Some structural properties of Markov and Rational Arrival Processes

Miklós Telek
Department of Telecommunications, Technical University of Budapest,
H-1521 Budapest, Hungary, telek@hit.hme.hu

Abstract

The structural properties of the moments of Markov arrival processes (MAPs) and Rational arrival processes (RAPs) are considered in this paper. We investigate how many and which moments can characterize these processes and show that redundant RAPs/MAPs of order n are characterized by less than n^2 independent moments.

Keywords: Markov arrival process, Rational arrival processes, moments based characterization, Hankel matrix.

1 Introduction

Markov arrival processes (MAPs) [11, 10, 8] are versatile point processes with flexibility to approximate real life (possibly correlated) data. The popularity of MAPs come from the fact that efficient numerical procedures [9, 4] are available for the numerical analysis of complex systems with MAP input and/or service process and the analysis of those models is supported with nice stochastic interpretations. As it is shown in [3] practically the same efficient numerical procedures can be used for the analysis of complex systems with the more general rational arrival processes (RAPs) [1] as input and/or service process, but in this case it is without a nice stochastic interpretation.

Similar to the relation of phase type (PH) [9, 12] and matrix exponential (ME) [2] distributions the set of MAPs of a given order is a subset of the set of RAPs of the same order (in case of order 2 the two sets are identical [5] and in case of higher orders MAPs form a real subset). In this paper we investigate RAP and MAP properties which are consequences of their matrix exponential nature and we do not devote particular attention to the question if there exists a MAP or a RAP of order n with a given set of properties.

MAPs and RAPs are commonly defined by a matrix-pair and it is well established how to obtain the basic properties (like joint density function and joint moments of inter-arrival times, lag correlation, etc.) based on a matrix-pair representation [9]. Here we focus on the opposite problem. We investigate how to characterize a MAP/RAP based on a given set of moments. The matrix-pair representation of MAPs/RAPs is known to contain more parameters than the number of independent parameters [13]. We investigate how many parameters characterize a MAP/RAP and the nature of these parameters.

The paper is organized as follows. The next section summarizes the main properties of MAPs and RAPs. In Section 3 we show that no finite set of moments can completely characterize a MAP/RAP of unknown (unbounded) order. Section 4 presents elements of moments based characterization for the case when the order is bounded. Classifications of low order cases are summarized in Section 5 and 6, and some properties of the higher order cases are presented in Section 7. Finally, an example of a redundant MAP arises from a queueing application is presented in Section 8.

2 Preliminaries

A MAP/RAP has representations of different sizes. We start by introducing an important tool (the Hankel matrix) to evaluate the order of a representation (which might be different from the size of the representation) and than we summarize basic MAP/RAP properties.

Definition 1. *The matrix composed by the elements of the series* $\{a_0, a_1, a_2, \ldots\}$ *as*

$$\mathbf{H}(\{a_0, a_1, a_2, \ldots\}) = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1}$$

is referred to as **Hankel matrix** *(1).*

According to this definition $H_{i,j}(\{a_0, a_1, \ldots\}) = a_{i+j}$, where the rows and the columns of $\mathbf{H}$ are numbered from 0.

Definition 2. *The* **Hankel order** *of the series* $\{a_0, a_1, \ldots\}$*, denoted as* $\mathrm{HO}(\{a_0, a_1, \ldots\})$*, is the rank of the Hankel matrix* $\mathbf{H}(\{a_0, a_1, \ldots\})$*. I.e.,* $\mathrm{HO}(\{a_0, a_1, \ldots\}) = \mathrm{rank}(\mathbf{H}(\{a_0, a_1, \ldots\}))$.

For a detailed introduction on MAP and RAP we refer, e.g., to [9] and [1]. Below we summarize only the main results. Throughout the paper we consider continuous time MAPs and RAPs.

A MAP is a point process which is modulated by a continuous time Markov chain (CTMC). MAPs are usually defined by two matrices. $\mathbf{D_0}$ contains the transition rates of the CTMC without an event (or arrival in case of an arrival process) and $\mathbf{D_1}$ describes the ones with an event. $\mathbf{D_0}$ is supposed to be non-singular and $\mathbf{D} = \mathbf{D_0} + \mathbf{D_1}$ is the generator of the CTMC modulating the point process. Since $\mathbf{D}$ is a generator matrix, its row sums are equal to zero, i.e., $\mathbf{D}\mathbb{1} = \mathbf{0}$, where $\mathbb{1}$ ($\mathbf{0}$) denotes the column vector of ones (zeros) of appropriate size. Consequently, $\mathbf{D_0}\mathbb{1} = -\mathbf{D_1}\mathbb{1}$.

In the analysis of MAPs, the state of the background CTMC (commonly referred to as “phase”) at arrival instants plays an important role. The state of the background CTMC at consecutive arrivals is referred to as the *process embedded at arrival instants*. The embedded process is a discrete time Markov chain (DTMC) with transition probability matrix $\mathbf{P} = (-\mathbf{D_0})^{-1}\mathbf{D_1}$. From $\mathbf{D_0}\mathbb{1} = -\mathbf{D_1}\mathbb{1}$ we also have $\mathbf{P}\mathbb{1} = \mathbb{1}$. The stationary probability vector of the embedded process, π, is the solution of the linear system $\pi\mathbf{P} = \pi, \pi\mathbb{1} = 1$. In steady state, the inter-arrival time is PH distributed with initial probability vector π and transient generator $\mathbf{D_0}$.

In case of MAPs the listed matrices obey structural restrictions. The elements of $\mathbf{D_1}, \mathbf{P}, \pi$ and the non-diagonal elements of $\mathbf{D_0}, \mathbf{D}$ are non-negative, and the diagonal elements of $\mathbf{D_0}, \mathbf{D}$ are negative. By relaxing these structural properties the set of arrival processes are generalized to the class of RAPs, whose stationary inter-arrival time is ME distributed.

The properties of MAPs or RAPs defined by a pair of matrices $(\mathbf{D_0}, \mathbf{D_1})$ are computed in the same ways. The cumulative distribution function (cdf), the probability density function (pdf) the Laplace transform, the moments and the reduced moments of the stationary inter-arrival time are $F_X(x) = P(X < x) = 1 - \pi e^{\mathbf{D_0}x}\mathbb{1}$, $f_X(x) = \pi e^{\mathbf{D_0}x}\mathbf{D_1}\mathbb{1}$, $f_X^*(s) = E(e^{-sX}) = \pi(s\mathbf{I} - \mathbf{D_0})^{-1}\mathbf{D_1}\mathbb{1}$, $E(X^k) = k!\,\pi(-\mathbf{D_0})^{-k}\mathbb{1}$, $\mu_k = E(X^k)/k! = \pi(-\mathbf{D_0})^{-k}\mathbb{1}$, where $\pi\mathbf{P} = \pi, \pi\mathbb{1} = 1$ and $\mathbf{P} = (-\mathbf{D_0})^{-1}\mathbf{D_1}$.

Definition 3. *The* **size** *of a PH/ME distribution with representation* $(\pi, \mathbf{D_0})$ *is the size of the square matrix* $\mathbf{D_0}$.

Definition 4. *The* **order** *of a PH/ME distribution is the Hankel order of its reduced moments series, i.e.,* $\mathrm{HO}(\{\mu_0, \mu_1, \mu_2, \ldots\})$.

The inter-arrival times in MAPs/RAPs are not independent. The joint density function of the inter-arrival times $X_0, X_1, \ldots, X_k$ is $f(x_0, x_1, \ldots, x_k) = \pi e^{\mathbf{D_0}x_0}\mathbf{D_1}e^{\mathbf{D_0}x_1}\mathbf{D_1}\ldots e^{\mathbf{D_0}x_k}\mathbf{D_1}\mathbb{1}$, and the $k+1$-tuple joint moment of the $s_0 = 0 < s_1 < s_2 < \cdots < s_k$-th inter arrival times is

$$E(X_0^{i_0}X_{s_1}^{i_1}\ldots X_{s_k}^{i_k}) = \pi i_0!(-\mathbf{D_0})^{-i_0}\mathbf{P}^{a_1}i_1!(-\mathbf{D_0})^{-i_1}\ldots\ \mathbf{P}^{a_k}i_k!(-\mathbf{D_0})^{-i_k}\mathbb{1}, \tag{2}$$

where $a_i = s_i - s_{i-1}$. The $k+1$-tuple reduced joint moment is

$$\gamma_{i_0,i_1,\ldots,i_k}^{(a_1,\ldots,a_k)} = \frac{E(X_0^{i_0}X_{s_1}^{i_1}\ldots X_{s_k}^{i_k})}{i_0!\ldots i_k!} = \pi(-\mathbf{D_0})^{-i_0}\mathbf{P}^{a_1}(-\mathbf{D_0})^{-i_1}\ldots\ \mathbf{P}^{a_k}(-\mathbf{D_0})^{-i_k}\mathbb{1}. \tag{3}$$

To simplify the notation we use $\gamma_{i,j}^{(1)} = \gamma_{i,j}$ and $\mathbf{E} = (-\mathbf{D_0})^{-1}$ in the sequel.

Definition 5. *The* **size** *of a MAP/RAP with representation* $(\mathbf{D_0}, \mathbf{D_1})$ *is the size of the square matrix* $\mathbf{D_0}$.

The matrix representation of a MAP/RAP is not unique. There are infinitely many matrix pairs representing the same process. E.g., $(\mathbf{D_0}, \mathbf{D_1})$ and $(\mathbf{B}^{-1}\mathbf{D_0}\mathbf{B}, \mathbf{B}^{-1}\mathbf{D_1}\mathbf{B})$ represents the same MAP/RAP, if $\mathbf{B}$ is a non-singular square matrix such that $\mathbf{B}\mathbb{1} = \mathbb{1}$. This similarity transformation maintains the size of the representation, but there are transformations between representations of different sizes [7].

Definition 6. *A MAP/RAP with representation* $(\mathbf{D_0}, \mathbf{D_1})$ *is* **minimal** *if there is no representation of the same process with smaller size. In this case the* $(\mathbf{D_0}, \mathbf{D_1})$ *representation is referred to as* **minimal representation**.

Definition 7. *The size of the minimal representation is referred to as the* **order** *of the MAP/RAP.*

How to find the order and a minimal representation of a MAP/RAP based on its matrix representation is discussed in [7].

Definition 8. *A MAP/RAP is* **non-redundant** *if the order of the stationary inter-arrival time distribution is identical with the order of the MAP/RAP and* **redundant** *otherwise.*

The moments based characterization of non-redundant MAPs/RAPs is available in [13, 6]. The most important features are the following:

- a non-redundant MAP/RAP of order n is characterized by n^2 parameters,
- the joint density of two consecutive inter-arrivals of a non-redundant MAP/RAP characterizes the process,
- the first $2n-1$ moments of the inter-arrival time, μ_i, $i = 1, 2, \ldots, 2n-1$, and the first $(n-1)^2$ joint moments of two consecutive inter-arrivals γ_{ij}, $i, j = 1, 2, \ldots, n-1$, of an order n non-redundant MAP/RAP (referred to as *basic moments set*) fully characterize the process.

