Analysis of Inhomogeneous Markov Reward Models

M. Telek[1], A. Horváth[2], G. Horváth[1]

[1] *Department of Telecommunications, Technical University of Budapest, 1521 Budapest, Hungary*
{telek,hgabor}@webspn.hit.bme.hu
[2] *Dipartimento di Informatica, Università di Torino, Torino, Italy*
horvath@di.unito.it

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ABSTRACT

The majority of computational methods applied for the analysis of homogeneous Markov reward models (MRMs) are not applicable for the analysis of inhomogeneous MRMs. By the nature of inhomogeneous models, only forward differential equations can be used to describe the model behaviour.

In this paper we provide forward partial differential equations describing the distribution of reward measures of inhomogeneous MRMs. Based on this descriptions, we introduce the set of ordinary differential equations that describes the behaviour of the moments of reward measures when it is possible. This description of moments allows the effective numerical analysis of rather large inhomogeneous MRMs.

A numerical example demonstrates the application of inhomogeneous MRMs in practice and the numerical behaviour of the introduced analysis technique.

1. Introduction

The extension of discrete state stochastic processes with a continuous variable depending on the past history of the stochastic process resulted in one of the most powerful tools of performance analysis, the reward model [12]. The main classifying features of reward models are the type of the underlying discrete state stochastic process, the type of reward accumulation and the kind of reward loss at state transitions. In the majority of cases the underlying process is a continuous time Markov chain (CTMC) [7, 11, 8, 6, 5], but there are results for reward models with underlying semi Markov process [3, 4] and Markov regenerative process [17]. When the reward increases at a given rate, r_i, during the sojourn of the underlying process in state i is referred to as rate reward accumulation and when the reward increases instantly at a state transition of the underlying process is referred to as impulse reward accumulation [13]. In some

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reward models the reward function may decrease at state transitions. The reward models without reward loss are called preemptive resume (prs) models, the models with complete reward loss at state transitions are referred to as preemptive repeat (different or identical) [3] and there are models with partial reward loss [2] as well. The two main performance parameters of reward models are the distribution of accumulated reward [7] (the value of the reward function at time t) and the distribution of completion time [11] (the time needed to accumulate a given amount of reward).

In this paper we consider reward models with underlying inhomogeneous CTMC with rate and impulse reward accumulation and without reward loss (prs). To the best of our knowledge all previous works of the field assumed a time homogeneous underlying stochastic process and based on this assumption the majority of the papers provided a Markov renewal theory based analytical description of reward measures. Unfortunately, Markov renewal theory is not applicable for the analysis of inhomogeneous stochastic processes and as a consequence the majority of the effective computational methods (e.g., the ones based on randomization) are not applicable.

A different analytical approach is needed to describe the behaviour of inhomogeneous reward models and to evaluate their reward measures. By the nature of inhomogeneous reward models only forward partial differential equations (PDEs) can be used to describe the model behaviour. The description of the distribution of accumulated reward with forward partial differential equations is available in the literature [15], but in this paper we provide the complete set of forward partial differential equations to describe all reward measures mentioned above with the particular inhomogeneous system behaviour where both the underlying process and the reward rate function depend on the time and the level of the accumulated reward and we extend the basic set of equations to the cases of rate plus impulse reward accumulation and states with zero reward rate.

Among the numerical procedures available for the analysis of homogeneous Markov reward models, the computational complexity of the methods that calculate the distribution of reward measures is much higher than the complexity of the ones calculate only the moments of reward measures. Here, we show that the analysis of the moments of inhomogeneous reward models is also simpler, in some cases, because the behaviour of moments of reward measures can be described by ordinary differential equations (ODE).

The rest of the paper is organized as follows. Section 2 introduces the basic model behaviour. Section 3 and 4 provide the distribution of accumulated reward and completion time, respectively. The analysis of the moments of these reward measures are provided in Section 5 and 6. Section 7 provides the analytical description of inhomogeneous MRMs with rate and impulse reward accumulation. A numerical example is introduced in Section 8 and finally the paper is concluded in Section 9.

2. Inhomogeneous Markov Reward Models

Let the structure state process, $\{Z(t), t \geq 0\}$, be an inhomogeneous continuous time Markov chain on state space $S = \{1, 2, ..., N\}$ with generator $\boldsymbol{Q}(t, w) = \{Q_{ij}(t, w)\}$ $(0 \leq Q_{ij}(t, w) < \infty)$ and initial probability vector π. The generator of the $\{Z(t), t \geq 0\}$ process depends on both the time (t) and the level of the accumulated reward (w), such that $Q_{ij}(t, w)$ is irreducible for $t, w \geq 0$. At time t and accumulated reward w the state transition probability matrix

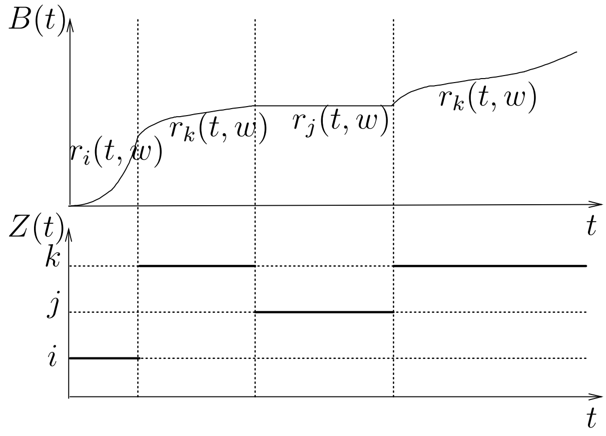

Figure 1. Rate reward accumulation in inhomogeneous MRMs

$\boldsymbol{P}(t) = \{P_{ij}(t)\}$ ($P_{ij}(t) = Pr(Z(t) = j \mid Z(0) = i)$) and the transient state probability vector $p(t) = \{p_i(t)\}$ ($p_i(t) = Pr(Z(t) = i)$) satisfy the forward differential equations

$$\frac{d}{dt}\boldsymbol{P}(t) = \boldsymbol{P}(t)\boldsymbol{Q}(t,w) \quad \text{with initial condition } \boldsymbol{P}(0) = \boldsymbol{I} \tag{1}$$

and

$$\frac{d}{dt}p(t) = p(t)\boldsymbol{Q}(t,w) \quad \text{with initial condition } p(0) = \pi \ , \tag{2}$$

where $\boldsymbol{I}$ is the identity matrix. Unfortunately, the dependence of $\boldsymbol{Q}(t,w)$ on the reward level (w) prevents us to analyze the transient behaviour of $Z(t)$ (based on (1) or (2)) independent of the reward accumulation process. Instead, we need to analyze the joint distribution of the system state and the reward level for this class of processes, as it is provided below.

Whenever the CTMC stays in state i at time t and the level of accumulated reward is w, reward is accumulated at rate $0 \leq r_i(t,w) < \infty$. When the CTMC undergoes a transition from state i to an other state the accumulated reward is maintained (preemptive resume case). (The case of rate and impulse reward accumulation is discussed in Section 7.) $B(t)$ denotes the amount of accumulated reward at time t. The dynamics of the $\{B(t), t \geq 0\}$ process can be described as follows (see Figure 1):

$$\frac{dB(t)}{dt} = r_{Z(t)}(t,w) \quad \text{and} \quad B(0) = 0 \ . \tag{3}$$

By this definition $B(t)$ is continuous and monotonically increasing. An important consequence of this property is that the analysis of reward measures at time t and reward level w (e.g., $Pr(B(t) < w)$) is based only on the evaluation of the $(0,t) \times (0,w)$ region. The maximum of the reward rate over this relevant region is $r_{max} = \max_{i \in S, \tau \in (0,t), x \in (0,w)} r_i(\tau, x) < \infty$.

