep these orders to a minimum, i.e., two states in our case.

Several (acyclic) PH representations of order 2 - prevalently for the continuous case - have been published, but are either based on the first two moments only (e.g., [18, 11, 7]) or are restricted to the hyperexponential situation (i.e., the squared coefficient of variation of the considered continuous distribution is greater than 1.0 [20, 2]). Another analytic method matches three feasible moments into a four-state continuous PH representation [9]. Note that - although two-moment queueing approximations are quite common - they may lead to serious errors, especially when the squared coefficient of variation is high. Empirical studies (see e.g., [2]) conclude that consideration of the third moment can compensate for large parts of these deviations. In this paper, moment fitting will lead to canonical representations of acyclic continuous or discrete PH distributions of second order, in short ACPH(2) and ADPH(2) respectively. These canonical forms [6, 4] are unique and minimal (in terms of number of parameters) representations of respective acyclic PH distributions while retaining the full generality of acyclic PH distributions up to the given order, i.e., order 2 in our case. Still, the moments of ACPH(2) and ADPH(2) distributions are subject to a few restrictions. With respect to the second moments, the squared coefficients of variation (defined as the variance of the distribution divided by the squared mean ($= {f_1}^2$)) must be greater than or equal to 0.5 for ACPH(2) [1] and for ADPH(2) the squared coefficients of variation must be greater than or equal to $0.5 - \frac{1}{f_1}$ if $2 \leq f_1$ or to $2 \cdot (f_1 - 1)$ if $1 \leq f_1 < 2$ [19].

One goal of this paper is to present - for both the continuous and discrete case - the bounds of the third moment as a function of the first two, namely in the respective full range of the squared coefficient of variation (including the hypoexponential/hypogeometric region). ACPH(2) distributions will be introduced in the next section, which also shows how the mentioned bounds are derived. Section 2 will be paralleled by an analogous treatment for ADPH(2) distributions in Section 3. In Section 4, we outline how the bounds of the previous sections may be utilized in a fitting procedure that eventually matches given first three moments to canonical forms of ACPH(2) and ADPH(2) distributions. A direct comparison of the continuous and discrete procedures in this section reveals the analogies and peculiarities of both cases. Section 5 conclude the paper.

2 The canonical ACPH(2) distribution and moment bounds

Generally, the random variable X associated with an arbitrary continuous PH distribution function $F_X(t)$ represents the time to absorption in a finite continuous-time Markov chain (with s transient states), or more formally: $F_X(t) = 1 - \boldsymbol{\alpha} e^{\boldsymbol{T} t}\boldsymbol{e}$. The nonsingular $(s \times s)$-matrix $\boldsymbol{T}$ denotes the generator of the transient Markov chain ($(\boldsymbol{T})_{ii} \leq 0$ for $1 \leq i \leq s$, $(\boldsymbol{T})_{ij} \geq 0$ for $i \neq j$ so that $(\boldsymbol{T}\boldsymbol{e})_i \leq 0$, but $\boldsymbol{T}\boldsymbol{e} \neq \boldsymbol{0}$). The s-dimensional vector $\boldsymbol{\alpha}$ is the initial distribution and $\boldsymbol{e}$ is the s-dimensional vector of ones. Note that the tuple $(\boldsymbol{\alpha}, \boldsymbol{T})$ completely characterizes the continuous PH distribution with power moments

$$m_i = E[X^i] = i!\, \boldsymbol{\alpha} \,(-\boldsymbol{T})^{-i} \boldsymbol{e} \ . \tag{1}$$

In this paper, we focus on the following specific class of continuous PH distributions: First, we consider the subclass of acyclic distributions, which admits minimal representations called canonical forms [6]. These distributions can be encoded by acyclic graphs so that $\boldsymbol{T}$ is an upper triangular matrix (with an appropriate ordering of the s states). Second, we study ACPH distributions of order 2, i.e., $s = 2$. The canonical representation $(\boldsymbol{\alpha}, \boldsymbol{T})$ is then given by

$$\boldsymbol{\alpha} = (p, 1-p) \quad \text{and} \quad \boldsymbol{T} = \begin{vmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{vmatrix} \ , \tag{2}$$

where $0 \leq p \leq 1$ and $0 < \lambda_1 \leq \lambda_2$. Figure 1 shows the related graph, where the filled circle depicts the absorbing state.