We are going to show that none of these features are valid, in general, for redundant MAPs/RAPs. Our main goal is to investigate the properties of MAPs/RAPs based on their moments (from now on, the general term *moments* refers to the set of reduced moments and reduced joint moments including double, triple, etc. joint moments). The moments based characterization of MAPs/RAPs is motivated by the fact that the matrix representation contains too many parameters ($2n^2$), while a non-redundant MAP/RAP is fully characterized by n^2 moments (referred to as *basic moments set*). d

3 Properties of MAP/RAP moments

In the previous section the introduction of MAP/RAPs assumes the existence of a matrix representation. In this section we assume that all moments of a MAP/RAP is available and restate some of the basic MAP/RAP properties based on this set of information. The main tool for checking these properties is the Hankel order of moments series.

3.1 Hankel order of moments series

Theorem 1. *If the Hankel order of a moments series is u, then the order of the MAP/RAP is greater or equal to u.*

Proof. We prove the statement for $\{\gamma_{i,0,j}^{(k,\ell)}, \gamma_{i,1,j}^{(k,\ell)}, \gamma_{i,2,j}^{(k,\ell)}, \ldots\}$. The proof for any other moments series follows the same pattern. The elements of the moments series $\{\gamma_{i,0,j}^{(k,\ell)}, \gamma_{i,1,j}^{(k,\ell)}, \gamma_{i,2,j}^{(k,\ell)}, \ldots\}$ can be written as

$$\gamma_{i,n,j}^{(k,\ell)} = \pi \mathbf{E}^i \mathbf{P}^k \mathbf{E}^n \mathbf{P}^\ell \mathbf{E}^j \mathbb{1}, \quad n = 0, 1, 2, \ldots,$$

and the associated Hankel matrix can be decomposed as

$$\mathbf{H}(\underline{\gamma}) = \begin{bmatrix} \pi\mathbf{E}^i\mathbf{P}^k\mathbf{E}^0 \\ \pi\mathbf{E}^i\mathbf{P}^k\mathbf{E}^1 \\ \pi\mathbf{E}^i\mathbf{P}^k\mathbf{E}^2 \\ \cdots \end{bmatrix} \left[\ \mathbf{E}^0\mathbf{P}^\ell\mathbf{E}^j\mathbb{1} \ \middle|\ \mathbf{E}^1\mathbf{P}^\ell\mathbf{E}^j\mathbb{1} \ \middle|\ \mathbf{E}^2\mathbf{P}^\ell\mathbf{E}^j\mathbb{1} \ \middle|\ \ldots\ \right], \tag{4}$$

where the first matrix is composed by row vectors and the second one by column vectors of the size of the representation. Based on this decomposition the order of the process could not be less than u. □

3.2 Minimal size and redundancy

From Definition 4 and 8 we have the following corollaries.

Corollary 1. *A MAP/RAP is redundant if there exists a series of moments* $\underline{\gamma} = \{\gamma_{i,0,j}^{(k,\ell)}, \gamma_{i,1,j}^{(k,\ell)}, \gamma_{i,2,j}^{(k,\ell)}, \ldots\}$ *such that* $\mathrm{HO}(\underline{\gamma}) > \mathrm{HO}(\underline{\mu})$ *where* $\underline{\mu} = \{\mu_0, \mu_1, \mu_2, \ldots\}$.

Corollary 2. *The order of a MAP/RAP is the maximum of the Hankel orders of all moments series.*

Next we investigate how many moments need to be analyzed for finding a moments series $\underline{\gamma}$ for which $\mathrm{HO}(\underline{\gamma}) > \mathrm{HO}(\underline{\mu})$ and how to find a moments series of maximal Hankel order. Unfortunately, in general, these questions cannot be answered based on a finite number of moments. To prove this we present a MAP/RAP structure which exhibits a low order MAP/RAP behavior based on the analysis of low order moments and a higher order behavior based on the analysis of higher order moments. To do that we first introduce the extended representation of MAPs/RAPs.

3.3 Extended representation

Let $(\check{\mathbf{D}}_\mathbf{0}, \check{\mathbf{D}}_\mathbf{1})$ be a matrix representation of a MAP/RAP where the matrices define the process according to (2)-(3), i.e., the closing vector is $\mathbb{1}$ and the initial vector is the solution of $\check{\pi}(-\check{\mathbf{D}}_\mathbf{0})^{-1}\check{\mathbf{D}}_\mathbf{1} = \check{\pi}\check{\mathbf{P}} = \check{\pi}, \check{\pi}\mathbb{1} = 1$. To study further properties of moments we need representations with general initial and closing vector. A similarity transformation

$$\check{\pi} \Rightarrow \delta = \check{\pi}\mathbf{B}^{-1}, \check{\mathbf{D}}_\mathbf{0} \Rightarrow \mathbf{D_0} = \mathbf{B}\check{\mathbf{D}}_\mathbf{0}\mathbf{B}^{-1}, \check{\mathbf{D}}_\mathbf{1} \Rightarrow \mathbf{D_1} = \mathbf{B}\check{\mathbf{D}}_\mathbf{1}\mathbf{B}^{-1}, \mathbb{1} \Rightarrow h = \mathbf{B}\mathbb{1},$$

with non-singular matrix $\mathbf{B}$ (where $\mathbf{B}\mathbb{1}$ can be different from $\mathbb{1}$) results in a representation with general closing vector. For notational convenience we use $\mathbf{E}$ and $\mathbf{P}$ instead of $\mathbf{D_0}$ and $\mathbf{D_1}$ in the extended representation.

Definition 9. *We refer to the four-tuple* $(\delta, \mathbf{E}, \mathbf{P}, h)$ *as* **extended representation** *of a MAP/RAP. The elements of the extended representation satisfy*

$$\delta\mathbf{P} = \delta, \quad \mathbf{P}h = h, \quad \delta h = 1. \tag{5}$$

3.4 MAPs/RAPs with identical low order behavior

Consider a MAP/RAP of size u_1 with extended representation $(\delta, \mathbf{E}, \mathbf{P}, h)$ and a MAP/RAP with extended representation $(\delta', \mathbf{E}', \mathbf{P}', h')$, where $\delta' = \{\delta, 0, 0, \ldots, 0, 0\}$,

$$\mathbf{E}' = \begin{bmatrix} \mathbf{E} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \ \mathbf{P}' = \begin{bmatrix} \mathbf{P} & p_v & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ p_h & & & & 0 \end{bmatrix}, h' = \begin{bmatrix} h \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

The size of the second MAP/RAP is u_1+u_2 and the size of the blocks of the representation are $u_1, 1, 1, \ldots, 1, 1$, respectively. To fulfill $\delta'\mathbf{P}' = \delta'$ and $\mathbf{P}'h' = h'$, p_v and p_h are such that $\delta p_v = 0$ and $p_h h = 0$.

Corollary 3. *The μ_i moment for $i \in \mathbb{N}$, the $\gamma^{(k,\ell)}_{i,n,j}$ moment for $i, j, n \in \mathbb{N}$ and $k + \ell \le u_2$, and the $\gamma^{(a_1,\dots,a_k)}_{i_0,i_1,\dots,i_k}$ moment for $i_s \in \mathbb{N}$, $s \in \{0, 1, \dots, k\}$, $\sum_{u=1}^{k} a_u \le u_2$ are identical for the processes with representation $(\delta, \mathbf{E}, \mathbf{P}, h)$ and $(\delta', \mathbf{E}', \mathbf{P}', h')$.*

Proof. We first prove the corollary for $\gamma^{(k,\ell)}_{i,n,j}$ with $k + \ell \le u_2$. Prime indicates the parameters of the process with representation $(\delta', \mathbf{E}', \mathbf{P}', h')$.

$$
\begin{aligned}
\gamma'^{(k,\ell)}_{i,n,j} &= \underbrace{\delta'\mathbf{E}'^{i}}\mathbf{P}'^{k}\mathbf{E}'^{n}\mathbf{P}'^{\ell}\underbrace{\mathbf{E}'^{j}h'} \\
&= [\delta\mathbf{E}^{i}, 0, 0, \dots, 0, 0]\mathbf{P}'^{k}\mathbf{E}'^{n}\mathbf{P}'^{\ell}\begin{bmatrix}\mathbf{E}^{j}h \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0\end{bmatrix} = [\delta\mathbf{E}^{i}\mathbf{P}^{k}, \underbrace{\star, \dots, \star}_{k}, 0, \dots, 0]\mathbf{E}'^{n}\begin{bmatrix}\mathbf{P}^{\ell}\mathbf{E}^{j}h \\ 0 \\ \vdots \\ 0 \\ \star \\ \vdots \\ \star\end{bmatrix}\Big\}\ell \quad = \\
&= \delta\mathbf{E}^{i}\mathbf{P}^{k}\mathbf{E}^{n}\mathbf{P}^{\ell}\mathbf{E}^{j}h = \gamma^{(k,\ell)}_{i,n,j},
\end{aligned}
$$

where $\star$ refers to potentially non-zero vector elements. The proof for the other cases (μ_i, $\gamma^{(a_1,\dots,a_k)}_{i_0,i_1,\dots,i_k}$) follows the same pattern, since the structure of the initial and closing vector depends only on the number of multiplications with $\mathbf{P}'$. The multiplications with $\mathbf{E}'$ does not modify the zero structure of the initial and closing vectors. □

As a consequence of Corollary 3 the full characterization of a MAP/RAP requires the analysis of both, low and high order, moments. If we are not given an upper bound of the order of an MAP/RAP then it is impossible to obtain its order based on any bounded moments set. In the opposite case it is possible to characterize the process based on a finite moments set and the next section is devoted to this task.

4 Moments based characterization of MAPs/RAPs

In order to investigate the properties of moments series we study how those moments series are related with a particular representation. Without loss of generality, we assume that the considered representation is minimal. As a consequence the identical eigenvalues of $\mathbf{D_0}$ (or equivalently $\mathbf{E}$) belong to the same Jordan block. It is because any representation with identical eigenvalues in different Jordan blocks can be transformed to a smaller representation. Furthermore we are going to utilize the fact that the eigenvalues of $\mathbf{D_0}$ are non-zero. First we introduce a useful representation which expresses important structural properties of MAP/RAP moments series, than we investigate how to obtain a moments series with maximal Hankel order.

