Finite queues at the limit of saturation

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***Abstract*—A wide range of real life systems are modeled by queueing systems with finite capacity buffers. There are well established numerical procedures for the analysis of these queueing models when the load is lower or higher than the system capacity, but these numerical methods become unstable as the load gets close to the system capacity.**

We present simple modifications of the standard computational methods which remain numerically stable at saturation as well.

We consider two specific Markov models: finite quasi birth death (QBD) process and finite Markov fluid queue (MFQ). The first one describes the behavior of queueing systems with discrete buffer content, while the second one describes the behavior of queueing systems with continuous buffer content.

The stationary solution of a finite QBD process is a combination of two matrix geometric series while the stationary fluid density of a finite MFQ is a combination of two matrix exponential functions. Apart of this there are several further similarities between the discrete and continuous buffer models at saturation. The proposed solution method exploits the similarities of the models.

Key words: finite quasi birth death processes, matrix geometric solution, Markov fluid model, matrix analytic methods.

I. Introduction

Intuitively it is quite clear that infinite buffer queueing systems remain stable as long as the system load is below the system capacity. It is also widely accepted that finite buffer systems remain stable also when the system load is higher than the system capacity. This second statement suggests that finite buffer systems can be easily analyzed for any load level. In contrast, it turns out that standard solution methods suffer from severe numerical instabilities at the region where the load is close to the system capacity. It is interesting to note that analysis methods of finite buffer queueing systems used for the dimensioning of telecommunication network components are typically used for evaluating models close to saturation.

Apart of this practical issue, there is a commonly applied analysis approach in the theoretical papers dealing with finite buffer queueing systems. According to this approach the discussion is restricted to the case when the load is below the system capacity and it is commonly left for the reader to invert the buffer content process if the load is higher. Unfortunately, this approach does not help when the load is equal to the system capacity.

In this paper we consider two well established models of finite buffer queueing systems, quasi-birth-and-death processes with finite levels and Markov fluid queues with finite fluid buffers. Both models have been extensively studied in the literature (see e.g., [1] [2] [3] [4] [5] [6] for finite QBD models and [7] [8] [9] [10] for finite MFQ models) for the case when the load is different from the capacity, but we have not met with the analysis of these models at the point of saturation.

The rest of the paper is organized as follows. Section II and III are devoted to the discussion of the two main models, quasi birth death (QBD) process and finite Markov fluid queue (MFQ), respectively. These big sections have similar substructures. The first subsections introduce the considered stochastic model, the second the commonly applied solution methods, the third the problems at saturation, the fourth the proposed solution methods. After that Section IV demonstrates the numerical properties of the standard and the proposed analysis methods. The paper is concluded in Section V.

II. Queueing system with discrete buffer

A. Finite quasi birth death process

A QBD is a Markov chain with a regular transition structure [11], [12]. The states are grouped into *level*s of identical size and state transitions are possible from a given state to the states of the previous, the same and the next level. In this paper we assume that QBDs have level independent transition structure. When the level process of a QBD with level size s has an upper bound at level m the generator matrix takes the form:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|} \hline \mathbf{L}' & \mathbf{F} & & & \\ \hline \mathbf{B} & \mathbf{L} & \ddots & & \\ \hline & \mathbf{B} & \ddots & \mathbf{F} & \\ \hline & & \ddots & \mathbf{L} & \mathbf{F} \\ \hline & & & \mathbf{B} & \mathbf{L}^{"} \\ \hline \end{array}, \tag{1}$$

where the matrix blocks are of size $s \times s$. In this case the stationary equations expressed in level sized blocks are

$$\pi_0 \mathbf{L}' + \pi_1 \mathbf{B} = \mathbf{0}, \tag{2}$$

$$\pi_{n-1}\mathbf{F} + \pi_n \mathbf{L} + \pi_{n+1}\mathbf{B} = \mathbf{0}, \qquad 1 \le n \le m-1, \tag{3}$$

$$\pi_{m-1}\mathbf{F} + \pi_m \mathbf{L}^{"} = \mathbf{0}, \tag{4}$$

and the normalizing equation is

$$\sum_{n=0}^{m} \pi_n \mathbb{1} = 1.$$

	$d<0$	$d=0$	$d>0$
$\mathbf{R}$	$Sp(\mathbf{R})<1$	$Sp(\mathbf{R})=1$	$Sp(\mathbf{R})=1$
$\mathbf{S}$	$Sp(\mathbf{S})=1$	$Sp(\mathbf{S})=1$	$Sp(\mathbf{S})<1$
$\mathbf{G}$	$Sp(\mathbf{G})=1$	$Sp(\mathbf{G})=1$	$Sp(\mathbf{G})<1$
$\widehat{\mathbf{G}}$	$Sp(\widehat{\mathbf{G}})<1$	$Sp(\widehat{\mathbf{G}})=1$	$Sp(\widehat{\mathbf{G}})=1$