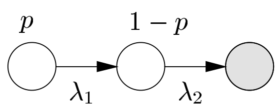

Figure 1: Canonical form of ACPH(2) distribution

Of course, the power moments can be computed directly from (1), but it might be more intuitive to have a look at the Laplace transform of the random variable X:

$$G_X(s) = E[e^{-sX}] = p\frac{\lambda_1}{s+\lambda_1}\frac{\lambda_2}{s+\lambda_2} + (1-p)\frac{\lambda_2}{s+\lambda_2} \ .$$

The first three power moments of X are:

$$m_1 = E[X] = -\frac{d}{ds}G_X(s)|_{s=0} = \frac{\lambda_1 + p\,\lambda_2}{\lambda_1\lambda_2} \quad (3)$$

$$m_2 = E[X^2] = \frac{d^2}{ds^2}G_X(s)|_{s=0} = \frac{2\,({\lambda_1}^2 + p\,\lambda_1\lambda_2 + p\,{\lambda_2}^2)}{{\lambda_1}^2{\lambda_2}^2} \quad (4)$$

$$m_3 = E[X^3] = -\frac{d^3}{ds^3}G_X(s)|_{s=0} = \frac{6\,({\lambda_1}^3 + p\,{\lambda_1}^2\lambda_2 + p\,\lambda_1{\lambda_2}^2 + p\,{\lambda_2}^3)}{{\lambda_1}^3{\lambda_2}^3} \quad (5)$$

Having gone from the distribution parameters p, λ_1, λ_2 to the power moments m_1, m_2, m_3, we would now like to find the reverse way (and succeed therein in Section 4). First of all, we observe that not any arbitrary triple (m_1, m_2, m_3) can be transformed back to some valid parameter set $(p, \lambda_1, \lambda_2)$. For example, nonpositive values for m_1 will obviously render the triple infeasible (since ACPH(2) distributions describe nonnegative random variables). Analogously, the other moments are bounded - possibly from more than one side. For the second moment, Aldous and Shepp provided the (order-independent) result that "the least variable phase-type distribution is Erlang" [1]. In other words and for $s = 2$, the squared coefficient of variation c_X^2 of an ACPH(2) distribution must satisfy:

$$c_X^2 = \frac{m_2}{{m_1}^2} - 1 \geq 0.5 \quad \Leftrightarrow \quad m_2 \geq 1.5\,{m_1}^2 \ .$$

Since the ACPH(2) class contains the Erlang-2 distribution ($p = 1, \lambda_1 = \lambda_2$), this bound is tight. It can be obtained from the formulae (3) and (4) by equating to 0 the derivative of m_2 with respect to m_1 (after having exploited the structural information $p = 1, \lambda_1 = \lambda_2$). Similarly, the bounds for the third moment m_3 can be found, where it turns out however that the bound behavior strongly depends on the precise value of c_X^2 or - expressed alternatively - on the relationship between the first two power moments. Figure 2 illustrates the typical features of the third-moment bounds for a fixed value $m_1 = \frac{4}{3}$ ($= m$ in the figure). While for $c_X^2 > 1$ only a lower bound exists, both a lower and an upper bound limit m_3 to a rather small region for $0.5 \leq c_X^2 \leq 1$.

Table 1 gives the derived functions of the bounds along with the respectively employed structural information in the last column. This information documents which types of ACPH(2) distributions attain the specific bounds. At $(c_X^2 = 1, m_3 = 6\,{m_1}^3)$, we have a singular point: At this point the one-dimensional exponential distribution with parameter $\lambda_2 = \frac{1}{m_1}$ ($p = 0, \lambda_1 =$ irrelevant) fulfills the conditions of the coordinates[1]. This point lies on the dotted line of Figure 2 defined by

$$c = 3\,{m_2}^2 - 2\,m_1 m_3 = 0 \quad \Leftrightarrow \quad m_3 = \frac{3}{2}\,{m_1}^3(c_X^2 + 1)^2 \ ,$$

[1]Note that there are infinitely many ACPH(2) representation of the exponential distribution, but the minimal unique canonical representation of this distribution is the one-dimensional exponential.