4.1 Jordan representation

Let $(\pi, \mathbf{E}, \mathbf{P}, \mathbb{1})$ be an extended representation of a MAP/RAP and $\mathbf{E} = \mathbf{B}^{-1}\mathbf{\Lambda}\mathbf{B}$ the Jordan decomposition of $\mathbf{E}$. Applying a similarity transformation with matrix $\mathbf{B}$ on $(\pi, \mathbf{E}, \mathbf{P}, \mathbb{1})$ we obtain

$$(\pi\mathbf{B}^{-1}, \mathbf{B}\mathbf{E}\mathbf{B}^{-1}, \mathbf{B}\mathbf{P}\mathbf{B}^{-1}, \mathbf{B}\mathbb{1}) = (\delta, \mathbf{\Lambda}, \hat{\mathbf{P}}, h),$$

where from (5) we have $\delta\hat{\mathbf{P}} = \delta, \hat{\mathbf{P}}h = h, \delta h = 1$.

Definition 10. *The* **Jordan representation** *of a MAP/RAP is an extended representation $(\delta, \mathbf{\Lambda}, \hat{\mathbf{P}}, h)$ where $\mathbf{\Lambda}$ is a Jordan matrix.*

4.2 Moments series of maximal Hankel order

The computation of the moments based on the Jordan representation,

$$\gamma^{(a_1,\ldots,a_k)}_{i_0,i_1,\ldots,i_k} = \frac{E(X_0^{i_0} X_{s_1}^{i_1} \ldots X_{s_k}^{i_k})}{i_0! \ldots i_k!} = \delta \mathbf{\Lambda}^{i_0} \hat{\mathbf{P}}^{a_1} \mathbf{\Lambda}^{i_1} \ldots \ \hat{\mathbf{P}}^{a_k} \mathbf{\Lambda}^{i_k} h \tag{6}$$

is an efficient tool to visualize the rank limitation of the Hankel matrix of moments series. Similar to (4) the Hankel matrix of the moments series $\underline{\gamma} = \{\gamma^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_{\ell-1},0,i_{\ell+1},\ldots,i_k}, \gamma^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_{\ell-1},1,i_{\ell+1},\ldots,i_k}, \gamma^{(a_1,\ldots,a_k)}_{i_0,\ldots,i_{\ell-1},2,i_{\ell+1},\ldots,i_k}, \ldots\}$ can be decomposed as

$$\mathbf{H}(\underline{\gamma}) = \begin{bmatrix} \delta \mathbf{\Lambda}^{i_0} \hat{\mathbf{P}}^{a_1} \ldots \mathbf{\Lambda}^{i_{\ell-1}} \hat{\mathbf{P}}^{a_\ell} \mathbf{\Lambda}^0 \\ \delta \mathbf{\Lambda}^{i_0} \hat{\mathbf{P}}^{a_1} \ldots \mathbf{\Lambda}^{i_{\ell-1}} \hat{\mathbf{P}}^{a_\ell} \mathbf{\Lambda}^1 \\ \ldots \end{bmatrix} \left[\ \mathbf{\Lambda}^0 \hat{\mathbf{P}}^{a_{\ell+1}} \mathbf{\Lambda}^{i_{\ell+1}} \ldots \hat{\mathbf{P}}^{a_k} \mathbf{\Lambda}^{i_k} h \ \middle| \ \mathbf{\Lambda}^1 \hat{\mathbf{P}}^{a_{\ell+1}} \mathbf{\Lambda}^{i_{\ell+1}} \ldots \hat{\mathbf{P}}^{a_k} \mathbf{\Lambda}^{i_k} h \ \middle| \ \ldots \ \right], \tag{7}$$

where the first matrix is composed by row vectors and the second one by column vectors of the size of the representation. For the sake of simplicity, we discuss the cases with and without a real Jordan block in $\mathbf{\Lambda}$ separately.

Matrix $\mathbf{\Lambda}$ is diagonal: A diagonal matrix $\mathbf{\Lambda}$ means that the eigenvalues of $\mathbf{\Lambda}$ are different (because there is no Jordan block larger than 1 and there are no identical Jordan blocks in a minimal representation) and non-zero.

In this case the non-zero elements of the row vector $\underline{u} = \delta \mathbf{\Lambda}^{i_0} \hat{\mathbf{P}}^{a_1} \ldots \mathbf{\Lambda}^{i_{\ell-1}} \hat{\mathbf{P}}^{a_\ell}$ determine the rank of the first matrix in (7). More precisely, the rank of the first matrix equals to the number of non-zero elements of $\underline{u}$. Similarly the rank of the second matrix in (7) equals to the number of non-zero elements of the column vector $\underline{v} = \hat{\mathbf{P}}^{a_{\ell+1}} \mathbf{\Lambda}^{i_{\ell+1}} \ldots \hat{\mathbf{P}}^{a_k} \mathbf{\Lambda}^{i_k} h$. The rank of the product equals to the overlapping non-zero entries of these vectors, i.e., the non-zero entries of $\underline{u} \odot \underline{v}^T$, where $\odot$ denotes the element-wise multiplication.

If both, δ and h, are composed of non-zero elements then, the rank of the moments series $\{\mu_0, \mu_1, \ldots\}$ is identical with the size of the representation (which is minimal), i.e., the MAP is non-redundant. For the analysis of the redundant cases, when δ or h contains at least one zero element, we differentiate structural and random zero elements in the $\underline{u}$ type row vectors and in the $\underline{v}$ type column vectors according to the following definition.

Let $\mathbf{A}$ and $\mathbf{B}$ be real or complex valued matrixes of size $n \times m$ and $m \times k$, respectively. The i, j element of $\mathbf{A}^\star$ ($\mathbf{B}^\star$) is 0 if $\mathbf{A}_{ij} = 0$ ($\mathbf{B}_{ij} = 0$) and $\star$ otherwise. Using the following multiplication and summation rules

a	b	$a+b$	ab
0	0	0	0
0	$\star$	$\star$	0
$\star$	0	$\star$	0
$\star$	$\star$	$\star$	$\star$

one can compute the $\star$-0 structure of the product $\mathbf{AB}$ denoted as $\mathbf{A}^\star \mathbf{B}^\star$.

Definition 11. *Those zero elements of* $\mathbf{AB}$ *whose associated* $\mathbf{A}^\star \mathbf{B}^\star$ *elements are zero as well are referred to as* structural zero *elements, while the zero elements of* $\mathbf{AB}$ *whose associated* $\mathbf{A}^\star \mathbf{B}^\star$ *elements are* $\star$ *are referred to as (structurally not determined or)* random zero *elements.*

The main difficulty of moments based MAP/RAP characterization comes from the random zeros in $\underline{u}$ and $\underline{v}$ type vectors. The occurrence of structural zero elements is not that hard to characterize, because they depend on the non-zero structures of δ, h, $\mathbf{\Lambda}$ and $\hat{\mathbf{P}}$. For example, if $\mathbf{\Lambda}$ is diagonal, then the $\star$-0 structure of $\underline{u}$ and $\underline{v}$ type vectors are invariant for multiplication with matrix $\mathbf{\Lambda}$. E.g., if $\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $\delta = [\star, 0]$ then $\delta \mathbf{\Lambda} = [\star, 0]$. Consequently, the zero structure of $\hat{\mathbf{P}}$ and the number of multiplications with $\hat{\mathbf{P}}$ determine the structural zero elements of the $\underline{u}$ and $\underline{v}$ type vectors in this case.

Matrix $\mathbf{\Lambda}$ with real Jordan block: The main difference of this case from the case of diagonal matrix $\mathbf{\Lambda}$ is that the non-zero structure of the $\underline{u}$ and $\underline{v}$ type vectors are not invariant for multiplication with $\mathbf{\Lambda}$ due to the presence of real Jordan blocks. For example, if $\mathbf{\Lambda} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ and $\underline{u} = [\star, 0]$ then $\underline{u}\mathbf{\Lambda} = [\star, \star]$. In this case the structural properties of both matrixes, $\mathbf{\Lambda}$ and $\hat{\mathbf{P}}$, play role in the occurrence of structural zero elements.

The moments series of maximal order is obtained when the overlapping non-zero elements of a $\underline{u}$ and a $\underline{v}$ type vectors are maximal. The random zeros which might show up in the $\underline{u}$ and $\underline{v}$ type vectors prevents efficient search methods for finding $\underline{u}$ and $\underline{v}$ type vectors with maximal overlapping non-zero elements and unfortunately one needs to perform exhaustive search.

4.3 Some MAP/RAP properties obtained form moments series

The moments series of maximal Hankel order defines the order of the MAP/RAP. Several related properties can be checked based on the the order of the MAP/RAP. First of all, the relation of the order with the Hankel order of the moments series $\{\mu_0, \mu_1, \ldots\}$ determines if the MAP/RAP is redundant.

If the MAP/RAP is non-redundant and $\mathbf{\Lambda}$ is diagonal then neither δ nor h contain a zero element. If the MAP is redundant then at least one of them contains a zero element, and the way as the structural zeros disappear from the $\underline{u}$ and $\underline{v}$ type vectors due to multiplications with $\mathbf{\Lambda}$ and $\hat{\mathbf{P}}$ characterizes various interesting MAP properties, e.g., the number of (independent) parameters.

The lower order moments series allow us to investigate the structural properties of the $\underline{u}$ and $\underline{v}$ type vectors. Let $\underline{\mu_i} = \{\mu_0, \mu_1, \ldots\}$, $\underline{\gamma_{i1}^{(1)}} = \{\gamma_{01}^{(1)}, \gamma_{11}^{(1)}, \gamma_{21}^{(1)}, \ldots\}$, $\underline{\gamma_{1i}^{(1)}} = \{\gamma_{10}^{(1)}, \gamma_{11}^{(1)}, \gamma_{12}^{(1)}, \ldots\}$ and $\underline{\gamma_{1i1}^{(11)}} = \{\gamma_{101}^{(11)}, \gamma_{111}^{(11)}, \gamma_{121}^{(11)}, \ldots\}$. If $\mathrm{HO}\big(\underline{\mu_i}\big) = \mathrm{HO}\Big(\underline{\gamma_{i1}^{(1)}}\Big)$ then from $\mu_i = \delta\mathbf{\Lambda}^i h$ and $\gamma_{i1}^{(1)} = \delta\mathbf{\Lambda}^i\hat{\mathbf{P}}\mathbf{\Lambda}h$ we have that either h does not contain zero element or $\hat{\mathbf{P}}\mathbf{\Lambda}h$ does not contain more non-zero elements in those positions where the associated δ element is non-zero. In the opposite case, when $\mathrm{HO}\big(\underline{\mu_i}\big) < \mathrm{HO}\Big(\underline{\gamma_{i1}^{(1)}}\Big)$, h contains a zero element at a position where the associated δ element is non-zero and a multiplication of h with $\mathbf{\Lambda}$ and $\hat{\mathbf{P}}$, eliminates at least one of such zero elements of h. Similar conclusions can be obtained from the relation of $\mathrm{HO}\big(\underline{\mu_i}\big)$, $\mathrm{HO}\Big(\underline{\gamma_{i1}^{(1)}}\Big)$, $\mathrm{HO}\Big(\underline{\gamma_{1i}^{(1)}}\Big)$, $\mathrm{HO}\Big(\underline{\gamma_{1i1}^{(11)}}\Big)$ and so on. Unfortunately, randomly occurring zeros in $\underline{u}$ and $\underline{v}$ type vectors might also cause that $\mathrm{HO}\big(\underline{\mu_i}\big) > \mathrm{HO}\Big(\underline{\gamma_{i1}^{(1)}}\Big)$, which inhibits several general statements on the MAP/RAP properties.