TABLE I
DRIFT RELATED PROPERTIES OF FINITE QBD PROCESSES

Throughout this paper we assume that the considered Markov models are irreducible and it implies that there is a unique stationary distribution, which satisfies the stationary equations.

B. Standard solution method

The stationary behavior of finite QBD models has been studied in several papers (see e.g. [1] and the references therein) and the major conclusion is that the stationary solution is a linear combination of two geometric series starting from the two bounds of the level process. That is

$$\pi_n = \alpha \mathbf{R}^n + \beta \mathbf{S}^{m-n}, \quad \forall 0 \leq n \leq m, \tag{5}$$

where matrix $\mathbf{R}$ and $\mathbf{S}$ are the minimal non-negative solutions of the matrix equations:

$$\mathbf{F} + \mathbf{R}\mathbf{L} + \mathbf{R}^2\mathbf{B} = \mathbf{0} \tag{6}$$

$$\mathbf{B} + \mathbf{S}\mathbf{L} + \mathbf{S}^2\mathbf{F} = \mathbf{0} \tag{7}$$

Let γ be the stationary solution of the phase process at the regular levels $(1, \ldots, m-1)$. γ is the solution of $\gamma(\mathbf{B}+\mathbf{L}+\mathbf{F}) = 0$ and $\gamma\mathbb{1} = 1$. The drift of the QBD process is $d = \gamma(-\mathbf{B}+\mathbf{F})\mathbb{1}$. The sign of the drift determines some essential properties of $\mathbf{R}$ and $\mathbf{S}$ [11], [12]. We have the following cases:

- negative drift: $Sp(\mathbf{R}) < 1$ and $Sp(\mathbf{S}) = 1$,
- positive drift: $Sp(\mathbf{R}) = 1$ and $Sp(\mathbf{S}) < 1$,
- zero drift: $Sp(\mathbf{R}) = 1$ and $Sp(\mathbf{S}) = 1$,

where $Sp(\mathbf{R})$ denotes the spectral radius of matrix $\mathbf{R}$. The counterparts matrices of $\mathbf{R}$ and $\mathbf{S}$ are defined by $\mathbf{G} = (-\mathbf{L} - \mathbf{R}\mathbf{B})^{-1}\mathbf{B}$ and $\widehat{\mathbf{G}} = (-\mathbf{L} - \mathbf{S}\mathbf{F})^{-1}\mathbf{F}$, respectively. $\mathbf{G}$ and $\widehat{\mathbf{G}}$ have nice probabilistic interpretations. The drift related properties of $\mathbf{G}$ and $\widehat{\mathbf{G}}$ are as follows [11], [12]:

- negative drift: $\mathbf{G}$ is stochastic and $\widehat{\mathbf{G}}$ is sub-stochastic,
- positive drift: $\mathbf{G}$ is sub-stochastic and $\widehat{\mathbf{G}}$ is stochastic,
- zero drift: $\mathbf{G}$ is stochastic and $\widehat{\mathbf{G}}$ is stochastic as well.

These main drift related properties of finite QBD processes are summarized in Table I. In all the cases of this table when the spectral radius equals to one there is a single eigenvalue on the unit cycle it is $+1$ and its multiplicity is one and all other eigenvalues are inside the unit disk. The eigenvector associated with this eigenvalue plays important role in the sequel. When $\mathbf{G}$ or $\widehat{\mathbf{G}}$ is stochastic then $\mathbb{1}$ is the right eigenvector associated with eigenvalue one, which is visible from $\mathbf{G}\mathbb{1} = \mathbb{1}$ or $\widehat{\mathbf{G}}\mathbb{1} = \mathbb{1}$. When the spectral radius of $\mathbf{R}$ is one then the left eigenvector of $\mathbf{R}$ associated with eigenvalue one, ξ, satisfies