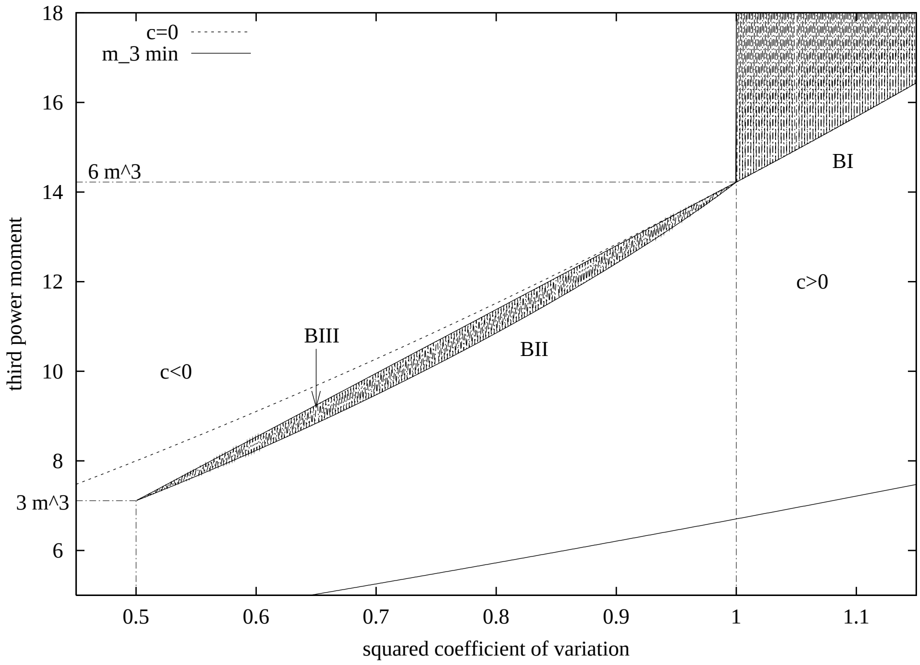

Figure 2: Third-moment bounds for ACPH(2) distribution with $m\,(= m_1) = \frac{4}{3}$

mom.	condition	bounds	ACPH(2)
1.		$0 < m_1 < \infty$	-
2.(c_X^2)		$0.5 \le c_X^2 < \infty$	-
3.	$0.5 \le c_X^2 \le 1$	$3\,{m_1}^3(3\,c_X^2 - 1 + \sqrt{2}\,(1-c_X^2)^{\frac{3}{2}}) \le m_3 \le 6\,{m_1}^3 c_X^2$	$\lambda_1 = \lambda_2$ (BII) $p = 1$ (BIII)
	$1 < c_X^2$	$\frac{3}{2}\,{m_1}^3(1+c_X^2)^2 \le m_3 (< \infty)$	$\lambda_2 \to \infty$ (BI)

Table 1: Bounds for the first three moments of the ACPH(2) distributions

which coincides with the lower bound in $c_X^2 \in (1, \infty)$. The importance of this dotted curve which separates the regions $c > 0$ and $c < 0$ will be discussed in Section 4. The lowest curve in Figure 2 marks the general lower bound for the third moment of any distribution on the nonnegative axis [20], where

$$m_1 \le {m_2}^{\frac{1}{2}} \le {m_3}^{\frac{1}{3}} \quad \Leftrightarrow \quad m_3 \ge {m_1}^3 (1 + c_X^2)^{\frac{3}{2}} \ .$$

Despite the obvious restrictions on the first three moments of ACPH(2) distributions, this subclass of continuous PH(2) distributions preserves an utmost flexibility in the sense that the presented bounds are identical with those of the more general class of matrix-exponential distributions [12].

3 The canonical ADPH(2) distribution and moment bounds

For the discrete case, we very much proceed along the same lines as for the continuous case - with the main difference that the factorial moments take the role of the power moments. As we will see, the bound behavior naturally bears similarities, but becomes a bit more involved. Again, we start by specializing the general notation (see [4]) of the discrete PH distributions to the canonical form of acyclic discrete PH distributions of order 2:

$$\boldsymbol{\alpha} = (p, 1-p) \quad \text{and} \quad \boldsymbol{B} = \begin{vmatrix} 1-\beta_1 & \beta_1 \\ 0 & 1-\beta_2 \end{vmatrix} , \tag{6}$$

where $0 \le p \le 1$ and $0 < \beta_1 \le \beta_2 \le 1$. Figure 3 displays the transient discrete-time Markov chain associated with this canonical representation. The discrete time to absorption (in unit time steps) will be denoted by the random variable N.