At time t and reward level w one can partition the state space S according to the sign of the reward rate associated with the states, $S^+(t,w) = \{i : i \in S, r_i(t,w) > 0\}$ and

$S^0(t,w) = \{i : i \in S, r_i(t,w) = 0\}$. To simplify the subsequent discussion and to avoid involved details of special cases (e.g., $r_i(t,0) > 0$ and $r_i(t,w) = 0, \forall w > 0$), we assume that the S^+, S^0 division of the state space is independent of t and w. The subsequent analysis technique allows to relax this restriction, but the careful discussion of all special cases would be too lengthy.

3. Analysis of accumulated reward

The joint distribution of the structure state and the accumulated reward at time t is defined as

$$Y_{ij}(t,w) = Pr(B(t) \le w, Z(t) = j \mid Z(0) = i)$$

and its derivative, the joint reward density function, is

$$y_{ij}(t,w) = \frac{\partial}{\partial w} Y_{ij}(t,w) = \lim_{\Delta \to 0} \frac{1}{\Delta} Pr(w \le B(t) < w + \Delta, Z(t) = j \mid Z(0) = i) \ .$$

The associated matrices are $\boldsymbol{Y}(t,w) = \{Y_{ij}(t,w)\}$ and $\boldsymbol{y}(t,w) = \{y_{ij}(t,w)\}$.

Theorem 1. *$\boldsymbol{y}(t,w)$ satisfies the following partial differential equation*

$$\frac{\partial}{\partial t}\boldsymbol{y}(t,w) + \frac{\partial}{\partial w}\boldsymbol{y}(t,w)\ \boldsymbol{R}(t,w) = \boldsymbol{y}(t,w)\ \boldsymbol{Q}(t,w) \tag{4}$$

with initial conditions

- $y_{ij}(0,w) = \delta_{ij}\ \Omega(w)$,
- $y_{ij}(t,0) = 0$ *for* $i \in S^+$ *or* $j \in S^+$ *and*
- *the initial condition* $y_{ij}(t,0)$ *for* $i,j \in S^0$ *is obtained from*

$$\frac{d}{dt} y_{ij}(t,0) = \sum_{k \in S} y_{ik}(t,0)\ Q^0_{kj}(t,0) \text{ with initial condition } y_{ij}(0,w) = \delta_{ij}\ \Omega(w),$$

where δ_{ij} is the Kronecker delta ($\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \ne j$), $\Omega(w)$ is the Dirac impulse function ($\Omega(w) = 0$ for $w \ne 0$ and $\int \Omega(w) = 1$), $\boldsymbol{R}(t,w)$ is the diagonal matrix of the reward rates ($\boldsymbol{R}(t,w) = \mathrm{diag}\langle r_i(t,w)\rangle$) and $\boldsymbol{Q^0}(t,w)$ is obtained from $\boldsymbol{Q}(t,w)$ by setting $Q^0_{ij}(t,w) = 0$ for $\forall i \in S^+$ and $\forall j \in S^+$.

***Proof* 1.** *The forward argument that describes the evolution of $y_{ij}(t,w)$ over the $(t, t+\Delta)$ interval is*

$$\begin{aligned} y_{ij}(t+\Delta, w) = \ & (1 + Q_{jj}(t, w - \Delta r_j(t,w))\Delta)\ y_{ij}(t, w - \Delta r_j(t,w)) + \\ & \sum_{k \in S, k \ne j} Q_{kj}(t, w - c_{kj}\Delta)\Delta\ y_{ik}(t, w - c_{kj}\Delta) + o(\Delta), \end{aligned} \tag{5}$$

where $c_{kj} \le r_{max} < \infty$. Algebraic manipulations and the $\Delta \to 0$ transition results

$$\frac{\partial y_{ij}(t,w)}{\partial t} + r_j(t,w)\frac{\partial y_{ij}(t,w)}{\partial w} = \sum_{k \in S} y_{ik}(t,w)\ Q_{kj}(t,w). \tag{6}$$

Eq. (4) is the matrix form of (6). The initial conditions are obtained by the definition of $y_{ij}(t,w)$ and by the fact that $y_{ij}(t,0) > 0$ iff $Z(\tau) \in S^0$ for $\forall \tau \in (0,t)$.

Note that, due to the dependence of $\boldsymbol{Q}(t,w)$ and $\boldsymbol{R}(t,w)$ on the reward level, it is easier to describe the reward density function instead of the reward distribution function.

Analysis of the transient vector Having the initial distribution of the background process, π, the vector of the transient reward measure $\bar{y}(t,w) = \{\bar{y}_i(t,w)\}$, where

$$\bar{y}_i(t,w) = \lim_{\Delta\to 0} \frac{1}{\Delta} Pr(w \le B(t) < w+\Delta, Z(t)=i) \ ,$$

can be obtained from the following corollary.

Corollary 2. *The transient distribution of the accumulated reward satisfies the following partial differential equation*

$$\frac{\partial}{\partial t}\bar{y}(t,w) + \frac{\partial}{\partial w}\bar{y}(t,w)\ \boldsymbol{R}(t,w) = \bar{y}(t,w)\ \boldsymbol{Q}(t,w). \tag{7}$$

with initial conditions

- $\bar{y}_i(0,w) = \pi_i\ \Omega(w)$,
- $\bar{y}_i(t,0) = 0$ *for* $i \in S^+$ *and*
- *the initial condition* $\bar{y}_i(t,0)$ *for* $i \in S^0$ *is obtained from*

$$\frac{d}{dt}\bar{y}_i(t,0) = \sum_{k\in S} \bar{y}_k(t,0) Q^0_{ki}(t,0) \text{ with initial condition } \bar{y}_i(0,w) = \pi_i\ \Omega(w)\ .$$

***Proof* 2.** *Corollary 2 is obtained from Theorem 1 by multiplying the matrix equations with vector* π *from the left and applying* $\bar{y}(t,w) = \pi\ \boldsymbol{y}(t,w)$.

In the rest of this paper we focus only on matrix measures (like $\boldsymbol{y}(t,w)$), and do not provide the associated transient vector measures (like $\bar{y}(t,w)$). Also for the rest of the matrix equations presented below the equations of the vector measures can be obtained by multiplying the matrix equations with π from the left. The introduction of the vector measures is important for effective numerical analysis. We always recommend to implement the vector equations in numerical procedures, because its memory requirement and computational complexity are much less.

4. Analysis of completion time

The user oriented performance measure of stochastic reward models is the distribution of time needed to complete a given task. This random time is commonly referred to as completion time. The completion time of a task with w "work requirement", $C(w)$, is defined as the first time when the accumulated reward ($B(t)$) reaches the required reward level (w):

$$C(w) = \min(t : B(t) \ge w).$$

Due to the monotonicity of $B(t)$ (in case of loss-less or "preemptive resume" reward accumulation) there is an obvious relation between the distribution of the accumulated reward and the completion time:

$$Pr(C(w) \le t) = Pr(B(t) \ge w) = 1 - Pr(B(t) < w) \tag{8}$$

In those cases when the required performance measure is the distribution of the completion time (independent of the state at completion) the related accumulated reward analysis can be applied using (8).