$$\xi(\mathbf{F} + \mathbf{R}\mathbf{L} + \mathbf{R}^2\mathbf{B}) = \xi(\mathbf{F} + \mathbf{L} + \mathbf{B}) = \mathbf{0}$$

according to (6). Consequently, if this eigenvector is normalized then $\gamma = \xi$ and $\gamma\mathbf{R} = \gamma$. Similarly, the left eigenvector of $\mathbf{S}$ associated with eigenvalue one, ω, satisfies

$$\omega(\mathbf{B} + \mathbf{S}\mathbf{L} + \mathbf{S}^2\mathbf{F}) = \omega(\mathbf{B} + \mathbf{L} + \mathbf{F}) = \mathbf{0}$$

from which we have $\gamma\mathbf{S} = \gamma$ according to (7).

The combined matrix geometric solution in (5) satisfies the stationary equations

$$\pi_{n-1}\mathbf{F} + \pi_n\mathbf{L} + \pi_{n+1}\mathbf{B} = \mathbf{0}$$

for $1 \leq n \leq m-1$. The unknown vectors, α and β, are obtained from the remaining equations as the solution of the linear system:

$$[\alpha \mid \beta] \begin{array}{|c|c|} \hline \mathbf{L}' + \mathbf{R}\mathbf{B} & \mathbf{R}^{m-1}\left(\mathbf{F} + \mathbf{R}\mathbf{L}^{"}\right) \\ \hline \mathbf{S}^{m-1}\left(\mathbf{S}\mathbf{L}' + \mathbf{B}\right) & \mathbf{S}\mathbf{F} + \mathbf{L}^{"} \\ \hline \end{array} = [\mathbf{0} \mid \mathbf{0}] \ , \tag{8}$$

with normalizing condition

$$\alpha \sum_{n=0}^{m} \mathbf{R}^n\mathbb{1} + \beta \sum_{n=0}^{m} \mathbf{S}^n\mathbb{1} = 1. \tag{9}$$

C. Finite QBD in saturation

The solution of (8) gets to be non trivial when the drift is zero according to the following properties.

Theorem 1. *If the drift is 0 then the rank of matrix*

$$\mathbf{M} = \begin{array}{|c|c|} \hline \mathbf{L}' + \mathbf{R}\mathbf{B} & \mathbf{R}^{m-1}\left(\mathbf{F} + \mathbf{R}\mathbf{L}^{"}\right) \\ \hline \mathbf{S}^{m-1}\left(\mathbf{S}\mathbf{L}' + \mathbf{B}\right) & \mathbf{S}\mathbf{F} + \mathbf{L}^{"} \\ \hline \end{array}$$

is not greater than $2s-2$.

Proof: First we show that all the four $s \times s$ blocks of $\mathbf{M}$ have 0 row sum. From the fact that the drift is 0 we have $\mathbf{G}\mathbb{1} = \widehat{\mathbf{G}}\mathbb{1} = \mathbb{1}$. From the fact that $\mathbf{Q}$ in (1) is a generator matrix with zero row sum we have $\mathbf{L}'\mathbb{1} = -\mathbf{F}\mathbb{1}$ and $\mathbf{L}^{"}\mathbb{1} = -\mathbf{B}\mathbb{1}$ and from the relation of the characteristic matrices of QBD processes we have $\mathbf{R}\mathbf{B} = \mathbf{F}\mathbf{G}$ and similarly $\mathbf{S}\mathbf{F} = \mathbf{B}\widehat{\mathbf{G}}$ [11], [12]. Using these we can write

$$(\mathbf{L}' + \mathbf{R}\mathbf{B})\mathbb{1} = (\mathbf{L}' + \mathbf{F}\mathbf{G})\mathbb{1} = (\mathbf{L}' + \mathbf{F})\mathbb{1} = 0,$$

$$\left(\mathbf{F} + \mathbf{R}\mathbf{L}^{"}\right)\mathbb{1} =$$

$$= (\mathbf{F} - \mathbf{R}\mathbf{B})\,\mathbb{1} = (\mathbf{F} - \mathbf{F}\mathbf{G})\,\mathbb{1} = (\mathbf{F} - \mathbf{F})\,\mathbb{1} = 0,$$

$$(\mathbf{S}\mathbf{L}' + \mathbf{B})\,\mathbb{1} = (-\mathbf{S}\mathbf{F} + \mathbf{B})\,\mathbb{1} = \left(-\mathbf{B}\widehat{\mathbf{G}} + \mathbf{B}\right)\mathbb{1} =$$

$$= (-\mathbf{B} + \mathbf{B})\,\mathbb{1} = 0,$$

$$(\mathbf{S}\mathbf{F} + \mathbf{L}^{"})\mathbb{1} = (\mathbf{B}\widehat{\mathbf{G}} - \mathbf{L}^{"})\mathbb{1} = (\mathbf{B} - \mathbf{L}^{"})\mathbb{1} = 0.$$