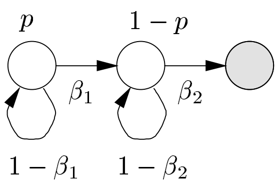

Figure 3: Canonical form of ADPH(2) distribution

Power moments might be derived directly from the probability mass function $f_N(k) = P\{N = k\} = \boldsymbol{\alpha}\boldsymbol{B}^{k-1}(\boldsymbol{I} - \boldsymbol{B})\boldsymbol{e}$ ($\boldsymbol{I}$ is the two-dimensional identity matrix) or indirectly via the factorial moments. These can be conveniently computed from the generator function of N

$$G_N(z) = E[z^N] = p\,\frac{\beta_1 z}{1-(1-\beta_1)z}\,\frac{\beta_2 z}{1-(1-\beta_2)z} + (1-p)\frac{\beta_2 z}{1-(1-\beta_2)z}$$

resulting in:

$$f_1 = E[N] = \frac{d}{dz}G_N(z)|_{z=1} = \frac{\beta_1 + \beta_2 p}{\beta_1 \beta_2} \tag{7}$$

$$f_2 = E[N(N-1)] = \frac{d^2}{dz^2}G_N(z)|_{z=1} = \frac{2\left({\beta_1}^2(1-\beta_2) + p\,\beta_1\,\beta_2 + p\,{\beta_2}^2(1-\beta_1)\right)}{{\beta_1}^2\,{\beta_2}^2}$$

$$f_3 = E[N(N-1)(N-2)] = \frac{d^3}{dz^3}G_N(z)|_{z=1} = \tag{8}$$

$$= \frac{6\left({\beta_1}^3(1-\beta_2)^2 + p\,\beta_1\,\beta_2\,(\beta_1 - 2\,\beta_1\,\beta_2 + \beta_2) + p\,{\beta_2}^3(1-\beta_1)^2\right)}{{\beta_1}^3\,{\beta_2}^3} \tag{9}$$

In this paper, the first three factorial moments serve as the starting point on our way from such a partial description of a discrete random variable to the parameter specification

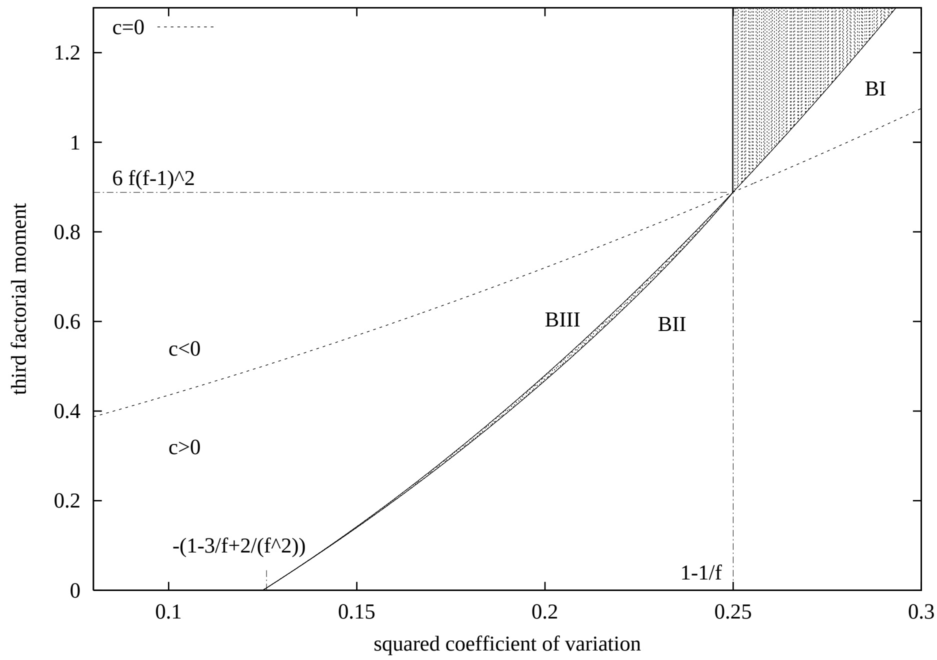

Figure 4: Third-moment bounds for ADPH(2) distribution with $f\ (= f_1) = \frac{4}{3}\ (< 2)$

(p, β_1, β_2) of the ADPH(2) canonical form. Also the moment bounds are given in the context of factorial moments. In [19], it was shown that the feasible range of the first factorial moment $f_1 \geq 1$ must be divided into two sections, in which the minimum squared coefficient of variation (scv: $c_N^2 = \frac{E[N^2]}{E[N]^2} - 1 = \frac{f_2 + f_1 - {f_1}^2}{{f_1}^2}$) follows different laws - both explicitly f_1-dependent though (see Table 2). These two ranges ($1 \leq f_1 < 2$ and $2 \leq f_1$) also have an effect on the third-moment behavior.