Unfortunately, this general relation of the accumulated reward and the completion time does not help in evaluating the joint distribution of the completion time and the system state at completion

$$F_{ij}(t,w) = Pr(C(w) < t, Z(C(w)) = j \mid X(0) = i)$$

and its derivative with respect to t, the joint density of the completion time

$$f_{ij}(t,w) = \frac{d}{dt}F_{ij}(t,w) = \lim_{\Delta\to 0}\frac{1}{\Delta}Pr(t \le C(w) < t+\Delta, Z(C(w)) = j \mid X(0) = i).$$

The matrices composed by these elements are $\boldsymbol{F}(t,w) = \{F_{ij}(t,w)\}$ and $\boldsymbol{f}(t,w) = \{f_{ij}(t,w)\}$.

The state dependent distribution of the completion time of homogeneous MRMs can be described by a set of backward differential equations, but backward differential equations can not capture the behaviour of inhomogeneous background process. That is why we need forward differential equations. The following theorem provides the relation of the distributions of completion time and the distribution of accumulated reward.

Theorem 3. *The state dependent distribution of the completion time is related to the distribution of the accumulated reward by means of the following equation:*

$$\boldsymbol{f}(t,w) = \boldsymbol{y}(t,w)\boldsymbol{R}(t,w)\ . \tag{9}$$

***Proof* 3.** *Based on the definition of the state dependent completion time distribution*

$$\begin{aligned}
&F_{ij}(t+\Delta,w) - F_{ij}(t,w) = \\
&\quad Pr(\textit{no state transition in } (t,t+\Delta))\ (Y_{ij}(t,w) - Y_{ij}(t,w-r_j(t,w)\Delta)) + \\
&\quad Pr(\textit{one state transition in } (t,t+\Delta))\ \cdot \\
&\qquad Pr(w - b_j\Delta < B(t) < w, Z(t) \neq j, Z(t+\Delta) = j \mid X(0) = i) + \\
&\quad \sigma(\Delta)\ ,
\end{aligned} \tag{10}$$

where $b_j \le r_{max} < \infty$. Dividing (10) by Δ we have:

$$\frac{F_{ij}(t+\Delta,w) - F_{ij}(t,w)}{\Delta} = \frac{Y_{ij}(t,w) - Y_{ij}(t,w-r_j(t,w)\Delta)}{\Delta} + \sigma(\Delta)\ , \tag{11}$$

From which the theorem is obtained by letting $\Delta \to 0$.

Theorem 3 provides the relation of the completion time and the accumulated reward measures. The following subsection discusses the direct analysis of the completion time.

4.1. Completion time with strictly positive reward rates

Theorem 4. *If the reward rates are strictly positive $\boldsymbol{f}(t,w)$ satisfies the following forward partial differential equation:*

$$\frac{\partial}{\partial t}\boldsymbol{f}(t,w)\boldsymbol{R}^{-1}(t,w) + \frac{\partial}{\partial w}\boldsymbol{f}(t,w) = \boldsymbol{f}(t,w)\boldsymbol{R}^{-1}(t,w)\boldsymbol{Q}(t,w)\ , \tag{12}$$

with initial conditions $f_{ij}(t,0) = \delta_{ij}\Omega(t)$ and $F_{ij}(0,w) = 0$.

***Proof* 4.** *We consider the evolution of $B(t)$ between w and $w+\Delta$:*

$$\begin{aligned} f_{ij}(t,w+\Delta) &= (1+Q_{jj}(t,w)\ \tfrac{\Delta}{r_j(t,w)})\ f_{ij}(t-\tfrac{\Delta}{r_j(t,w)},w) \\ &+ \sum_{k\in S,k\neq j} Q_{kj}(t,w)\ \tfrac{\Delta}{r_k(t,w)}\ f_{ik}(t-d_{kj}(t,w)\Delta,w) \\ &+\ \sigma(\Delta)\ , \end{aligned} \tag{13}$$

where $0<\min\Big(\frac{1}{r_j(t,w)},\frac{1}{r_k(t,w)}\Big)\le d_{kj}(t,w)\le\max\Big(\frac{1}{r_j(t,w)},\frac{1}{r_k(t,w)}\Big)<\infty$. The first term of the rhs. represents the case when the structure state process does not change state between reward levels w and $w+\Delta$, i.e., it stays in state j for the whole $\frac{\Delta}{r_j(t,w)}+\sigma(\Delta)$ long interval. The second term represents the case when level w is reached in state k and there is one state transition before reaching level $w+\Delta$, and finally, the third term captures the cases with more than two state transitions and the error of the first two terms. Starting from (13) standard analysis steps provide (12).

4.2. Completion time with positive and zero reward rates

When both the S^+ and the S^0 part of the state space are non-empty the reward level increases only during the sojourn in S^+ and it remains constant during the sojourn in S^0. The completion can occur only in S^+. Without loss of generality, we order the states such that $i<j,\forall i\in S^+,\forall j\in S^0$. With this ordering, we partition the generator of the structure state process, the reward rate matrix and the completion time matrix as

$$\boldsymbol{Q}(t,w)=\begin{array}{|c|c|}\hline \boldsymbol{Q^+}(t,w) & \boldsymbol{Q^{+0}}(t,w)\\\hline \boldsymbol{Q^{0+}}(t,w) & \boldsymbol{Q^0}(t,w)\\\hline\end{array}\ ,\ \boldsymbol{R}(t,w)=\begin{array}{|c|c|}\hline \boldsymbol{R^+}(t,w) & \boldsymbol{0}\\\hline \boldsymbol{0} & \boldsymbol{0}\\\hline\end{array}\ ,\ \boldsymbol{f}(t,w)=\begin{array}{|c|c|}\hline \boldsymbol{f^+}(t,w) & \boldsymbol{0}\\\hline \boldsymbol{f^{0+}}(t,w) & \boldsymbol{0}\\\hline\end{array}$$

Theorem 5. *If the reward rates are positive and zero the state dependent distribution of the completion time satisfies the following equations:*

$$\begin{aligned} &\frac{\partial}{\partial t}\boldsymbol{f^+}(t,w)\boldsymbol{R^+}^{-1}(t,w)+\frac{\partial}{\partial w}\boldsymbol{f^+}(t,w)=\boldsymbol{f^+}(t,w)\boldsymbol{R^+}^{-1}(t,w)\boldsymbol{Q^+}(t,w)+ \\ &\int_{\tau=0}^{t}\boldsymbol{f^+}(\tau,w)\ \boldsymbol{R^+}^{-1}(\tau,w)\ \boldsymbol{Q^{+0}}(\tau,w)\ \exp\Big(\int_{u=\tau}^{t}\boldsymbol{Q^0}(u,w)\ du\Big)\,\boldsymbol{Q^{0+}}(t,w)\ d\tau \end{aligned} \tag{14}$$

with initial conditions $f^+_{ij}(t,0)=\delta_{ij}\Omega(t)$ and $f^+_{ij}(0,w)=0$.