The following sections investigate the properties of the simplest cases, the rank 2 and the rank 3 MAPs/RAPs in details.

5 Characterization of order 2 MAPs/RAPs

In this section we show that all order 2 MAP/RAPs are non-redundant. Similar to the previous section we differentiate the cases with and without real Jordan block in $\mathbf{\Lambda}$.

When $\mathbf{\Lambda}$ is diagonal, $\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, we can freely interchange the eigenvalues in $\mathbf{\Lambda}$ and the associated vector and matrix elements in δ, h and $\hat{\mathbf{P}}$ and the obtained representation remains to be a Jordan representation. We have the following meaningful cases (after a potential reordering of the eigenvalues).

a) $\delta = \{\star, \star\}$, $h^T = \{\star, \star\}$,
b) $\delta = \{\star, 0\}$, $h^T = \{\star, \star\}$,
c) $\delta = \{\star, \star\}$, $h^T = \{\star, 0\}$,
d) $\delta = \{\star, 0\}$, $h^T = \{\star, 0\}$,

Case a) is the non-redundant case. It is discussed in [13, 6]. Case b), c) and d) have lower orders. In case b), without loss of generality, we assume $\delta = \{\delta_1, 0\}$, $h^T = \{1, 1\}$. From $\delta h = 1$ we have $\delta_1 = 1$. From $\delta\hat{\mathbf{P}} = \delta$ and $\hat{\mathbf{P}}h = h$ we have $\hat{\mathbf{P}} = \begin{bmatrix} 1 & 0 \\ p_{21} & 1 - p_{21} \end{bmatrix}$. The zero structures of $\delta = \{1, 0\}$, $\mathbf{\Lambda}$ and $\hat{\mathbf{P}}$ result in that any multiplication of δ with $\mathbf{\Lambda}^i$ and $\hat{\mathbf{P}}^j$ maintains the non-zero structure of the $\underline{u}$ type vectors, i.e.,

$\delta\mathbf{\Lambda}^{i_1}\hat{\mathbf{P}}^{j_1}\mathbf{\Lambda}^{i_2}\hat{\mathbf{P}}^{j_2}\ldots = \{\star, 0\}$, $\forall i_1, j_1, i_2, j_2, \ldots \geq 0$. As a consequence the rank of any moments series (in (4)) equal to one. That is, the process is a rank 1 MAP, a Poisson process. The rank of case c) and d) can be obtained in a similar way.

When $\mathbf{\Lambda}$ has a real Jordan block, $\mathbf{\Lambda} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, we cannot interchange the vector elements and need to consider an extended set of cases:

a) $\delta = \{\star, \star\}$, $h^T = \{\star, \star\} \quad \Rightarrow \quad \delta\mathbf{\Lambda} = \{\star, \star\}$, $(\mathbf{\Lambda}h)^T = \{\star, \star\}$,
b) $\delta = \{\star, 0\}$, $h^T = \{\star, \star\} \quad \Rightarrow \quad \delta\mathbf{\Lambda} = \{\star, \star\}$, $(\mathbf{\Lambda}h)^T = \{\star, \star\}$,
c) $\delta = \{\star, \star\}$, $h^T = \{\star, 0\} \quad \Rightarrow \quad \delta\mathbf{\Lambda} = \{\star, \star\}$, $(\mathbf{\Lambda}h)^T = \{\star, 0\}$,
d) $\delta = \{\star, 0\}$, $h^T = \{\star, 0\} \quad \Rightarrow \quad \delta\mathbf{\Lambda} = \{\star, \star\}$, $(\mathbf{\Lambda}h)^T = \{\star, 0\}$,
e) $\delta = \{0, \star\}$, $h^T = \{\star, \star\} \quad \Rightarrow \quad \delta\mathbf{\Lambda} = \{0, \star\}$, $(\mathbf{\Lambda}h)^T = \{\star, \star\}$
f) $\delta = \{\star, \star\}$, $h^T = \{0, \star\} \quad \Rightarrow \quad \delta\mathbf{\Lambda} = \{\star, \star\}$, $(\mathbf{\Lambda}h)^T = \{\star, \star\}$,
g) $\delta = \{0, \star\}$, $h^T = \{\star, 0\} \quad \Rightarrow \quad \delta\mathbf{\Lambda} = \{0, \star\}$, $(\mathbf{\Lambda}h)^T = \{\star, 0\}$,
h) $\delta = \{\star, 0\}$, $h^T = \{0, \star\} \quad \Rightarrow \quad \delta\mathbf{\Lambda} = \{\star, \star\}$, $(\mathbf{\Lambda}h)^T = \{\star, \star\}$,
i) $\delta = \{0, \star\}$, $h^T = \{0, \star\} \quad \Rightarrow \quad \delta\mathbf{\Lambda} = \{0, \star\}$, $(\mathbf{\Lambda}h)^T = \{\star, \star\}$.

Among these cases a), b), e), h) represent non-redundant order 2 MAPs/RAPs and all the other cases result in a lower order processes (case g) is order 0, which is not a process, and the other cases are order 1. Consequently, all order 2 MAPs/RAPs are non-redundant. To study the properties of redundant MAPs/RAPs we need to consider at least order 3 processes.

6 Characterization of order 3 MAPs/RAPs

From the previous section we also learned that instead of the zero structure of δ and h the zero structure of $\delta\mathbf{\Lambda}^{n-1}$ and $\mathbf{\Lambda}^{n-1}h$, where n is the size of the largest Jordan block of $\mathbf{\Lambda}$, decides the structural properties of the MAP/RAP. We have following relevant cases of order 3 MAPs/RAPs with diagonal $\mathbf{\Lambda}$:

a) $\delta\mathbf{\Lambda}^{n-1} = \{\star, \star, \star\}$, $(\mathbf{\Lambda}^{n-1}h)^T = \{\star, \star, \star\}$,
b) $\delta\mathbf{\Lambda}^{n-1} = \{\star, \star, 0\}$, $(\mathbf{\Lambda}^{n-1}h)^T = \{\star, \star, \star\}$,
c) $\delta\mathbf{\Lambda}^{n-1} = \{\star, 0, \star\}$, $(\mathbf{\Lambda}^{n-1}h)^T = \{\star, \star, 0\}$,
d) $\delta\mathbf{\Lambda}^{n-1} = \{\star, \star, 0\}$, $(\mathbf{\Lambda}^{n-1}h)^T = \{\star, \star, 0\}$.

The other cases are either lower order or can be obtained by replacing the role of δ and h or by reordering the eigenvalues in $\mathbf{\Lambda}$. A matrix $\mathbf{\Lambda}$ with real Jordan block prevents the interchange of the roles of δ and h, but apart of that these four cases represents the cases with real Jordan blocks as well.

Case a) is the non-redundant case, which is known to be defined by $3^2 = 9$ independent parameters [6, 13]. An independent set of parameters from which all moments and a matrix representation can be computed is $\mu_1, \ldots, \mu_5$ moments and the $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}$ joint moments. This set of moments composes the *basic moments set* of case a).

In the remaining 3 cases we obtain MAPs/RAPs with essentially different properties. These MAPs/RAPs are defined by less than 9 independent parameters and the sets of independent moments are different from the one of the non-redundant case.

6.1 Case b): $\delta\mathbf{\Lambda}^{n-1} = \{\star, \star, 0\}$, $(\mathbf{\Lambda}^{n-1}h)^T = \{\star, \star, \star\}$

We analyze the case by considering the effect of the zero structure of $\delta\mathbf{\Lambda}^{n-1}$ and $(\mathbf{\Lambda}^{n-1}h)^T$ on the rank of the moments matrices. Using this information we look for a set of moments and a set of equations based on which all other moments of the MAP/RAP can be computed. From $\delta\mathbf{\Lambda}^{n-1} = \{\star, \star, 0\}$ the rank of

$$\mathbf{M_1} = \begin{bmatrix} \delta\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}^1 \\ \delta\mathbf{\Lambda}^2 \\ \delta\mathbf{\Lambda}^3 \\ \vdots \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}^0 h \mid \mathbf{\Lambda}^1 h \mid \mathbf{\Lambda}^2 h \end{bmatrix} = \begin{pmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \\ \mu_3 & \mu_4 & \mu_5 \\ \vdots & \vdots & \vdots \end{pmatrix},$$

is 2 and the determinant of any 3×3 sub-matrix of $\mathbf{M_1}$ is zero[1]. $\det \begin{pmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{pmatrix} = 0$ allows us to determine μ_4 based on μ_1, μ_2, μ_3 and recursively all higher moments can be obtained in a similar way.

The lower right sub-matrix of $\mathbf{M_1}$, separated by the horizontal and vertical lines, contains the unknowns which can be determined base on the known moments presented in the complementer part of the matrix. A moment which shows up in both parts is an unknown when it is in the lower right sub-matrix and it is considered to be known otherwise. For example, μ_4 as unknown is determined by the known moments μ_1, μ_2, μ_3 and in a consecutive step μ_5 as unknown is determined by the known moments μ_1, μ_2, μ_3, μ_4. The meaning of the indicated lower right sub-matrix is going to be the same in the consecutive moments matrices.

Computing γ_{ij}: The rank of

$$\mathbf{M_2} = \begin{bmatrix} \delta \\ \delta\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda} \end{bmatrix} \left[\ \mathbf{\Lambda}^0 h \ \middle|\ \mathbf{\Lambda}^1 h \ \middle|\ \mathbf{\Lambda}^2 h \ \middle|\ \mathbf{\Lambda}^3 h \ \middle|\ \dots\ \right] = \left(\begin{array}{ccccc} 1 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_1 & \gamma_{11} & \gamma_{12} & \gamma_{13} & \cdots \\ \cline{4-5} \mu_2 & \gamma_{12} & \gamma_{13} & \multicolumn{1}{|c}{\gamma_{14}} & \cdots \end{array} \right),$$

is not reduced by the structure of order 3 type b) processes. In this way, based on the 4×4 sub-matrices of $\mathbf{M_2}$ we can compute γ_{1j} for $j \geq 4$ from $\mu_1, \mu_2, \mu_3, \dots$ and $\gamma_{11}, \gamma_{12}, \gamma_{13}$.