For $f_1 = \frac{4}{3}\ (< 2)$ - the same value as for the mean m_1 in Figure 2 - the third factorial moment f_1 is plotted over the squared coefficient of variation c_N^2. Although the shapes of the feasible regions of Figures 4 and 2 have much in common, several important differences are identified: First, the low-variability (here hypogeometric) range is not fixed (as to $(0.5, 1.0)$ for ACPH(2) distributions), but lies within boundaries which depend on f_1 ($1 \leq f_1 < 2$):

$$\frac{\langle f_1 \rangle (1 - \langle f_1 \rangle)}{{f_1}^2} = \frac{(f_1 - 1)(2 - f_1)}{{f_1}^2} = -(1 - \frac{3}{f_1} + \frac{2}{{f_1}^2}) \leq c_N^2 < 1 - \frac{1}{f_1} \quad , \tag{10}$$

where $\langle f_1 \rangle$ denotes the fractional part of f_1, i.e., $\langle f_1 \rangle = f_1 - \lfloor f_1 \rfloor = f_1 - 1$ (since $1 < f_1 < 2$). Note that as f_1 approaches 1 or 2, the lower bound on the (nonnegative) squared coefficient of variation vanishes, i.e., $c_N^2 \geq 0$ in the limit. For $f_1 \to 1$, the ADPH(2) distributions converge towards the unit-step deterministic distribution ($p = 0, \beta_2 = 1$), while for $f_1 = 2$ (actually part of case $2 \leq f_1$), the respective minimum $c_N^2 = 0$ yields the deterministic distribution with $E[N] = 2$ ($p = 1, \beta_1 = \beta_2 = 1$). Our choice of $f_1 = \frac{4}{3}$ in Figure 4 imposes the strictest lower bound on c_N^2 in the range $1 \leq f_1 < 2$, i.e., the minimum squared

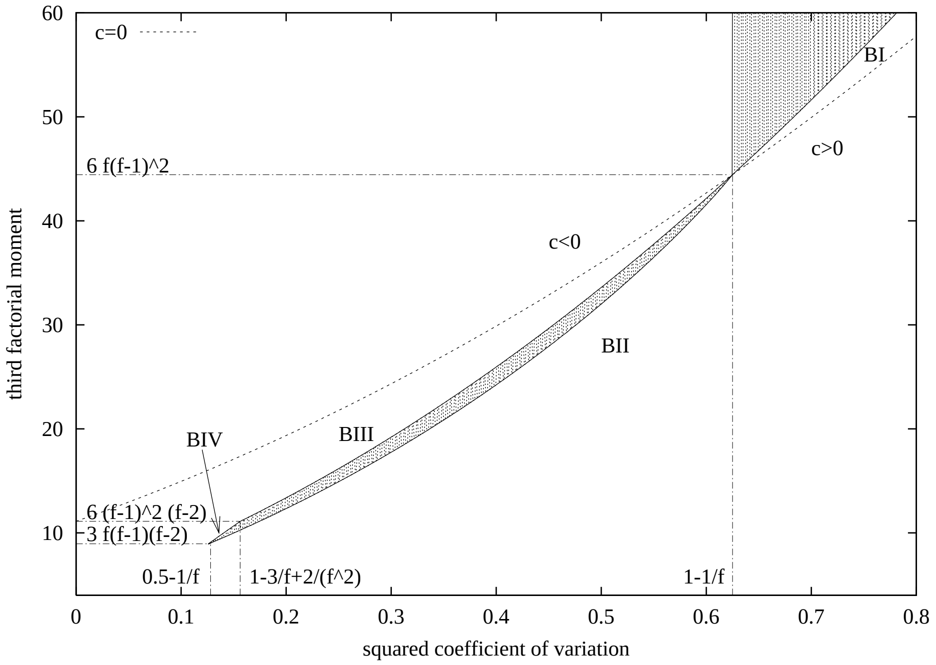

Figure 5: Third-moment bounds for ADPH(2) distribution with $f\ (= f_1) = \frac{8}{3}\ (> 2)$

coefficient of variation is maximal and equal to $c_N^2 = \frac{1}{8}$. The third factorial moment starts from zero at the minimum coefficient of variation and increases to $6\, f_1\, (f_1 - 1)^2$ for $c_N^2 \to 1 - \frac{1}{f_1}$, where only relatively little variation is tolerated in between.