***Proof* 5.** *To prove the theorem we need the following notations:*

- *$\alpha(t,w)$: is the time of the first state transition after t which drives the system to S^+, i.e., $\alpha(t,w)=\min\{\tau\mid\tau\ge t,Z(\tau)\in S^+,B(t)=w\}$,*
- *$g_{\ell j}(t,w,\tau)$ ($\ell\in S^0,j\in S^+$) is the final state dependent density of the sojourn time in S^0 starting from t:*

$$g_{\ell j}(t,w,\tau)=\lim_{\Delta\to 0}\frac{1}{\Delta}Pr(\tau\le\alpha(t,w)-t<\tau+\Delta,Z(\alpha(t,w))=j|Z(t)=\ell,B(t)=w)\ .$$

The $\boldsymbol{g}(t,w,\tau)=\{g_{\ell j}(t,w,\tau)\}$ matrix can be computed as

$$\boldsymbol{g}(t,w,\tau)=\exp\Big(\int_{u=t}^{t+\tau}\boldsymbol{Q^0}(u,w)\ du\Big)\,\boldsymbol{Q^{0+}}(t+\tau,w)\ . \tag{15}$$

For $i,j \in S^+$ the evolution of $B(t)$ between w and $w+\Delta$ is:

$$
\begin{aligned}
f_{ij}(t, w+\Delta) &= (1 + Q_{jj}(t,w)\ \tfrac{\Delta}{r_j(t,w)})\ f_{ij}(t - \tfrac{\Delta}{r_j(t,w)}, w) \\
&+ \sum_{k \in S^+, k \neq j} Q_{kj}(t,w)\ \tfrac{\Delta}{r_k(t,w)}\ f_{ik}(t - d_{kj}(t,w)\Delta, w) \\
&+ \int_{\tau=0}^{t} \sum_{k \in S^+,} \sum_{\ell \in S^0} Q_{k\ell}(t-\tau, w)\ \tfrac{\Delta}{r_k(t-\tau,w)}\ f_{ik}(t-\tau-d_{kj}(t-\tau,w)\Delta, w)\ g_{\ell j}(t-\tau, w, \tau)\ d\tau \\
&+\ \sigma(\Delta)\ ,
\end{aligned}
\tag{16}
$$

where $0 < d_{kj}(\tau, w) < \infty$ for $0 \le \tau \le t$.

The first term of the rhs represents the case with no state transition between hitting level w and $w+\Delta$. The second term stands for the case when reward level w is reached in state k at time $t - d_{kj}(t,w)\Delta$ and there is a state transition from k to j before reaching level $w+\Delta$. The third term captures the case when level w is reached in state k at time $t-\tau-d_{kj}(t-\tau,w)\Delta$, there is a state transition to S^0 before reaching level $w+\Delta$ and the structure state process returns to S^+ by visiting state j first. The fourth term capture the error of the previous ones.

Considering (15), the $\Delta \to 0$ limit and the matrix representation of (16) results in the Theorem.

For the analysis of $\boldsymbol{f^{0+}}(t,w)$ we introduce the completion time of the delayed accumulation process:

$$\check{F}_{ij}(t,w,\tau) = Pr(C(w) < t, Z(C(w)) = j \mid X(0) = i, B(u) = 0, 0 < u \le \tau),$$

$\check{f}_{ij}(t,w,\tau) = \dfrac{d}{dt}\check{F}_{ij}(t,w,\tau)$, and their matrices $\check{\boldsymbol{F}}(t,w) = \{\check{F}_{ij}(t,w)\}$ and $\check{\boldsymbol{f}}(t,w) = \{\check{f}_{ij}(t,w)\}$.

Theorem 6. *Starting from S^0, the completion time is:*

$$\boldsymbol{f^{0+}}(t,w) = \int_{\tau=0}^{t} \exp\left(\int_{u=0}^{\tau} \boldsymbol{Q^0}(u,0)\ du\right) \boldsymbol{Q^{0+}}(\tau,0)\ \check{\boldsymbol{f}}^{+}(t,w,\tau)\ d\tau \tag{17}$$

where $\check{\boldsymbol{f}}^{+}(t,w,\tau)$ is defined by

$$
\begin{aligned}
&\frac{\partial}{\partial t}\check{\boldsymbol{f}}^{+}(t,w,\tau)\boldsymbol{R^{+}}^{-1}(t,w) + \frac{\partial}{\partial w}\check{\boldsymbol{f}}^{+}(t,w,\tau) = \check{\boldsymbol{f}}^{+}(t,w,\tau)\boldsymbol{R^{+}}^{-1}(t,w)\boldsymbol{Q^{+}}(t,w) + \\
&\int_{\tau=0}^{t} \check{\boldsymbol{f}}^{+}(\tau,w,\tau)\ \boldsymbol{R^{+}}^{-1}(\tau,w)\ \boldsymbol{Q^{+0}}(\tau,w)\ \exp\left(\int_{u=\tau}^{t} \boldsymbol{Q^0}(u,w)\ du\right) \boldsymbol{Q^{0+}}(t,w)\ d\tau
\end{aligned}
\tag{18}
$$

with initial conditions $f^{+}_{ij}(t,0,\tau) = \delta_{ij}\Omega(t-\tau)$ and $f^{+}_{ij}(0,w,\tau) = 0$.

***Proof* 6.** *The first sojourn time in S^0 is $\alpha(0,0)$ with final state dependent density matrix $\boldsymbol{g}(0,0,\tau)$. After the first visit in S^0, a delayed accumulation starts from a state of S^+ described by $\check{\boldsymbol{f}}^{+}(t,w,\tau)$.*

The evolution of $\check{\boldsymbol{f}}^{+}(t,w,\tau)$ is identical with the evolution of $\boldsymbol{f}^{+}(t,w)$ and only their initial conditions differ because the initial condition defines the starting time of the reward accumulation.

4.3. Probabilistic interpretation and dual processes

The comparison of (4) and (12) and the associated initial conditions (keeping in mind that $S = S^+$ in (12)) shows that the accumulated reward and the completion time are the solution of very similar partial differential equations. Interchanging the role of t and w variables allows us to transform accumulated reward problems to completion time problems and vice-versa. For example, interchanging the role of t and w in (12), introducing $\hat{\boldsymbol{f}}(t, w) = \boldsymbol{f}(w, t)$, and similarly $\hat{\boldsymbol{Q}}(t, w)$, $\hat{\boldsymbol{R}}(t, w)$, results in:

$$\frac{\partial}{\partial t}\hat{\boldsymbol{f}}(t, w) + \frac{\partial}{\partial w}\hat{\boldsymbol{f}}(t, w)\hat{\boldsymbol{R}}^{-1}(t, w) = \hat{\boldsymbol{f}}(t, w)\hat{\boldsymbol{R}}^{-1}(t, w)\hat{\boldsymbol{Q}}(t, w) \ , \tag{19}$$

with initial conditions $\hat{f}_{ij}(0, w) = \delta_{ij}\Omega(w)$ and $\hat{f}_{ij}(t, 0) = 0$. The resulting partial differential equation is equivalent to (4), which means that the analysis of the completion time is equivalent to the analysis of the accumulated reward of a modified inhomogeneous Markov reward model with generator $\hat{\boldsymbol{R}}^{-1}(t, w)\hat{\boldsymbol{Q}}(t, w)$ and reward rate matrix $\hat{\boldsymbol{R}}^{-1}(t, w)$. The probabilistic interpretation of this duality is provided in several papers e.g., [1, 18].

If S^0 is not empty, i.e., there are periods of time while the reward level remains constant, this means that the dual process has jumps during the sojourn in S^0. These jumps are commonly referred to as impulse reward.