The set of vectors $V = \{v(x_1, x_2)\}$, $x_1, x_2 \in \mathbb{R}$, where $v(x_1, x_2) = x_1 \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ \mathbb{1} \end{bmatrix}$, satisfies $\mathbf{M}\ v(x_1, x_2) = 0$, i. e., the null space of $\mathbf{M}$ is at least rank 2. ∎

Theorem 2. *If the drift is 0 then* $(\gamma, -\gamma)\mathbf{M} = (0, 0)$.

Proof: From $\gamma\mathbf{R} = \gamma\mathbf{S} = \gamma$ it follows that

$$[\gamma \mid -\gamma] \begin{array}{|c|c|} \hline \mathbf{L}' + \mathbf{RB} & \mathbf{R}^{m-1}\left(\mathbf{F} + \mathbf{RL}^{"}\right) \\ \hline \mathbf{S}^{m-1}\left(\mathbf{SL}' + \mathbf{B}\right) & \mathbf{SF} + \mathbf{L}^{"} \\ \hline \end{array}$$

$$= \left[\gamma\left(\mathbf{L}' + \mathbf{B}\right) - \gamma\left(\mathbf{L}' + \mathbf{B}\right) \mid \gamma\left(\mathbf{F} + \mathbf{L}^{"}\right) - \gamma\left(\mathbf{F} + \mathbf{L}^{"}\right)\right]$$
$$= [\mathbf{0} \mid \mathbf{0}]$$

■

The following corollary is a straightforward consequence of Theorem 2.

Corollary 1. *If the drift is 0,* $[\alpha \mid \beta]$ *is a solution of* (8) *and* c *is a constant then* $[\alpha + c\gamma \mid \beta - c\gamma]$ *is also a solution of* (8).

Theorem 3. *If the drift is 0,* $[\alpha \mid \beta]$ *is a solution of* (8) *and* π_n *(*$n \in \{0, \ldots, m\}$*) computed from* (5) *is non-zero, then* $[\alpha' \mid \beta'] = [\alpha + c\gamma \mid \beta - c\gamma]$ *and* (5) *results in the same stationary distribution.*

Proof:

$$\alpha'\mathbf{R}^n + \beta'\mathbf{S}^{m-n} = (\alpha + c\gamma)\mathbf{R}^n + (\beta - c\gamma)\mathbf{S}^{m-n} =$$
$$= \alpha\mathbf{R}^n + c\gamma + \beta\mathbf{S}^{m-n} - c\gamma = \alpha\mathbf{R}^n + \beta\mathbf{S}^{m-n} = \pi_n,$$

■

Theorem 1 and 3 compose a bad news good news pair. The main message of Theorem 1 is that (8) with normalizing condition (9) does not have a unique solution. On the other hand Theorem 3 says that almost any solution (except the ones proportional to $[\gamma \mid -\gamma]$) of (8) is a good solution for computing the stationary probabilities.

We still need to show that the rank of the null space of $\mathbf{M}$ is not greater than 2.

Theorem 4. *If the drift is 0, the rank of matrix* $\mathbf{M}$ *is* $2s - 2$.

Proof: From Theorem 1 we have $rank(\mathbf{M}) \leq 2s - 2$. Here we show that if $rank(\mathbf{M}) < 2s - 2$ then there are more than one set of vectors satisfying (2) - (4), which is in contrast with the unique stationary solution.