With f_1 entering the range $2 \le f_1$, the expression $(1 - \frac{3}{f_1} + \frac{2}{{f_1}^2})$ (see formula (10)) turns nonnegative and - as indicated above - the lower bound of the squared coefficient of variation is replaced by $0.5 - \frac{1}{f_1}$. Nevertheless, the expression in brackets retains an important role also in the case $2 \le f_1$, which is illustrated by Figure 5 for the specific $f_1 = \frac{8}{3}$ (= f in the figure). Note that this doubled f_1 stipulates the same value for the minimum squared coefficient of variation as before. I.e., $0.5 - \frac{1}{f_1^{(2)}} = \frac{1}{8} = -(1 - \frac{3}{f_1^{(1)}} + \frac{2}{{f_1^{(1)}}^2})$, where $f_1^{(2)} = \frac{8}{3}$ and $f_1^{(1)} = \frac{4}{3}$.

In Figure 5, we observe that for $2 \le f_1$ - as opposed to the case $1 \le f_1 < 2$ - the third factorial moment does no longer reach down to zero at the minimum c_N^2. Furthermore, the upper bound in the low-variability range behaves differently for c_N^2 less or greater than $1 - \frac{3}{f_1} + \frac{2}{{f_1}^2} = \frac{(f_1-2)(f_1-1)}{{f_1}^2}$ (bounds BIV and BIII).

In both cases, $1 \le f_1 < 2$ and $2 \le f_1$ (i.e., $1 \le f_1$) - in analogy to ACPH(2) - a singular point occurs, now at $(c_N^2 = 1 - \frac{1}{f_1}, f_3 = 6\, f_1\, (f_1 - 1)^2)$ on the dotted lines

$$c = 3\, {f_2}^2 - 2\, f_1 f_3 = 0 \quad \Leftrightarrow \quad f_3 = \frac{3}{2}\, f_1\, \left(f_1\, (c_N^2 + 1) - 1\right)^2 \quad .$$

The canonical representation of this point is the geometric distribution with parameter $\beta_2 = \frac{1}{f_1}$ ($p = 0, \beta_1 =$ irrelevant). In the hypergeometric range (i.e., $c_N^2 > 1 - \frac{1}{f_1}$), only

mom.	condition	bounds	ADPH(2)
1.		$1 \le f_1 < \infty$	-
2.(c_N^2)	$1 \le f_1 < 2$	$\frac{(2-f_1)(f_1-1)}{{f_1}^2} \le c_N^2 < \infty$	-
	$2 \le f_1$	$0.5 - \frac{1}{f_1} \le c_N^2 < \infty$	-
3.	$1 \le f_1 < 2$		
	$\frac{(2-f_1)(f_1-1)}{{f_1}^2} \le c_N^2 < 1 - \frac{1}{f_1}$	$g \le f_3$ $\le \frac{3\,f_2(f_2-2\,f_1+2)}{2\,(f_1-1)}$	$\beta_1 = \beta_2$ (BII) $\beta_2 = 1$ (BIII)
	$2 \le f_1$		
	$0.5 - \frac{1}{f_1} \le c_N^2 < \frac{(f_1-2)(f_1-1)}{{f_1}^2}$	$g \le f_3$ $\le 6\,{f_1}^2(f_1-1)c_N^2$	$\beta_1 = \beta_2$ (BII) $p = 1$ (BIV)
	$\frac{(f_1-2)(f_1-1)}{{f_1}^2} \le c_N^2 < 1 - \frac{1}{f_1}$	$g \le f_3$ $\le \frac{3\,f_2(f_2-2\,f_1+2)}{2\,(f_1-1)}$	$\beta_1 = \beta_2$ (BII) $\beta_2 = 1$ (BIII)
	$1 \le f_1$		
	$1 - \frac{1}{f_1} \le c_N^2$	$\frac{3\,f_2(f_2-2\,f_1+2)}{2\,(f_1-1)} \le f_3(< \infty)$	$\beta_2 = 1$ (BI)

Table 2: Bounds for the first three moments of the ADPH(2) distributions

lower bounds exist for the third factorial moment for any feasible value of f_1 (see Figures 4 and 5). The exact formulae of the discussed bounds can be found in Table 2. They were derived in a similar manner as in the continuous case - again exploiting the structural information listed in the last column. To enhance the readability of Table 2, we left the variable f_2 in some expressions (instead of substituting it by $f_2 = {f_1}^2(c_N^2+1) - f_1$) and introduced the following auxiliary variable

$$g = \frac{6}{(2\,f_1 + \sqrt{2\,d})^3}\left(f_1\,(2\,f_1 + \sqrt{2\,d})(3\,f_2 + 2\,f_1)(f_2 - 2\,f_1 + 2) - 2\,{f_2}^2\,(f_2 - \sqrt{2\,d})\right) \quad ,$$

where $d = 2\,{f_1}^2 - 2\,f_1 - f_2$. Variable d and the previously defined c will also appear in the moment fitting procedure to be outlined in the next section.