4.4. Random work requirement

When the work requirement of a particular job is a random variable, W, with distribution $G(w) = Pr(W \leq w)$, the completion time is defined as:

$$C_W = \min(t : B(t) = W),$$

In this case the joint distribution of the completion time and the state at completion is

$$Pr(C_W < t, Z(C_W) = j \mid Z(0) = i) = \int_0^\infty F_{ij}(t, w) \ dG(w),$$

and the distribution of the completion time is

$$Pr(C_W < t) = \int_0^\infty \pi \boldsymbol{F}(t, w)\mathbf{1} \ dG(w) = 1 - \int_0^\infty \pi \boldsymbol{Y}(t, w)\mathbf{1} \ dG(w),$$

where $\mathbf{1}$ is the column vector of ones.

In the general cases when the transition rates of the structure state process and the reward rates depends on both the time and the reward level, the numerical solutions of the partial differential equations (4) and (12) or (14) offers the only possible way for the numerical analysis of inhomogeneous Markov reward models. The following sections discuss those special cases when more effective numerical approaches can be applied for the solution these models.

5. Analysis of the moments of accumulated reward

The distribution of the accumulated reward can be evaluated based on Theorem 1 using numerical partial differential equation solution methods. The computational complexity of these solution methods increases exponentially with the number of states in S. To make possible the analysis of accumulated reward of large inhomogeneous Markov reward models in the special case when $\boldsymbol{Q}(t,w)$ and $\boldsymbol{R}(t,w)$ do not depend on the reward level (i.e., $\boldsymbol{Q}(t,w) = \boldsymbol{Q}(t)$ and $\boldsymbol{R}(t,w) = \boldsymbol{R}(t), w > 0$), we analyze the distribution of accumulated reward through its moments. Let $V_{ij}^{(n)}(t)$ denote the nth moment of the state dependent accumulated reward, where

$$V_{ij}^{(n)}(t) = \int_{w=0}^{\infty} w^n y_{ij}(t,w) dw = \int_{w=0}^{\infty} w^n dY_{ij}(t,w)$$

and the associated matrix is $\boldsymbol{V}^{(n)} = \{V_{ij}^{(n)}(t)\}$. Note that this definition is valid for $n = 0$ as well. $V_{ij}^{(0)}(t)$ is the state transition probability $Pr(Z(t) = j \mid Z(0) = i)$.

Theorem 7. *The nth moment ($n \geq 1$) of the accumulated reward satisfies the following ordinary differential equation*

$$\frac{d}{dt}\ \boldsymbol{V}^{(n)}(t) = n\ \boldsymbol{V}^{(n-1)}(t)\ \boldsymbol{R}(t) + \boldsymbol{V}^{(n)}(t)\ \boldsymbol{Q}(t), \tag{20}$$

with initial condition $\boldsymbol{V}^{(n)}(0) = \boldsymbol{0}, \forall n \geq 1$ and $\boldsymbol{V}^{(0)}(0) = \boldsymbol{I}$.

Proof 7. *Let $Y_{ij}^*(t,v) = \int_{w=0}^{\infty} e^{-vw} y_{ij}(t,w)\ dw = \int_{w=0}^{\infty} e^{-vw} dY_{ij}(t,w)$ denote the Laplace transform of $y_{ij}(t,w)$. The nth moment of the accumulated reward can be obtained from $Y_{ij}^*(t,v)$ as*

$$\boldsymbol{V}^{(n)}(t) = (-1)^n\ \frac{d^n}{dv^n} \boldsymbol{Y}^*(t,v) \bigg|_{v \to 0}\ . \tag{21}$$

The Laplace transform of (4) with respect to w is

$$\frac{\partial}{\partial t} \boldsymbol{Y}^*(t,v) + \Big(v \boldsymbol{Y}^*(t,v) - \boldsymbol{y}(t,0) \Big) \boldsymbol{R}(t) = \boldsymbol{Y}^*(t,v)\ \boldsymbol{Q}(t). \tag{22}$$

In (22), the $\boldsymbol{y}(t,0)\boldsymbol{R}(t)$ term vanishes because $y_{ij}(t,0) \neq 0$ iff $r_i \in S^0$. The nth derivative of (22) with respect to v is

$$\frac{\partial^n}{\partial v^n}\ \frac{\partial}{\partial t} \boldsymbol{Y}^*(t,v) + v\ \frac{\partial^n}{\partial v^n} \boldsymbol{Y}^*(t,v)\ \boldsymbol{R}(t) + n\ \frac{\partial^{n-1}}{\partial v^{n-1}} \boldsymbol{Y}^*(t,v)\ \boldsymbol{R}(t) = \frac{\partial^n}{\partial v^n} \boldsymbol{Y}^*(t,v)\ \boldsymbol{Q}(t). \tag{23}$$

Using (21), eq. (20) is obtained from (23) as $v \to 0$.

Considering only homogeneous background processes there is a similar relation between (20) and the randomization based iterative method presented in [18] as between the numerical partial differential equation solution of (1) and the randomization method for the transient analysis of CTMCs.

In the numerical analysis of the moments of accumulated reward, it is of crucial importance that the $\boldsymbol{Q}(t)$ and $\boldsymbol{R}(t)$ coefficient matrices are independent of w. This makes the Laplace transformation of (4) with respect to w compact. For the same reason, the analysis of completion time can not be conducted in a similar way because it requires a Laplace transformation of (12) with respect to t.

If $\boldsymbol{Q}(t,w)$ and $\boldsymbol{R}(t,w)$ are independent of the reward level w, the moments of the accumulated reward can be computed in a computationally effective way using (20), but the same approach is not applicable for the analysis of the completion time. If only the distribution of the completion time (independent of the structure state at time t) is the performance measure of interest one can apply the following two step procedure:

- estimate the distribution of the accumulated reward based on the obtained moments (e.q., using the method implemented in MRMSolve [14]),
- and from this estimate calculate the distribution of the completion time using (8).

6. Analysis of the moments of completion time

In contrast to the previous section, the moments of the completion time can be calculated in a computationally effective way when the generator of the structure state process and the reward rates depend only on the reward level (w) and they are independent of the system time t (i.e., $\boldsymbol{Q}(t,w) = \boldsymbol{Q}(w)$ and $\boldsymbol{R}(t,w) = \boldsymbol{R}(w)$ $t > 0$).

Let $D_{ij}^{(n)}(w)$ denote the nth moment of the state dependent completion time of w work requirement.

$$D_{ij}^{(n)}(w) = \int_{t=0}^{\infty} t^n f_{ij}(t,w)dt = \int_{t=0}^{\infty} t^n dF_{ij}(t,w) \quad \text{and} \quad \boldsymbol{D}^{(n)} = \{D_{ij}^{(n)}(t)\}.$$

For $n = 0$, $D_{ij}^{(0)}(w) = Pr(Z(C(w)) = j \mid Z(0) = i)$.

Theorem 8. *With strictly positive reward rates the nth moment ($n \geq 1$) of the completion time satisfies the following ordinary differential equation*

$$\frac{d}{dw}\, \boldsymbol{D}^{(n)}(w) = n\, \boldsymbol{D}^{(n-1)}(w)\; \boldsymbol{R}^{-1}(w) + \boldsymbol{D}^{(n)}(w)\; \boldsymbol{R}^{-1}(w)\; \boldsymbol{Q}(w), \tag{24}$$

with initial conditions $\boldsymbol{D}^{(n)}(0) = \boldsymbol{0}, n \geq 1$ and $\boldsymbol{D}^{(0)}(0) = \boldsymbol{I}$.