From $\delta\mathbf{\Lambda}^i = \{\star, \star, 0\}$ we have that the rank of

$$\mathbf{M_3} = \begin{bmatrix} \delta\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}^1 \\ \delta\mathbf{\Lambda}^2 \\ \vdots \end{bmatrix} \left[\ h \ \middle|\ \mathbf{\Lambda} h \ \middle|\ \hat{\mathbf{P}}\mathbf{\Lambda} h \ \middle|\ \hat{\mathbf{P}}\mathbf{\Lambda}^2 h \ \middle|\ \cdots\ \right] = \left(\begin{array}{ccccc} 1 & \mu_1 & \mu_1 & \mu_2 & \cdots \\ \mu_1 & \mu_2 & \gamma_{11} & \gamma_{12} & \cdots \\ \cline{3-5} \mu_2 & \mu_3 & \multicolumn{1}{|c}{\gamma_{21}} & \gamma_{22} & \cdots \\ \vdots & \vdots & \multicolumn{1}{|c}{\vdots} & \vdots & \ddots \end{array} \right),$$

is 2 and the determinant of any 3×3 sub-matrix of $\mathbf{M_3}$ is zero. We can compute γ_{ij} for $i \geq 2$ based on $\mu_1, \mu_2, \mu_3, \dots$ and γ_{1j} for $j \geq 1$.

Computing $\gamma_{ij}^{(2)}$: Similarly, the rank of

$$\mathbf{M_4} = \begin{bmatrix} \delta \\ \delta\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}^2 \end{bmatrix} \left[\ \mathbf{\Lambda}^0 h \ \middle|\ \mathbf{\Lambda}^1 h \ \middle|\ \mathbf{\Lambda}^2 h \ \middle|\ \mathbf{\Lambda}^3 h \ \middle|\ \dots\ \right] = \left(\begin{array}{ccccc} 1 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_1 & \gamma_{11} & \gamma_{12} & \gamma_{13} & \cdots \\ \cline{4-5} \mu_1 & \gamma_{11}^{(2)} & \gamma_{12}^{(2)} & \multicolumn{1}{|c}{\gamma_{13}^{(2)}} & \cdots \end{array} \right),$$

is not reduced by the structure of order 3 type b) processes. From the 4×4 sub-matrices of $\mathbf{M_4}$ we can compute $\gamma_{1j}^{(2)}$ for $j \geq 3$ from $\mu_1, \mu_2, \mu_3, \dots, \gamma_{11}, \gamma_{12}, \gamma_{13}, \dots$ and $\gamma_{11}^{(2)}, \gamma_{12}^{(2)}$.

Similar to $\mathbf{M_3}$, due to the structure of the initial matrix the rank of

$$\begin{bmatrix} \delta\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}^1 \\ \delta\mathbf{\Lambda}^2 \\ \vdots \end{bmatrix} \left[\ h \ \middle|\ \mathbf{\Lambda} h \ \middle|\ \hat{\mathbf{P}}^2\mathbf{\Lambda}^j h\ \right] = \left(\begin{array}{ccc} 1 & \mu_1 & \mu_1 \\ \mu_1 & \mu_2 & \gamma_{1j}^2 \\ \cline{3-3} \mu_2 & \mu_3 & \multicolumn{1}{|c}{\gamma_{2j}^2} \\ \vdots & \vdots & \multicolumn{1}{|c}{\vdots} \end{array} \right)$$

is 2 and using its 3×3 blocks we can obtain γ_{ij}^2 for $i \geq 2$ based on $\mu_1, \mu_2, \mu_3, \dots$ and γ_{1j}^2.

Computing $\gamma_{ij}^{(k)}$: Now we show that all $\gamma_{ij}^{(k)}$ moments can be obtained based on μ_1, μ_2, μ_3 and $\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{11}^{(2)}, \gamma_{12}^{(2)}$. Let us assume that the γ_{ij}^{ℓ}, $i, j \geq 1$, $\ell < k$ moments are known. From the 4×4 sub-matrices of

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}^2 \end{bmatrix} \left[\ \mathbf{\Lambda}^0 h \ \middle|\ \mathbf{\Lambda}^1 h \ \middle|\ \mathbf{\Lambda}^2 h \ \middle|\ \hat{\mathbf{P}}^{k-2}\mathbf{\Lambda} h \ \middle|\ \hat{\mathbf{P}}^{k-2}\mathbf{\Lambda}^2 h \ \middle|\ \dots\ \right] = \left(\begin{array}{cccccc} 1 & \mu_1 & \mu_2 & \mu_1 & \mu_2 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \gamma_{11}^{(k-2)} & \gamma_{12}^{(k-2)} & \cdots \\ \mu_1 & \gamma_{11} & \gamma_{12} & \gamma_{11}^{(k-1)} & \gamma_{12}^{(k-1)} & \cdots \\ \cline{4-5} \mu_1 & \gamma_{11}^{(2)} & \gamma_{12}^{(2)} & \multicolumn{1}{|c}{\gamma_{11}^{(k)}} & \gamma_{12}^{(k)} & \cdots \end{array} \right),$$

[1]To avoid pathological cases in this section we exclude the rank degradation due to random zeros.

we can compute $\gamma_{1j}^{(k)}$ for $j \geq 1$ and from the 3×3 blocks of

$$\begin{bmatrix} \delta\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}^1 \\ \delta\mathbf{\Lambda}^2 \\ \vdots \end{bmatrix} \left[\; h \;\middle|\; \mathbf{\Lambda} h \;\middle|\; \hat{\mathbf{P}}^k \mathbf{\Lambda}^j h \;\right] = \begin{pmatrix} 1 & \mu_1 & \mu_1 \\ \mu_1 & \mu_2 & \gamma_{1j}^{(k)} \\ \mu_2 & \mu_3 & \gamma_{2j}^{(k)} \\ \vdots & \vdots & \vdots \end{pmatrix},$$

we obtain $\gamma_{ij}^{(k)}$ for $i \geq 2$.

Computing higher moments: Finally, any other higher moment can be obtained from computable lower order ones using the fact that the determinant of

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \\ \star \end{bmatrix} \left[\; \mathbf{\Lambda}^0 h \;\middle|\; \mathbf{\Lambda}^1 h \;\middle|\; \mathbf{\Lambda}^2 h \;\middle|\; \star \;\right] = \begin{pmatrix} 1 & \mu_1 & \mu_2 & \star \\ \mu_1 & \mu_2 & \mu_3 & \star \\ \mu_1 & \gamma_{11} & \gamma_{12} & \star \\ \star & \star & \star & \bullet \end{pmatrix}$$

is zero and applying a similar recursive procedure as the one in [6]. As an example we demonstrate the computation of $\gamma_{111}^{(11)}$:

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda} \end{bmatrix} \left[\; \mathbf{\Lambda}^0 h \;\middle|\; \mathbf{\Lambda}^1 h \;\middle|\; \mathbf{\Lambda}^2 h \;\middle|\; \hat{\mathbf{P}}\mathbf{\Lambda} h \;\right] = \begin{pmatrix} 1 & \mu_1 & \mu_2 & \mu_1 \\ \mu_1 & \mu_2 & \mu_3 & \gamma_{11} \\ \mu_1 & \gamma_{11} & \gamma_{12} & \gamma_{11}^{(2)} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{111}^{(11)} \end{pmatrix}.$$

We conclude that a *basic moments set* of order 3 type b) MAPs/RAPs is composed by 8 moments $\mu_1, \mu_2, \mu_3,\ \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{11}^{(2)}$ and $\gamma_{12}^{(2)}$ and any other moments of the process can be computed base on this basic moments set.

Generating a matrix representation: Having the moments we can generate a matrix representation of an order 3 type b) MAP/RAP. Based on the moments series γ_{1j} $j \geq 0$, obtained from

$$\begin{bmatrix} \delta\mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda}^1 \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda}^2 \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda}^3 \\ \vdots \end{bmatrix} \left[\; \mathbf{\Lambda}^0 h \;\middle|\; \mathbf{\Lambda}^1 h \;\middle|\; \mathbf{\Lambda}^2 h \;\middle|\; \mathbf{\Lambda}^3 h \;\middle|\; \dots \;\right] = \begin{pmatrix} \mu_1 & \gamma_{11} & \gamma_{12} & \gamma_{13} & \cdots \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \cdots \\ \gamma_{12} & \gamma_{13} & \gamma_{14} & \gamma_{15} & \cdots \\ \gamma_{13} & \gamma_{14} & \gamma_{15} & \gamma_{16} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{8}$$

we generate a row vector α and a matrix $\bar{\mathbf{E}}$ of size 3 such that $\gamma_{1j} = \alpha\bar{\mathbf{E}}^j\mathbb{1}$. One way to do it is to divide γ_{1j} by μ_1 to obtain $\mu'_j = \gamma_{1j}/\mu_1$. Consider the obtained μ'_j series as a moments series of a PH/ME distribution. Apply the same procedure as in [13] to obtain a vector matrix pair $(\alpha', \bar{\mathbf{E}})$ which provides $\mu'_j = \alpha'\bar{\mathbf{E}}^j\mathbb{1}$ and set $\alpha = \alpha'\mu_1$. With these α and $\bar{\mathbf{E}}$ we have

$$\mathbf{M_7} = \begin{bmatrix} \alpha\bar{\mathbf{E}}^0 \\ \alpha\bar{\mathbf{E}}^1 \\ \alpha\bar{\mathbf{E}}^2 \end{bmatrix} \left[\; \bar{\mathbf{E}}^0\mathbb{1} \;\middle|\; \bar{\mathbf{E}}^1\mathbb{1} \;\middle|\; \bar{\mathbf{E}}^2\mathbb{1} \;\right] = \begin{pmatrix} \mu_1 & \gamma_{11} & \gamma_{12} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & \gamma_{13} & \gamma_{14} \end{pmatrix}. \tag{9}$$

Comparing (8) with (9) shows that there is only a similarity transformation between α and $\delta\mathbf{\Lambda}\hat{\mathbf{P}}$, $\bar{\mathbf{E}}$ and $\mathbf{\Lambda}$, and $\mathbb{1}$ and h. Assuming that there is the same similarity transformation between $\bar{\mathbf{P}}$ and $\hat{\mathbf{P}}$ we have

$$\begin{bmatrix} \alpha\bar{\mathbf{E}}^0 \\ \alpha\bar{\mathbf{E}}^1 \\ \alpha\bar{\mathbf{E}}^2 \end{bmatrix} \bar{\mathbf{P}} \left[\; \bar{\mathbf{E}}^0\mathbb{1} \;\middle|\; \bar{\mathbf{E}}^1\mathbb{1} \;\middle|\; \bar{\mathbf{E}}^2\mathbb{1} \;\right] = \begin{pmatrix} \mu_1 & \gamma_{11}^{(2)} & \gamma_{12}^{(2)} \\ \gamma_{11} & \gamma_{111}^{(11)} & \gamma_{112}^{(11)} \\ \gamma_{12} & \gamma_{121}^{(11)} & \gamma_{122}^{(11)} \end{pmatrix}. \tag{10}$$