4 Method of moments for ACPH(2) and ADPH(2) distributions

The procedures of this section provide the best possible mapping of the first three moments of a generally distributed random variable into a PH representation of order 2 - in both the continuous and discrete setting. Though starting from power or factorial moments, respectively, the corresponding formulae for both cases resemble one another so strongly that the two methods of moments are treated in parallel. The moment bounds of the previous sections are crucial for these procedures in that they determine whether the given triple of moments is feasible or not.

ACPH(2)	**ADPH(2)**
Power moments	Factorial moments
$m_1 = E[X],\ m_2 = E[X^2],$ $m_3 = E[X^3]$	$f_1 = E[N],\ f_2 = E[N(N–1)],$ $f_3 = E[N(N–1)(N–2)]$
Auxiliary variables	
$d = 2\,{m_1}^2 - m_2,\ c = 3\,{m_2}^2 - 2\,m_1\,m_3$ $b = 3\,m_1\,m_2 - m_3$ $a = b^2 - 6\,c\,d$	$d = 2\,{f_1}^2 - 2\,f_1 - f_2,\ c = 3\,{f_2}^2 - 2\,f_1\,f_3$ $b = 3\,f_1\,f_2 - 6\,(f_1 + f_2 - {f_1}^2) - f_3$ $a = b^2 - 6\,c\,d$
Moments fitting	
$m_1,\ m_2,\ m_3 \to p,\ \lambda_1,\ \lambda_2$	$f_1,\ f_2,\ f_3 \to p,\ \beta_1,\ \beta_2$
if $c > 0$	
$p = \dfrac{-b + 6\,m_1\,d + \sqrt{a}}{b + \sqrt{a}}$ $\lambda_1 = \dfrac{b - \sqrt{a}}{c}, \lambda_2 = \dfrac{b + \sqrt{a}}{c}$	$p = \dfrac{-b + 6\,f_1\,d + \sqrt{a}}{b + \sqrt{a}}$ $\beta_1 = \dfrac{b - \sqrt{a}}{c}, \beta_2 = \dfrac{b + \sqrt{a}}{c}$
if $c < 0$	
$p = \dfrac{b - 6\,m_1\,d + \sqrt{a}}{-b + \sqrt{a}}$ $\lambda_1 = \dfrac{b + \sqrt{a}}{c}, \lambda_2 = \dfrac{b - \sqrt{a}}{c}$	$p = \dfrac{b - 6\,f_1\,d + \sqrt{a}}{-b + \sqrt{a}}$ $\beta_1 = \dfrac{b + \sqrt{a}}{c}, \beta_2 = \dfrac{b - \sqrt{a}}{c}$
if $c = 0$	
$p = 0,\ \lambda_1 = 0,\ \lambda_2 = \dfrac{1}{m_1}$ (exp.)	$p = 0,\ \beta_1 = 0,\ \beta_2 = \dfrac{1}{f_1}$ (geom.)

Table 3: Moment fitting with ACPH(2) and ADPH(2) distributions

Let us begin with the former situation (feasibility), in which all three moments fall into the related intervals within the derived boundaries. Solving each system of nonlinear algebraic equations - either (3) – (5) or (7) – (9) - for the parameters of the ACPH(2) or ADPH(2) distributions, respectively[2], one may finally arrive at the moment-fitting procedures of Table 3. In particular, the distinction of cases $c < 0, c = 0, c > 0$ can be graphically reproduced in Figures 2, 4 and 5. In this context, notice the congruent expressions for c in the discrete and continuous cases regardless of power or factorial moments.

[2]For example, we applied the Mathematica package with subsequent manipulations

We now turn to the situation with initially infeasible moments. Generally - and according to [9] -, there are essentially three approaches to handle this problem:

option 1: matching the first two moments instead of three

option 2: adjusting the moments to be matched

option 3: using alternative three-moment matching techniques usually (in our case definitely) leading to higher-order (PH) representations

The presented moment bounds for ACPH(2) and ADPH(2) distributions make option 2 superior over option 1. They enable us to select the optimal moment-boundary values to enforce feasibility. In practice, one will merely set the third moment to the closest boundary value (computed for feasible first two moments), if the third power/factorial moment exceeds the limits. Moment fitting then follows Table 3. If the second power/factorial moment does not comply with the moment bounds significantly, avoiding higher-order representations hardly seems reasonable. For example, analytic option 3 alternatives are discussed in [9, 7] for the continuous and in [4] for the discrete case.