***Proof* 8.** *Let $\boldsymbol{F}^*(s,w) = \int_{t=0}^{\infty} e^{-st} f_{ij}(t,w)\, dt = \int_{t=0}^{\infty} e^{-st} dF_{ij}(t,w)$. Starting from the Laplace transform of (12) with respect to t:*

$$\Big(s\boldsymbol{F}^*(s,w) - \boldsymbol{f}(0,w)\Big)\boldsymbol{R}^{-1}(w) + \frac{\partial}{\partial w}\boldsymbol{F}^*(s,w) = \boldsymbol{F}^*(s,w)\; \boldsymbol{R}^{-1}(w)\; \boldsymbol{Q}(w), \tag{25}$$

where $\boldsymbol{f}(0,w) = \boldsymbol{0}$ and evaluating the nth derivative at $s \to 0$ gives the theorem.

Theorem 9. *With positive and zero reward rates the nth moment ($n \geq 1$) of the completion time satisfies the following ordinary differential equation*

$$\begin{aligned} \frac{d}{dw}\, \boldsymbol{D^+}^{(n)}(w) &= n\, \boldsymbol{D^+}^{(n-1)}(w)\; \boldsymbol{R^+}^{-1}(w) + \boldsymbol{D^+}^{(n)}(w)\; \boldsymbol{R^+}^{-1}(w)\; \widehat{\boldsymbol{Q}}(w) \\ &+ \sum_{i=1}^{n} \binom{n}{i}\; i!\; \boldsymbol{D^+}^{(n-i)}(w)\; \boldsymbol{R^+}^{-1}(w)\; \boldsymbol{Q^{+0}}(w)\; (-\boldsymbol{Q^0})^{-(i+1)}\; \boldsymbol{Q^{0+}}(w) \end{aligned} \tag{26}$$

where $\widehat{\boldsymbol{Q}}(w) = \boldsymbol{Q^+}(w) - \boldsymbol{Q^{+0}}(w)\boldsymbol{Q^0}^{-1}(w)\boldsymbol{Q^{0+}}(w)$. The initial conditions are $\boldsymbol{D^+}^{(n)}(0) = \boldsymbol{0}, n \geq 1$ and $\boldsymbol{D^+}^{(0)}(0) = \boldsymbol{I}$.

***Proof* 9.** *The Laplace transform of (14) with respect to t is:*

$$\Big(s\boldsymbol{F}^{+*}(s,w)-\boldsymbol{f}^{+}(0,w)\Big)\boldsymbol{R}^{+^{-1}}(w)+\frac{\partial}{\partial w}\boldsymbol{F}^{+*}(s,w)=\boldsymbol{F}^{+*}(s,w)\ \boldsymbol{R}^{+^{-1}}(w)\ \boldsymbol{Q}^{+}(w)+$$
$$\boldsymbol{F}^{+*}(s,w)\ \boldsymbol{R}^{+^{-1}}(w)\ \boldsymbol{Q}^{+\mathbf{0}}(w)\Big(s\boldsymbol{I}-\boldsymbol{Q}^{\mathbf{0}}(w)\Big)^{-1}\boldsymbol{Q}^{\mathbf{0}+}(w), \tag{27}$$

where $\boldsymbol{f}^{+}(0,w)=\mathbf{0}$. The nth derivative of (27) at $s\to 0$ gives the theorem.

For the analysis of $\boldsymbol{D}^{\mathbf{0}+^{(n)}}(w)$, it is necessary to evaluate the moments of the sojourn time in S^0 and the moments of the completion time of the delayed accumulation process with differential equations similar to (24) and (26) and to compute the overall moments via cummulants.

In the case when $\boldsymbol{Q}(w)$ and $\boldsymbol{R}(w)$ depends on the reward level only, it is not possible to evaluate the distribution of the accumulated reward in this way. To approximate the accumulated reward, a distribution estimation and equation (8) needs to be applied.

7. Inhomogeneous MRMs with rate and impulse reward

Consider the same inhomogeneous Markov reward model with impulse reward accumulation at state transitions. The dynamic behaviour of the system is the following. Whenever the CTMC stays in state i at time t and accumulated reward level w, reward is accumulated at rate $0\le r_i(t,w)<\infty$:

$$\frac{dB(t)}{dt}=r_{Z(t)}(t,w) \quad \text{and} \quad B(0)=0\ . \tag{28}$$

When the structure state process undergoes a transition from state i to state j a random amount of reward $\chi_{ij}(t,w)$ (with distribution $C_{ij}(t,w,x)=Pr(\chi_{ij}(t,w)<x)$) is gained instantly:

$$B(t^{+})=B(t^{-})+\chi_{ij}(t,w). \tag{29}$$

By this definition $B(t)$ remains monotone increasing and all of its consequence remains valid (e.g., eq. (8)).

Matrix $\boldsymbol{C}(t,w,x)$ is defined based on $C_{ij}(t,w,x)$ such that the entries associated with state transitions without impulse reward and the diagonal entries equal to the unit step function.

Theorem 10. *In the case of inhomogeneous rate and impulse reward accumulation $\boldsymbol{y}(t,w)$ satisfies the following partial differential equation*

$$\frac{\partial}{\partial t}\boldsymbol{y}(t,w)+\frac{\partial}{\partial w}\boldsymbol{y}(t,w)\ \boldsymbol{R}(t,w)=\int_{x=0}^{w}\boldsymbol{y}(t,w-x)\ \boldsymbol{Q}(t,w-x)\circ d\boldsymbol{C}(t,w-x,x). \tag{30}$$

with the same initial conditions as in Theorem 1.

Note that the element-wise matrix multiplication operation (indicated by $\circ$) has higher precedence than the standard matrix multiplication in this paper.

***Proof* 10.** *The forward argument that describes the evolution of $y_{ij}(t,w)$ over the $(t,t+\Delta)$ interval is*

$$\begin{aligned} y_{ij}(t+\Delta,w) = \;& (1+Q_{jj}(t,w)\Delta)\; y_{ij}(t,w-\Delta r_j(t,w))+ \\ & \sum_{k\in S,k\neq j}\int_{x=0}^{w} Q_{kj}(t,w-x)\Delta\; y_{ik}(t,w-x)\; dC_{kj}(t,w-x,x)+ \\ & o\left(\Delta\right), \end{aligned} \tag{31}$$

The same steps as in the proof of Theorem 1 results in Theorem 10.

Theorem 11. *With strictly positive reward rates, the density of the completion time satisfies the following equation:*

$$\begin{aligned} &\frac{\partial}{\partial t}\boldsymbol{f}(t,w)\boldsymbol{R}^{-1}(t,w)+\frac{\partial}{\partial w}\boldsymbol{f}(t,w) = \\ &\int_{x=0}^{w}\boldsymbol{f}(t,w-x)\;\boldsymbol{R}^{-1}(t,w-x)\;\boldsymbol{Q}(t,w-x)\circ d\boldsymbol{C}(t,w-x,x)\;, \end{aligned} \tag{32}$$

with initial conditions $f_{ij}(t,0)=\delta_{ij}\;\Omega(t)$ and $f_{ij}(0,w)=0$.

***Proof* 11.** *The forward argument describing the behaviour of $f_{ij}(t,w)$ is the following*

$$\begin{aligned} & f_{ij}(t,w+\Delta) = (1+Q_{jj}(t,w)\;\tfrac{\Delta}{r_j(t,w)})\; f_{ij}(t-\tfrac{\Delta}{r_j(t,w)},w) \\ & +\sum_{k\in S,k\neq j}\int_{x=0}^{w} Q_{kj}(t,w-x)\;\tfrac{\Delta}{r_k(t,w-x)}\; f_{ik}(t-d_{kj}(t,w)\Delta,w-x)\; dC_{kj}(t,w-x,x) \\ & +\;\sigma(\Delta)\;, \end{aligned} \tag{33}$$

with $0<d_{kj}(t,w)<\infty$. The same analysis steps as before results Theorem 11.