From (5) we have

$$[\pi_0 \mid \pi_m] = [\alpha \mid \beta] \begin{array}{|c|c|} \hline \mathbf{I} & \mathbf{S}^m \\ \hline \mathbf{R}^m & \mathbf{I} \\ \hline \end{array}. \qquad (10)$$

First we show that the null space of matrix $\begin{array}{|c|c|} \hline \mathbf{I} & \mathbf{S}^m \\ \hline \mathbf{R}^m & \mathbf{I} \\ \hline \end{array}$ is one. Let $[\eta \mid \vartheta]$ be a solution of $[\mathbf{0} \mid \mathbf{0}] = [\eta \mid \vartheta] \begin{array}{|c|c|} \hline \mathbf{I} & \mathbf{S}^m \\ \hline \mathbf{R}^m & \mathbf{I} \\ \hline \end{array}$ then $\eta = -\vartheta\mathbf{R}^m$ and $\eta\mathbf{S}^m = -\vartheta$, from which $\eta = \eta\mathbf{S}^m\mathbf{R}^m$. Due to $Sp(\mathbf{R}) \leq 1$ and $Sp(\mathbf{S}) \leq 1$ the last equation can be satisfied only by the eigenvector associated with one, which is γ. This expression also indicates that matrix $\begin{array}{|c|c|} \hline \mathbf{I} & \mathbf{S}^m \\ \hline \mathbf{R}^m & \mathbf{I} \\ \hline \end{array}$ is non-singular if the drift is not zero.

In (10) $[\alpha \mid \beta]$ is an element of the zero space of $\mathbf{M}$. If the rank of the zero space of $\mathbf{M}$ is greater than two then rank of the possible $[\pi_0 \mid \pi_m]$ vectors satisfying the stationary equations is greater then one according to the rank of the matrix in (10), which is in contrast with the unique solution of the QBD process. ■

D. The proposed stationary analysis method

We propose a modification of the standard finite QBD solution method which is applicable for all drift values.

The typical stationary analysis of Markov models with ℓ states requires the solution of a linear system $\theta\mathbf{M} = \mathbf{0}$ with normalizing condition $\theta v = 1$, where the rank of $\mathbf{M}$ the $\ell \times \ell$ matrix is $\ell - 1$. In this case the vector equation $\theta\mathbf{M} = \mathbf{0}$ is composed by m scalar equations with m unknowns. Out of the m equations any $m - 1$ are independent. Replacing one equation of $\theta\mathbf{M} = \mathbf{0}$ with $\theta v = 1$ results in a determined system of equations with a unique solution. Unfortunately, this procedure is not applicable when the rank of $\mathbf{M}$ is less than $\ell - 1$. Due to the fact that the rank of $\mathbf{M}$ depends on the drift and might be less than $\ell - 1$ a different approach has to be applied.

The solutions of (8) fall into the null space of $\mathbf{M}$. In computational methods the null space is given by spanning vectors. There are several computational environments where the spanning vectors of the null space can be computed with built in functions. If it is not the case the spanning vectors of the null space can be computed from the singular value decomposition of $\mathbf{M}$ which is known to be a numerically stable method.

The following procedure gives a solution of (8) and (9) which is applicable for both, $d \neq 0$ and $d = 0$:

1) compute (the spanning vectors of) the null space of $\mathbf{M}$,
2) if the null space of $\mathbf{M}$ is composed by a single vector then take this vector as the required non-normalized solution of (8)
3) if the null space of $\mathbf{M}$ is composed by more than a single vector then check if the first vector is proportional to $[\gamma \mid -\gamma]$
4) if the first vector is not proportional to $[\gamma \mid -\gamma]$ then take the first vector as the required non-normalized solution of (8), otherwise take the second spanning vector of the null space.
5) normalize the given solution of (8) according to (9).

E. Computation of the normalizing constant

We need to compute $\sum_{k=0}^{m} \mathbf{R}^k$ in an efficient way. The computation of $\sum_{k=0}^{m} \mathbf{S}^k$ follows the same pattern. This problem is already considered at [1, page 1107], without noting the fact that the left eigenvector associated with one is γ when $Sp(\mathbf{R}) = 1$.

If $Sp(\mathbf{R}) < 1$, then the finite sum can be computed as

$$\sum_{k=0}^{m} \mathbf{R}^k = (\mathbf{I} - \mathbf{R}^{m+1})(\mathbf{I} - \mathbf{R})^{-1},$$

because $\mathbf{I} - \mathbf{R}$ is non-singular.