Due to the fact that $\mathbf{M_7}$ is rank 3 $\bar{\mathbf{P}}$ can be computed from (10) as

$$\bar{\mathbf{P}} = \begin{bmatrix} \alpha\bar{\mathbf{E}}^0 \\ \alpha\bar{\mathbf{E}}^1 \\ \alpha\bar{\mathbf{E}}^2 \end{bmatrix}^{-1} \begin{pmatrix} \mu_1 & \gamma_{11}^{(2)} & \gamma_{12}^{(2)} \\ \gamma_{11} & \gamma_{111}^{(11)} & \gamma_{112}^{(11)} \\ \gamma_{12} & \gamma_{121}^{(11)} & \gamma_{122}^{(11)} \end{pmatrix} \left[\ \bar{\mathbf{E}}^0\mathbb{1} \mid \bar{\mathbf{E}}^1\mathbb{1} \mid \bar{\mathbf{E}}^2\mathbb{1}\ \right]^{-1},$$

from which $\bar{\mathbf{D}}_\mathbf{0} = -\bar{\mathbf{E}}^{-1}$ and $\bar{\mathbf{D}}_\mathbf{1} = \bar{\mathbf{E}}^{-1}\bar{\mathbf{P}}$ is a matrix representation of the process.

Example 1. *An example of type b) order 3 matrix representation (with $h = \mathbb{1}$) is*

$$\mathbf{D_0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \mathbf{D_1} = \begin{pmatrix} 1/10 & 4/5 & 1/10 \\ 1/220 & 491/990 & -1/1980 \\ 2/5 & 3/5 & 1 \end{pmatrix}.$$

6.2 Case c): $\delta\mathbf{\Lambda}^{n-1} = \{\star, 0, \star\}$, $(\mathbf{\Lambda}^{n-1}h)^T = \{\star, \star, 0\}$

The most remarkable feature of case c) is that the Hankel order of the $\mu_i = \delta\mathbf{\Lambda}^i h = \delta_1\lambda_{11}^i h_1$ moments series is one, which means on the one hand that the stationary inter-arrival time is exponentially distributed and on the other hand that the rank of

$$\begin{bmatrix} \delta\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}^1 \\ \vdots \end{bmatrix} \left[\ \mathbf{\Lambda}^0 h \mid \mathbf{\Lambda}^1 h\ \right] = \begin{pmatrix} 1 & \mu_1 \\ \mu_1 & \mu_2 \\ \vdots & \vdots \end{pmatrix},$$

is one and the Hankel matrices containing this matrix as a sub-matrix (e.g. M_2 of case b)) cannot be used to compute higher moments due to rank degradation.
Computing γ_{ij}: Consequently we can apply the following rank 2 matrices to compute higher moments. From μ_i, $i \geq 1$ and $\gamma_{11}, \gamma_{21}, \gamma_{12}$ we can compute γ_{i1} $\forall i \geq 3$ using

$$\begin{bmatrix} \delta\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}^1 \\ \delta\mathbf{\Lambda}^2 \end{bmatrix} \left[\ h \mid \mathbf{\Lambda}^0\hat{\mathbf{P}}\mathbf{\Lambda}h \mid \mathbf{\Lambda}^1\hat{\mathbf{P}}\mathbf{\Lambda}h \mid \cdots\ \right] = \begin{pmatrix} 1 & \mu_1 & \gamma_{11} & \cdots \\ \mu_1 & \gamma_{11} & \gamma_{21} & \cdots \\ \mu_2 & \gamma_{21} & \gamma_{31} & \cdots \end{pmatrix},$$

and γ_{1j} $\forall j \geq 3$ using

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda}^1 \\ \vdots \end{bmatrix} \left[\ \mathbf{\Lambda}^0 h \mid \mathbf{\Lambda}^1 h \mid \mathbf{\Lambda}^2 h\ \right] = \begin{pmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \gamma_{11} & \gamma_{12} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

From μ_i, $i \geq 1$ and γ_{i1}, γ_{1j} $\forall i, j \geq 1$ we can compute γ_{ij}, for $\forall i, j \geq 2$ using

$$\begin{bmatrix} \delta\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}^1 \\ \delta\mathbf{\Lambda}^2 \\ \vdots \end{bmatrix} \left[\ \hat{\mathbf{P}}\mathbf{\Lambda}^0 h \mid \hat{\mathbf{P}}\mathbf{\Lambda}^1 h \mid \hat{\mathbf{P}}\mathbf{\Lambda}^2 h \mid \cdots\ \right] = \begin{pmatrix} 1 & \mu_1 & \mu_2 & \cdots \\ \mu_1 & \gamma_{11} & \gamma_{12} & \cdots \\ \mu_2 & \gamma_{21} & \gamma_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Computing $\gamma_{ij}^{(2)}$: From μ_i, $i \geq 1$, γ_{ij}, $i, j \geq 1$, and $\gamma_{11}^{(2)}$ we can compute $\gamma_{1j}^{(2)}$ for $\forall j \geq 2$ using

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}^2\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}^2\mathbf{\Lambda}^1 \\ \vdots \end{bmatrix} \left[\ \mathbf{\Lambda}^0 h \mid \mathbf{\Lambda}^1 h \mid \mathbf{\Lambda}^2 h\ \right] = \begin{pmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \gamma_{11} & \gamma_{12} \\ \mu_1 & \gamma_{11}^{(2)} & \gamma_{12}^{(2)} \\ \gamma_{11}^{(2)} & \gamma_{12}^{(2)} & \gamma_{13}^{(2)} \\ \vdots & \vdots & \vdots \end{pmatrix},$$

and from μ_i, $i \geq 1$, γ_{ij}, $i,j \geq 1$, and $\gamma_{1j}^{(2)}$ $\forall i \geq 1$ we can compute $\gamma_{ij}^{(2)}$ for $\forall i,j \geq 1$ using

$$\begin{bmatrix} \delta\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}^1 \\ \delta\mathbf{\Lambda}^2 \\ \vdots \end{bmatrix} \left[\ h \;\middle|\; \hat{\mathbf{P}}\mathbf{\Lambda}h \;\middle|\; \hat{\mathbf{P}}^2\mathbf{\Lambda}^1 h \;\middle|\; \hat{\mathbf{P}}^2\mathbf{\Lambda}^2 h \;\middle|\; \hat{\mathbf{P}}^2\mathbf{\Lambda}^3 h \;\middle|\; \cdots\ \right] = \begin{pmatrix} 1 & \mu_1 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \gamma_{11} & \gamma_{11}^{(2)} & \gamma_{12}^{(2)} & \gamma_{13}^{(2)} & \cdots \\ \mu_2 & \gamma_{21} & \gamma_{21}^{(2)} & \gamma_{22}^{(2)} & \gamma_{23}^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Computing $\gamma_{ij}^{(k)}$: Having μ_i, $i \geq 1$, γ_{ij}, $\gamma_{ij}^{(2)}$ $i,j \geq 1$, and $\gamma_{11}^{(3)}$ we can compute $\gamma_{ij}^{(3)}$ for $\forall i,j \geq 1$ in the same way as $\gamma_{ij}^{(2)}$ by replacing $\hat{\mathbf{P}}^2$ with $\hat{\mathbf{P}}^3$ in the matrices. In a similar manner having μ_i, $i \geq 1$, $\gamma_{ij}^{(k)}$ $i,j \geq 1$, $k = 1,2,3$, and $\gamma_{11}^{(4)}$ we can compute $\gamma_{ij}^{(4)}$ for $\forall i,j \geq 1$.

The higher $\gamma_{11}^{(k)}$ moments ($k \geq 5$) can be computed from the rank 3 matrix

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}^2 \end{bmatrix} \left[\ h \;\middle|\; \hat{\mathbf{P}}\mathbf{\Lambda}h \;\middle|\; \hat{\mathbf{P}}^2\mathbf{\Lambda}h \;\middle|\; \hat{\mathbf{P}}^3\mathbf{\Lambda}h \;\middle|\; \cdots\ \right] = \begin{pmatrix} 1 & \mu_1 & \mu_1 & \mu_1 & \cdots \\ \mu_1 & \gamma_{11} & \gamma_{11}^{(2)} & \gamma_{11}^{(3)} & \cdots \\ \mu_1 & \gamma_{11}^{(2)} & \gamma_{11}^{(3)} & \gamma_{11}^{(4)} & \cdots \\ \mu_1 & \gamma_{11}^{(3)} & \gamma_{11}^{(4)} & \gamma_{11}^{(5)} & \cdots \end{pmatrix},$$

and the associated $\gamma_{ij}^{(k)}$ $i,j \geq 1$ moments can be computed in the same way as $\gamma_{ij}^{(2)}$.

Up to this point we have computed all double joint moments ($E(X_0^i X_k^j)/i!j! = \gamma_{ij}^{(k)}$, $\forall i,j,k \geq 1$) based on 7 basic moments: μ_1, $\gamma_{11}, \gamma_{21}, \gamma_{12}$, $\gamma_{11}^{(2)}$, $\gamma_{11}^{(3)}$ and $\gamma_{11}^{(4)}$.