Reward level independent case

In the special case when $\boldsymbol{Q}(t,w)$, $\boldsymbol{R}(t,w)$ and $\boldsymbol{C}(t,w,x)$ are all independent of the reward level (i.e., $\boldsymbol{Q}(t,w)=\boldsymbol{Q}(t)$, $\boldsymbol{R}(t,w)=\boldsymbol{R}(t)$, $\boldsymbol{C}(t,w,x)=\boldsymbol{C}(t,x), w>0$) we can analyze the moments of accumulated reward in a computationally effective way. The nth moment of the impulse reward gained at the state transition from i to j at time t is $C_{ij}^{(n)}(t)=\int_{x=0}^{\infty}x^n dC_{ij}(t,x)$, and matrix $\boldsymbol{C}^{(n)}(t)$ is composed by the $C_{ij}^{(n)}(t)$ elements.

Theorem 12. *The nth moment ($n\geq 1$) of the accumulated reward can be calculated based on the following ordinary differential equation*

$$\frac{d}{dt}\;\boldsymbol{V}^{(n)}(t) = n\;\boldsymbol{V}^{(n-1)}(t)\;\boldsymbol{R}(t)+\sum_{i=0}^{n}\binom{n}{i}\;\boldsymbol{V}^{(n-i)}(t)\;\boldsymbol{Q}(t)\circ\boldsymbol{C}^{(i)}(t), \tag{34}$$

with initial condition $\boldsymbol{V}^{(n)}(0)=\boldsymbol{0}, n\geq 1$ and $\boldsymbol{V}^{(0)}(0)=\boldsymbol{I}$.

***Proof* 12.** *With $\boldsymbol{C}^*(t,v)=\int_{x=0}^{\infty}e^{-xv}d\boldsymbol{C}(t,x)$, the Laplace transform of (30) with respect to w is*

$$\frac{\partial}{\partial t}\boldsymbol{Y}^*(t,v)+\Big(v\boldsymbol{Y}^*(t,v)-\boldsymbol{y}(t,0)\Big)\boldsymbol{R}(t) = \boldsymbol{Y}^*(t,v)\;\boldsymbol{Q}(t)\circ\boldsymbol{C}^*(t,v). \tag{35}$$

The nth derivative of (35) and the $v\to 0$ limit results in Theorem 12.

Unfortunately, there is not similar effective numerical methods for the moments of the completion time, because it can not be calculated based on the moments of the impulse reward.

8. Numerical example

To demonstrate the application of inhomogeneous MRMs in practice we analyze the reward measures of the inhomogeneous queueing system presented in Figure 2. Users arrive to a queue according to a Poisson process with parameter λ. The queue comprises three servers whose service rate depends on the time elapsed since the start of the system. The service rate as the function of transient time is shown in Figure 3. The system has room for K users including those in service. The rate at which reward is accumulated is also a function of the elapsed time (pointed at by dashed arrows in Figure 2).

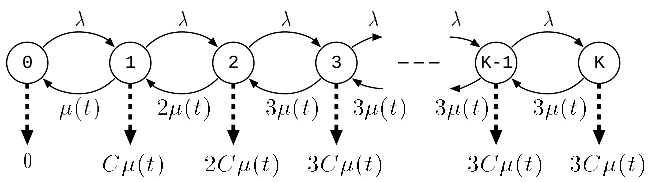

Figure 2. Queue with servers aging according to elapsed time

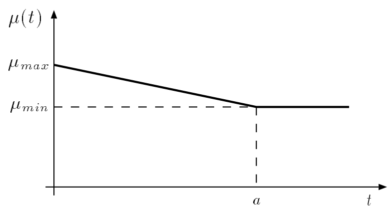

Figure 3. Function that describes the aging of the server

Additionally, two variants of the example are considered:

- The example is complemented with impulse rewards: the system gains 0.2, 0.4 or 0.6 units of reward with probability 1/3 each when a user leaves the queue.
- The aging of the servers depends on the amount of the performed work (i.e., on the accumulated reward) instead of the elapsed time. In the example, $\mu(t)$ is substituted by $\mu(w)$.

The distribution of the accumulated reward is computed by discretization of equation (4) in absence of impulse reward. In the presence of impulse reward equation (30) has to be solved. The distribution of the completion time is gained by solving equation (14). For the solution of these two-variable PDEs, we have adopted the method proposed in [9].

For the computation of the moments, equations (20), (26) and (34) have to be solved numerically. Since these are single variable ODEs without special difficulties, there are many methods to obtain their solution. To generate the results for our example we used TR-BDF2 algorithm with adaptive step size control, according to [16]. The general form of (20), (26) and (34) is the following:

$$\frac{d}{dx}\,\boldsymbol{U}(x) = \boldsymbol{U}(x)\ \boldsymbol{A}(x) + \boldsymbol{B}(x).$$

Using this general form, an elementary step of the TR-BDF2 method to compute the solution at x_{i+1} from the solution at x_i (with step size $h = x_{i+1} - x_i$) is composed by two parts: the TR part calculate the solution at $x_{i+\gamma} = x_i + \gamma h$

$$\boldsymbol{U}(x_{i+\gamma}) = \left(\boldsymbol{U}(x_i)\left(\boldsymbol{I} + \frac{h\gamma}{2}\boldsymbol{A}(x_i)\right) + \boldsymbol{B}(x_{i+\gamma}) + \boldsymbol{B}(x_i)\right) \cdot \left[\boldsymbol{I} - \frac{h\gamma}{2}\boldsymbol{A}(x_{i+1})\right]^{-1}.$$

and the BDF2 part calculates the solution at x_{i+1}

$$\boldsymbol{U}(x_{i+1}) = \left(\frac{1}{\gamma(2-\gamma)}\boldsymbol{U}(x_{i+\gamma}) - \frac{(1-\gamma)^2}{\gamma(2-\gamma)}\boldsymbol{U}(x_i) + h\frac{1-\gamma}{2-\gamma}\boldsymbol{B}(x_i)\right) \cdot \left[\boldsymbol{I} - h\frac{1-\gamma}{2-\gamma}\boldsymbol{B}(x_{i+1})\right]^{-1}.$$

For the adaptive step size control the local truncation error (LTE) is computed after each step [16]. If it exceeds a given threshold, the step size is divided by 2, otherwise it is increased by 1.2.

The example was solved with the following set of parameters:

$$\lambda = 1.0,\ K = 5,\ C = 0.5,\ \mu_{max} = 2.0,\ \mu_{min} = 1.0,\ a = 4.0.$$

For the case when the servers age according to the elapsed time, the distribution of accumulated reward for different transient time with and without impulse reward are given in Figure 4 and 6, respectively. The moments of the accumulated reward are depicted in Figure 5 and 7. In both cases the initial state was state 0.

For the case when the servers age according to the accumulated reward, the distribution of completion time for different reward levels are shown in Figure 8. The associated moments are depicted in Figure 9. The background process starts in state 1.

We implemented these computational methods in MATLAB. The solution of the two-variable PDEs (i.e. the calculation of the distributions) took approximately 5 hours on a machine running at 1 Ghz. The moments can be computed much faster, it needed a few minutes to calculate the results (keeping the local truncation error below $LTE = 10^{-6}$).