5 Conclusions

Acyclic PH distributions of order 2 allow very compact approximation of generally distributed random variables. For both continuous and discrete settings, bounds for the first three (power/factorial) moments are derived. This theoretical result is applied in analytic moment-fitting procedures. From three given moments, the three parameters of an ACPH(2) or ADPH(2) distribution are determined, which match these moments exactly (for feasible moments) or approximately in best effort (for infeasible moments). An obvious next step for the proposed techniques is to include the fourth and fifth moments in the fitting procedures. This would result in acyclic PH distributions of order 3. Finally, we stress once again the relevance of very low-order representations in stochastic modeling. When PH distributions are employed (e.g., for service, interarrival or repair times), the number of states in the model depends multiplicatively on the number of phases. The state-space explosion problem in analytical algorithms demands compact and effective distributional models.

References

[1] D. Aldous and L. Shepp. The least variable phase type distribution is Erlang. *Commun. Statist.-Stochastic Models*, 3:467–473, 1987.

[2] T. Altiok. On the phase-type approximations of general distributions. Technical report, IE Working Paper 84–113, 1984.

[3] A. Bobbio and A. Cumani. ML estimation of the parameters of a PH distribution in triangular canonical form. In G. Balbo and G. Serazzi, editors, *Computer Performance Evaluation*, pages 33–46. Elsevier Science, 1992.

[4] A. Bobbio, A. Horváth, M. Scarpa, and M. Telek. Acyclic discrete phase type distributions - Part 1: Properties and canonical forms. *Performance Evaluation*, 2002. (submitted for publication).

[5] A. Bobbio and M. Telek. A benchmark for PH estimation algorithms: results for Acyclic-PH. *Stochastic Models*, 10:661–677, 1994.

[6] A. Cumani. On the canonical representation of homogeneous Markov processes modelling failure-time distributions. *Microelectronics and Reliability*, 22:583–602, 1982.

[7] A. Heindl and M. Telek. MAP-based decomposition of tandem networks of ·/PH/1(/K) queues with MAP input. In *Proc. 11th GI/ITG Conference on Measuring, Modelling and Evaluation of Computer and Communication Systems*, pages 179–194, Aachen, Germany, 2001.

[8] A. Heindl and M. Telek. Output approximations of MAP/PH/1(/K) queues for an efficient network decomposition. 2002. submitted for publication.

[9] M. A. Johnson and M. R. Taaffe. Matching moments to phase distributions: Mixtures of Erlang distributions of common order. *Commun. Statist.-Stochastic Models*, 5(4):711–743, 1989.

[10] M. A. Johnson and M. R. Taaffe. Matching moments to phase distributions: nonlinear programming approaches. *Commun. Statist.-Stochastic Models*, 6(2):259–281, 1990.

[11] R. A. Marie. Calculating equilibrium probabilities for $\lambda(m)/C_k/1/N$ queues. In *Proc. Int. Symposium on Computer Performance Modeling*, 1980.

[12] K. Mitchell and A. van de Liefvoort. Approximation models of feed-forward G/G/1/N queueing networks with correlated arrivals. In *Proc. 4th Int. Workshop on Queueing Networks with Finite Capacity*, pages 32/1–12, Ilkley, UK, 2000. Networks UK.

[13] M. Neuts. *Matrix-Geometric Solutions in Stochastic Models*. John Hopkins University Press, 1981.

[14] M. Neuts. Two further closure properties of PH-distributions. *Asia-Pacific Journal of Operational Research*, 9:459–477, 1992.

[15] M. F. Neuts. Probability distributions of phase type. In *Liber amicorum Prof. Emeritus H. Florin*, pages 173–206. University of Louvain, 1975.

[16] C. A. O'Cinneide. Characterization of phase-type distributions. *Commun. Statist.-Stochastic Models*, 6(1):1–57, 1990.

[17] W. C. Parr and W. R. Schucany. Minimum distance and robust estimation. *Journal of the American Statistical Association*, 75:616–624, 1980.

[18] C. H. Sauer and K. M. Chandy. *Computer Systems Performance Modeling*. Prentice-Hall, 1981.

[19] M. Telek. Minimal coefficient of variation of discrete phase type distributions. In *3rd International Conference on Matrix-Analitic Methods in Stochastic models, MAM3*, pages 391–400, Leuven, Belgium, 2000. Notable Publications Inc.

[20] W. Whitt. Approximating a point process by a renewal process, I. Two basic methods. *Operations Research*, 30:125–147, 1982.