Computing triple and higher joint moments: Having all double joint moments and $\gamma_{111}^{(11)}$ we can further compute $\gamma_{111}^{(1j)}$ and $\gamma_{121}^{(1j)}$ from the rank 3 matrix

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}^2 \end{bmatrix} \left[\ h \;\middle|\; \hat{\mathbf{P}}\mathbf{\Lambda}h \;\middle|\; \hat{\mathbf{P}}^2\mathbf{\Lambda}h \;\middle|\; \mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda}h\ \right] = \begin{pmatrix} 1 & \mu_1 & \mu_1 & \gamma_{11} \\ \mu_1 & \gamma_{11} & \gamma_{11}^{(2)} & \gamma_{21} \\ \mu_1 & \gamma_{11}^{(2)} & \gamma_{11}^{(3)} & \gamma_{111}^{(11)} \\ \mu_1 & \gamma_{11}^{(3)} & \gamma_{11}^{(4)} & \gamma_{111}^{(21)} \end{pmatrix}.$$

Having these moments we further compute $\gamma_{i\ell j}^{(11)}$ from the rank 3 matrix

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}^2 \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda}^2 \\ \vdots \end{bmatrix} \left[\ h \;\middle|\; \mathbf{\Lambda}h \;\middle|\; \hat{\mathbf{P}}\mathbf{\Lambda}h \;\middle|\; \mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda}h\ \right] = \begin{pmatrix} 1 & \mu_1 & \mu_1 & \gamma_{11} \\ \mu_1 & \gamma_{11} & \gamma_{11}^{(2)} & \gamma_{111}^{(11)} \\ \mu_1 & \gamma_{11}^{(2)} & \gamma_{11}^{(3)} & \gamma_{111}^{(21)} \\ \gamma_{11} & \gamma_{12} & \gamma_{111}^{(11)} & \gamma_{121}^{(11)} \\ \gamma_{12} & \gamma_{13} & \gamma_{121}^{(11)} & \gamma_{131}^{(11)} \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Finally, we conclude that the *basic moments set* of order 3, type c) MAPs/RAPs is composed by 8 moments μ_1, $\gamma_{11}, \gamma_{12}, \gamma_{21}$, $\gamma_{11}^{(2)}, \gamma_{11}^{(3)}, \gamma_{11}^{(4)}$ and $\gamma_{111}^{(11)}$ and any other higher order moments of the process can be computed based on this basic moments set in the form

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \\ \star \end{bmatrix} \left[\ h \;\middle|\; \hat{\mathbf{P}}\mathbf{\Lambda}h \;\middle|\; \hat{\mathbf{P}}^2\mathbf{\Lambda}h \;\middle|\; \star\ \right] = \begin{pmatrix} 1 & \mu_1 & \mu_1 & \star \\ \mu_1 & \gamma_{11} & \gamma_{11}^{(2)} & \star \\ \mu_1 & \gamma_{11}^{(2)} & \gamma_{11}^{(3)} & \star \\ \star & \star & \star & \bullet \end{pmatrix}.$$

Generating a matrix representation: All the double joint moments series of order 3 type c) MAPs/RAPs can be computed from rank 2 matrices which means that the $\gamma_{1j}^{(k)}$ $j \geq 0$ and $\gamma_{i1}^{(k)}$ $i \geq 0$, moments series are at most order 2 for any k. To characterize the order 3 generator of the process we need to use a moments series which is computed from rank 3 matrices. $\gamma^{(11)}_{1j1}$ $j \geq 0$ is such a moments series. Taking this moments series we generate a row vector α and a matrix $\bar{\mathbf{E}}$ of size 3 such that $\gamma^{(11)}_{1j1} = \alpha \bar{\mathbf{E}}^j \mathbb{I}$. With these α and $\bar{\mathbf{E}}$ we have

$$\begin{bmatrix} \alpha\bar{\mathbf{E}}^0 \\ \alpha\bar{\mathbf{E}}^1 \\ \alpha\bar{\mathbf{E}}^2 \end{bmatrix} \left[\ \bar{\mathbf{E}}^0\mathbb{I} \mid \bar{\mathbf{E}}^1\mathbb{I} \mid \bar{\mathbf{E}}^2\mathbb{I}\ \right] = \begin{pmatrix} \gamma^{(2)}_{11} & \gamma^{(11)}_{111} & \gamma^{(11)}_{121} \\ \gamma^{(11)}_{111} & \gamma^{(11)}_{121} & \gamma^{(11)}_{131} \\ \gamma^{(11)}_{121} & \gamma^{(11)}_{131} & \gamma^{(11)}_{141} \end{pmatrix},$$

and

$$\begin{bmatrix} \alpha\bar{\mathbf{E}}^0 \\ \alpha\bar{\mathbf{E}}^1 \\ \alpha\bar{\mathbf{E}}^2 \end{bmatrix} \bar{\mathbf{P}} \left[\ \bar{\mathbf{E}}^0\mathbb{I} \mid \bar{\mathbf{E}}^1\mathbb{I} \mid \bar{\mathbf{E}}^2\mathbb{I}\ \right] = \begin{pmatrix} \gamma^{(3)}_{11} & \gamma^{(21)}_{111} & \gamma^{(21)}_{121} \\ \gamma^{(12)}_{111} & \gamma^{(111)}_{1111} & \gamma^{(111)}_{1121} \\ \gamma^{(12)}_{121} & \gamma^{(111)}_{1211} & \gamma^{(111)}_{1221} \end{pmatrix}.$$

Due to the fact that the Hankel matrix of the moments series of $\gamma^{(11)}_{1j1}$ is rank 3 $\bar{\mathbf{P}}$ can be computed as

$$\bar{\mathbf{P}} = \begin{bmatrix} \alpha\bar{\mathbf{E}}^0 \\ \alpha\bar{\mathbf{E}}^1 \\ \alpha\bar{\mathbf{E}}^2 \end{bmatrix}^{-1} \begin{pmatrix} \gamma^{(3)}_{11} & \gamma^{(21)}_{111} & \gamma^{(21)}_{121} \\ \gamma^{(12)}_{111} & \gamma^{(111)}_{1111} & \gamma^{(111)}_{1121} \\ \gamma^{(12)}_{121} & \gamma^{(111)}_{1211} & \gamma^{(111)}_{1221} \end{pmatrix} \left[\ \bar{\mathbf{E}}^0\mathbb{I} \mid \bar{\mathbf{E}}^1\mathbb{I} \mid \bar{\mathbf{E}}^2\mathbb{I}\ \right]^{-1},$$

from which $\bar{\mathbf{D}}_\mathbf{0} = -\bar{\mathbf{E}}^{-1}$ and $\bar{\mathbf{D}}_\mathbf{1} = \bar{\mathbf{E}}^{-1}\bar{\mathbf{P}}$ is a matrix representation of the process.

Example 2. *An example of type c) order 3 Jordan representation is*

$$\delta = (1, 0, 1/5), \quad \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \quad \hat{\mathbf{P}} = \begin{pmatrix} 1/5 & 4/5 & 1/10 \\ 1/10 & 9/10 & 3/10 \\ 4 & -4 & 1/1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Note that this representation satisfies (5).

6.3 Case d): $\delta\mathbf{\Lambda}^{n-1} = \{\star, \star, 0\}$, $(\mathbf{\Lambda}^{n-1}h)^T = \{\star, \star, 0\}$

Based on the properties of $\delta\mathbf{\Lambda}^{n-1}$ and $\mathbf{\Lambda}^{n-1}h$ the rank of

$$\begin{bmatrix} \delta\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}^1 \\ \delta\mathbf{\Lambda}^2 \\ \delta\mathbf{\Lambda}^3 \\ \vdots \end{bmatrix} \left[\ \mathbf{\Lambda}^0 h \mid \mathbf{\Lambda}^1 h \mid \mathbf{\Lambda}^2 h\ \right] = \begin{pmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \\ \mu_3 & \mu_4 & \mu_5 \\ \vdots & \vdots & \vdots \end{pmatrix},$$

is 2 and we can compute all μ_i, $i \geq 4$ moments based on μ_1, μ_2, μ_3.
Computing γ_{ij}: The rank of

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \end{bmatrix} \left[\ \mathbf{\Lambda}^0 h \mid \mathbf{\Lambda}^1 h \mid \mathbf{\Lambda}^2 h \mid \mathbf{\Lambda}^3 h \mid \dots\ \right] = \begin{pmatrix} 1 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_1 & \gamma_{11} & \gamma_{12} & \gamma_{13} & \cdots \end{pmatrix},$$

is 2 and we can compute γ_{1j} for $j \geq 2$ from $\mu_1, \mu_2, \mu_3, \dots$ and γ_{11}. Similarly, the rank of

$$\begin{bmatrix} \delta\mathbf{\Lambda}^0 \\ \delta\mathbf{\Lambda}^1 \\ \delta\mathbf{\Lambda}^2 \\ \vdots \end{bmatrix} \left[\ h \mid \mathbf{\Lambda}h \mid \hat{\mathbf{P}}\mathbf{\Lambda}h \mid \hat{\mathbf{P}}\mathbf{\Lambda}^2 h \mid \cdots\ \right] = \begin{pmatrix} 1 & \mu_1 & \mu_1 & \mu_2 & \cdots \\ \mu_1 & \mu_2 & \gamma_{11} & \gamma_{12} & \cdots \\ \mu_2 & \mu_3 & \gamma_{21} & \gamma_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

is 2 and we can compute γ_{ij} for $i \geq 2$ based on $\mu_1, \mu_2, \mu_3, \dots$ and γ_{1j}, $j \geq 1$.

Computing $\gamma_{ij}^{(k)}$: Having, $\gamma_{11}^{(2)}$ ($\gamma_{11}^{(3)}$) and repeating exactly the same steps as for computing γ_{1j} results in $\gamma_{ij}^{(2)}$ ($\gamma_{ij}^{(3)}$). For the computation of the $\gamma_{ij}^{(k)}$ moments for $k \geq 4$ the $\gamma_{11}^{(k)}$ moment can be obtained from the rank 3 matrix

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}^2 \end{bmatrix} \left[\, h \;\middle|\; \mathbf{\Lambda}h \;\middle|\; \hat{\mathbf{P}}\mathbf{\Lambda}h \;\middle|\; \hat{\mathbf{P}}^2\mathbf{\Lambda}h \;\middle|\; \dots \,\right] = \begin{pmatrix} 1 & \mu_1 & \mu_1 & \mu_1 & \cdots \\ \mu_1 & \mu_2 & \gamma_{11} & \gamma_{11}^{(2)} & \cdots \\ \mu_1 & \gamma_{11} & \gamma_{11}^{(2)} & \gamma_{11}^{(3)} & \cdots \\ \mu_1 & \gamma_{11}^{(2)} & \gamma_{11}^{(3)} & \gamma_{11}^{(4)} & \cdots \end{pmatrix}.$$

At this point we can compute all $\gamma_{ij}^{(k)}$ moments based on μ_1, μ_2, μ_3, and γ_{11}, $\gamma_{11}^{(2)}$, $\gamma_{11}^{(3)}$.