The time to compute the moments with $K = 50$ took less than 10 minutes for our MATLAB code with matrix ODE solution using full matrix multiplications.

The moment curves in Figure 5, 7 and 9 were evaluated in two ways, based on the distribution of the reward measures and based on the ODEs providing the moments directly. Apart of negligible numerical inaccuracy the results equal. The results of the introduced numerical methods have been compared with the results of a simulation tool [10], and we obtained perfect matching up to the first three meaningful digits.

We also implemented a computation method for the analysis of moments with fixed step size ODE solvers in C optimized to speed and memory requirement. This implementation uses only vector measures and as a consequence calculates only vector-matrix multiplications. We applied a sparse matrix representation of the generator matrix. To check the computational complexity of the analysis method we evaluated the same example for increasing values of K

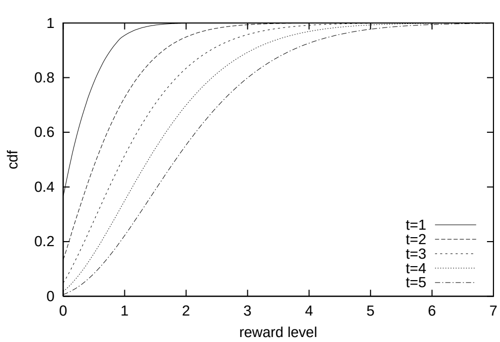

Figure 4. Distribution of accumulated reward for different values of time t

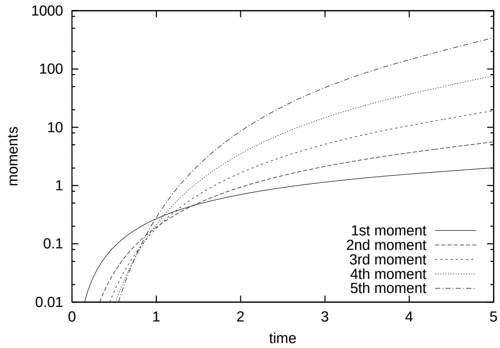

Figure 5. Moments of accumulated reward as a function of time t

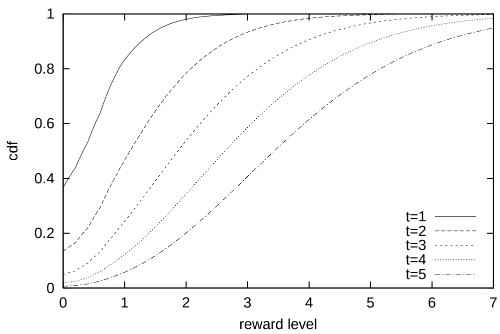

Figure 6. Distribution of accumulated reward for different time t with impulse rewards

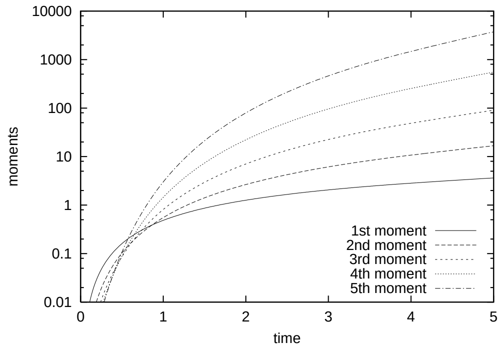

Figure 7. Moments of accumulated reward as a function of time t with impulse rewards

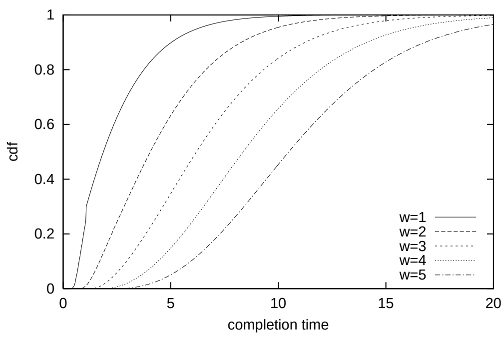

Figure 8. Distribution of completion time for different values of reward requirement

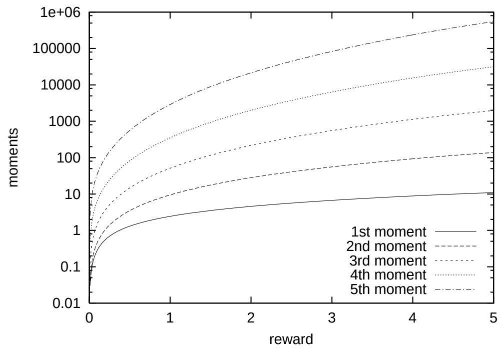

Figure 9. Moments of completion time as function of reward requirement

with this effective C implementation. A 2.4 Ghz windows machine with 512MB RAM computed the reward measures. The dependence of the execution time on K is depicted in Figure 10 In

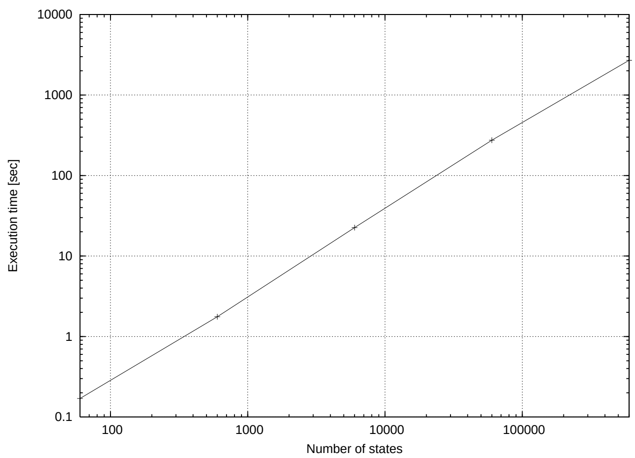

Figure 10. Execution time of the ODE solution method versus K

this example the MRM is contains $K+1$ states and $2K$ transitions.

With $K = 10^6$ the computation time of one elementary step of the ODE solution was around 1 second, and obtaining two moments of the accumulated reward in $t \in (0, 1)$ took about 20 minutes. The computation time is proportional to the length of the time interval (in case of fixed step size) and the number of moments. E.g. the computation time of the same example with $t \in (0, 5)$ and calculating 5 moments (indeed it means $5 + 1$ including the 0th moment) was 200 minutes. This example shows that the presented equations can be implemented in an efficient way, which allows the analysis of fairly large inhomogeneous MRMs.

9. Conclusion

In this paper we presented the analytical description of the reward measures of inhomogeneous Markov reward models. In contrast with the analytical description of homogeneous MRMs, only forward differential equations can be used to describe inhomogeneous MRMs. The distribution of reward measures are characterized by PDEs, but we provided ODEs for the analysis of the moments of reward measures when it was possible.

Similar to the analysis methods available for homogeneous MRMs, the computational complexity of calculating reward measure distributions is much higher than the one of calculating its moments. Using effective implementations the complexity of the methods to calculate the moments of inhomogeneous MRMs is comparable to the complexity of the ones that calculates the moments of homogeneous MRMs based on randomization [18], i.e., models with $\sim 10^6$ states can be analyzed.

A numerical example of an inhomogeneous queueing system is studied using the presented analytical description and the associated numerical methods. The results of the PDE and the ODE solvers and a simulator package showed a perfect matching.

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