Computing higher moments: Having additionally $\gamma_{111}^{(11)}$ allows us to compute all higher moments in the form

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \\ \star \end{bmatrix} \left[\, h \;\middle|\; \mathbf{\Lambda}h \;\middle|\; \hat{\mathbf{P}}\mathbf{\Lambda}h \;\middle|\; \star \,\right] = \begin{pmatrix} 1 & \mu_1 & \mu_1 & \star \\ \mu_1 & \mu_2 & \gamma_{11} & \star \\ \mu_1 & \gamma_{11} & \gamma_{11}^{(2)} & \star \\ \star & \star & \star & \bullet \end{pmatrix}.$$

For example the $\gamma_{imj}^{(k\ell)}$ moments can be computed from

$$\begin{bmatrix} \delta \\ \delta\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda} \\ \delta\mathbf{\Lambda}\hat{\mathbf{P}}^2 \end{bmatrix} \left[\, h \;\middle|\; \mathbf{\Lambda}h \;\middle|\; \hat{\mathbf{P}}\mathbf{\Lambda}h \;\middle|\; \mathbf{\Lambda}\hat{\mathbf{P}}\mathbf{\Lambda}h \;\middle|\; \hat{\mathbf{P}}^2\mathbf{\Lambda}h \,\right] = \begin{pmatrix} 1 & \mu_1 & \mu_1 & \gamma_{11} & \mu_1 \\ \mu_1 & \mu_2 & \gamma_{11} & \gamma_{21} & \gamma_{11} \\ \mu_1 & \gamma_{11} & \gamma_{11}^{(2)} & \gamma_{111}^{(11)} & \gamma_{11}^{(3)} \\ \gamma_{11} & \gamma_{12} & \gamma_{111}^{(11)} & \gamma_{121}^{(11)} & \gamma_{111}^{(12)} \\ \mu_1 & \gamma_{11}^{(2)} & \gamma_{11}^{(3)} & \gamma_{111}^{(21)} & \gamma_{11}^{(4)} \end{pmatrix}.$$

We conclude that a *basic moments set* of order 3 type d) MAPs/RAPs is composed by 7 moments $\mu_1, \mu_2, \mu_3, \gamma_{11}, \gamma_{11}^{(2)}, \gamma_{11}^{(3)}$ and $\gamma_{111}^{(11)}$ and any other moments of the process can be computed base on this basic moments set.

Generating a matrix representation: Similar to case c) the $\gamma_{ij}^{(k)}$ moments series are all obtained from rank 2 matrices. The first rank 3 moments series is $\gamma_{1i1}^{(11)}$, $i \geq 0$. Based on this moments series we can generate a $\bar{\mathbf{D}}_\mathbf{0}$, $\bar{\mathbf{D}}_\mathbf{1}$ matrix representation following exactly the same steps as in case c).

Example 3. *An example of type d) order 3 MAP is*

$$\mathbf{D_0} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3/5 & 1/10 \\ 0 & 1/15 & -17/30 \end{pmatrix}, \quad \mathbf{D_1} = \begin{pmatrix} 1/5 & 18/25 & 2/25 \\ 1/9 & 1/3 & 1/18 \\ 0 & 2/45 & 41/90 \end{pmatrix}.$$

6.4 Summary of order 3 MAP/RAP cases

Case	a)	b)	c)	d)
$\delta\mathbf{\Lambda}^{n-1}$	$\{\star,\star,\star\}$	$\{\star,\star,0\}$	$\{\star,0,\star\}$	$\{\star,\star,0\}$
$(\mathbf{\Lambda}^{n-1}h)^T$	$\{\star,\star,\star\}$	$\{\star,\star,\star\}$	$\{\star,\star,0\}$	$\{\star,\star,0\}$
number of param.	9	8	8	7
$\mathrm{HO}(\underline{\mu_i})$	3	2	1	2
$\left(\mathrm{HO}\left(\underline{\gamma_{i1}^{(1)}}\right), \mathrm{HO}\left(\underline{\gamma_{1i}^{(1)}}\right)\right)$	(3,3)	(2,3)	(2,2)	(2,2)
$\mathrm{HO}\left(\underline{\gamma_{1i1}^{(11)}}\right)$	3	3	3	3
required information	lag-1 joint mom.	lag-2 joint mom.	triple joint mom.	triple joint mom.
basic moments set	$\mu_1, \dots, \mu_5$, $\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}$	μ_1, μ_2, μ_3, $\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{11}^{(2)}, \gamma_{12}^{(2)}$	μ_1, $\gamma_{11}, \gamma_{12}, \gamma_{21}$, $\gamma_{11}^{(2)}, \gamma_{11}^{(3)}, \gamma_{11}^{(4)}, \gamma_{111}^{(11)}$	μ_1, μ_2, μ_3, $\gamma_{11}, \gamma_{11}^{(2)}, \gamma_{11}^{(3)}, \gamma_{111}^{(11)}$

7 Higher order MAPs/RAPs

Due to the low order of the above studied processes several peculiar properties do not show up in the previous section. The complexity of the higher order cases seems to be inconceivable, but our main message that the zero structure of the $\underline{u}$ and $\underline{v}$ type vectors characterizes the properties of moments series remains valid also for higher order processes.

In lack of general roles, we only demonstrate one of the peculiar features. We present a MAP/RAP of size 7 (if $\ddots$ stands for a single matrix element) with a special structure which cannot be characterized based on triple joint moments (or equivalently based on triple joint density of inter-arrival times), but requires the knowledge of higher-tuple moments (or joint density function). The Jordan representation of this MAP/RAP is $(\delta, \mathbf{\Lambda}, \hat{\mathbf{P}}, h)$, where $\delta = \{\star, \star, 0, 0, 0, 0, \ldots\}$,

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 1 & & & & & \\ & \lambda_1 & & & & & \\ & & \lambda_2 & 1 & & & \\ & & & \lambda_2 & & & \\ & & & & \lambda_3 & 1 & \\ & & & & & \lambda_3 & \\ & & & & & & \ddots \end{bmatrix}, \ \hat{\mathbf{P}} = \begin{bmatrix} \star & \star & \star & & & & \\ \star & \star & \star & & & & \\ \star & \star & \star & & & & \\ & & & \star & \star & & \\ & & & \star & \star & & \\ & & & & & \star & \star \\ & & & & & \star & \ddots \end{bmatrix}, \ h = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}.$$

The Hankel orders of the moments series of this MAP/RAP (without the presence of random zeros) are as follows, $\mathrm{HO}\left(\underline{\mu_i}\right) = 2$, $\mathrm{HO}\left(\underline{\gamma_{1i}^{(1)}}\right) = 3$, $\mathrm{HO}\left(\underline{\gamma_{11i}^{(11)}}\right) = 5$, $\mathrm{HO}\left(\underline{\gamma_{111i}^{(111)}}\right) = 7$. If $\ddots$ stands for the extension of the given structure to larger size then the Hankel order of all k-tuple moments series is less than or equal to $2k-1$. Consequently, if the size of the representation is 9 ($k = 5$), then only the 5-tuple moments series indicates the order of the MAP/RAP and all lower-tuple moments series have lower Hankel orders.

8 ETAQA approximation of the output process of a M/PH/1 queue

One of the cases when redundant MAP or RAP arises in applications is the ETAQA approximation of the output process of various queueing models [14]. The ETAQA approximation of the output process of a M/PH/1 queue with arrival rate 1, service time $\mathrm{PH}(\tau, \mathbf{T})$ with $\tau = \{0.5, 0.5\}$, $\mathbf{T} = \begin{pmatrix} -16 & 3 \\ 0 & -5 \end{pmatrix}$ results in the following MAP

$$\mathbf{D_0} = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -17 & 3 & 1 & 0 \\ 0 & 0 & 0 & -6 & 0 & 1 \\ 0 & 0 & 0 & 0 & -16 & 3 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{pmatrix}, \quad \mathbf{D_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 6.5 & 6.5 & 0 & 0 & 0 & 0 \\ 2.5 & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 6 & 0.5 & 0.5 \\ 0 & 0 & 2 & 2 & 0.5 & 0.5 \end{pmatrix}.$$

The Hankel order of the moments series of this MAP are $\mathrm{HO}\left(\underline{\mu_i}\right) = 3$, $\mathrm{HO}\left(\underline{\gamma_{i1}^{(1)}}\right) = 5$, $\mathrm{HO}\left(\underline{\gamma_{1i}^{(1)}}\right) = 3$, $\mathrm{HO}\left(\underline{\gamma_{i11}^{(11)}}\right) = 5$, $\mathrm{HO}\left(\underline{\gamma_{1i1}^{(11)}}\right) = 5$, $\mathrm{HO}\left(\underline{\gamma_{11i}^{(11)}}\right) = 3$, and all higher moments series are order 5 or less. All joint moments of this process can be obtained from 23 independent moments, which are, e.g., $r_1, \ldots, r_5$, $\gamma_{11}^{(1)}, \ldots, \gamma_{51}^{(1)}$, $\gamma_{12}^{(1)}, \ldots, \gamma_{52}^{(1)}$, $\gamma_{11}^{(2)}, \ldots, \gamma_{41}^{(2)}$, $\gamma_{12}^{(2)}, \ldots, \gamma_{42}^{(2)}$.

References

[1] S. Asmussen and M. Bladt. Point processes with finite-dimensional conditional probabilities. *Stochastic Processes and their Application*, 82:127–142, 1999.

[2] S. Asmussen and C. A. O'Cinneide. Matrix-exponential distributions. In *Encyclopedia of Statistical Science Update*, volume 2, pages 435–440, New York, 1997. Wiley.

[3] N. G. Bean and B. F. Nielsen. Quasi-birth-and-death processes with rational arrival process components. *Stochastic Models*, 26(3):309–334, 2010.

[4] D. A. Bini, B. Meini, S. Steffé, and B. Van Houdt. Structured Markov chains solver: software tools. In *SMCtools Workshop*, Pisa, Italy, 2006. ACM Press. http://www.win.ua.ac.be/∼vanhoudt/.

[5] L. Bodrog, A. Heindl, G. Horváth, and M. Telek. A markovian canonical form of second-order matrix-exponential processes. *European Journal of Operation Research*, 190:459–477, 2008.

[6] L. Bodrog, A. Horváth, and M. Telek. Moment characterization of matrix exponential and Markovian arrival processes. *Annals of Operations Research*, 160:51–68, 2008.

[7] P. Buchholz and M. Telek. On minimal representation of rational arrival processes. In *Madrid Conference on Queueing theory (MCQT)*, Toledo, Spain, June 2010. An extended version is under review at ANOR.

[8] S. R. Chakravarthy. *Markovian Arrival Processes*. Wiley, 2010. Published Online: 15 JUN 2010.

[9] G. Latouche and V. Ramaswami. *Introduction to matrix analytic methods in stochastic modeling*. SIAM, Philadelphia, 1999.

[10] D. M. Lucantoni. New results on the single server queue with a batch Markovian arrival process. *Commun. Statist.-Stochastic Models*, 7(1):1–46, 1991.

[11] M. Neuts. A versatile markovian point process. *Journal of Applied Probability*, 16:764–779, 1979.

[12] M. Neuts. *Matrix-Geometric Solutions in Stochastic Models*. John Hopkins University Press, Baltimore, MD, USA, 1981.

[13] M. Telek and G. Horváth. A minimal representation of Markov arrival processes and a moments matching method. *Performance Evaluation*, 64(9-12):1153–1168, Aug. 2007.

[14] Q. Zhang, A. Heindl, and E. Smirni. Characterizing the BMAP/MAP/1 departure process via the ETAQA truncation. *Stochastic Models*, 21(2-3):821–846, 2005.