If $Sp(\mathbf{R}) = 1$, then one is an eigenvalue of $\mathbf{R}$ and the associated left eigenvector is γ. The associated right eigenvector is a non-zero solution of $\mathbf{R}u = u$ and we define the diad associated with eigenvalue one as $\mathbf{\Pi} = \frac{u\gamma}{\gamma u}$. Now we separate the diad associated with eigenvalue one, $\mathbf{\Pi}$, and $\mathbf{R}$–$\mathbf{\Pi}$, whose spectral radius is strictly less than one and consequently $\mathbf{R}$–$\mathbf{\Pi}$ is non-singular. Due to the orthogonality of the eigenvectors we have $(\mathbf{R}-\mathbf{\Pi})\mathbf{\Pi} = \mathbf{\Pi}(\mathbf{R}-\mathbf{\Pi}) = \mathbf{0}$. Furthermore $\mathbf{\Pi}^i = \mathbf{\Pi}$, because the only nonzero eigenvalue of $\mathbf{\Pi}$ is one. Using this properties for $k \geq 1$ we have

$$\mathbf{R}^k = ((\mathbf{R}-\mathbf{\Pi}) + \mathbf{\Pi})^k =$$
$$(\mathbf{R}-\mathbf{\Pi})^k + \underbrace{(\mathbf{R}-\mathbf{\Pi})^{k-1}\mathbf{\Pi} + \ldots}_{\mathbf{0}} + \mathbf{\Pi}^k = (\mathbf{R}-\mathbf{\Pi})^k + \mathbf{\Pi}$$

We compute the finite sum based on this separation of $\mathbf{\Pi}$ and $\mathbf{R}-\mathbf{\Pi}$ as follows

$$\sum_{k=0}^{m} \mathbf{R}^k = \mathbf{I} + \sum_{k=1}^{m} \mathbf{R}^k = \mathbf{I} + \sum_{k=1}^{m} (\mathbf{R}-\mathbf{\Pi})^k + \sum_{k=1}^{m} \mathbf{\Pi} =$$
$$= \sum_{k=0}^{m} (\mathbf{R}-\mathbf{\Pi})^k + m\mathbf{\Pi} =$$
$$= \left(\mathbf{I} - (\mathbf{R}-\mathbf{\Pi})^{m+1}\right)\left(\mathbf{I} - (\mathbf{R}-\mathbf{\Pi})\right)^{-1} + m\mathbf{\Pi} \ .$$

III. Queueing system with fluid buffer

A. Markov fluid queue

The evolution of Markov fluid queue with single fluid buffer is determined by a discrete state of the environment and the continuous fluid level in the fluid buffer. The $Z(t) = \{M(t), X(t); t \geq 0\}$ process represents the state of the MFQ, where $M(t) \in \mathcal{S}$ is the (discrete) state of the environment and $X(t) \in [0, b]$ is the fluid level in the fluid buffer at time t, where b denotes the buffer size. The fluid level cannot be negative or greater than b. We define $\hat{\pi}_j(t,x)$, $\hat{p}_j(t,0)$ and $\hat{p}_j(t,b)$ to describe the transient fluid densities at fluid level x and the transient probability masses of the fluid distribution at idle and full buffer as follows

$$\hat{\pi}_j(t,x) = \lim_{\Delta \to 0} \frac{Pr(M(t) = j, x \leq X(t) < x + \Delta)}{\Delta} \ ,$$

$$\hat{p}_j(t,x) = Pr(M(t) = j, X(t) = x) \quad x = 0, b.$$

One of the main goal of the analysis of MFQ is to compute the stationary fluid density $\pi_j(x) = \lim_{t\to\infty} \hat{\pi}_j(t,x)$ and fluid mass at idle and full buffer $p_j(x) = \lim_{t\to\infty} \hat{p}_j(t,x)$, $x = 0, b$. The row vector $\pi(x) = \{\pi_j(x)\}$, satisfies [13]

$$\frac{d}{dx}\pi(x)\mathbf{R} = \pi(x)\mathbf{Q} \ , \tag{11}$$

where matrix $\mathbf{Q} = \{Q_{ij}\}$ is the transition rate matrix of the environment process, and the diagonal matrix $\mathbf{R} = \text{diag}\langle R_j \rangle$ is composed by the fluid rates $R_j, j \in \mathcal{S}$. R_j rate determines the rate at which the fluid level changes (increases when $R_j > 0$ or decreases when $R_j < 0$) when the environment is in state j.

Soares and Latouche [9] proposed a simplification of this general model such that the fluid rates are restricted to $+1$ and -1. MFQs with general fluid rates (including zero fluid rate) can be transformed to MFQs with fluid rate $+1$ and -1. Using this result we restrict our attention to finite buffer Markov fluid model with fluid rate $+1$ and -1 and buffer size b in this paper. Additionally, we assume that the states of the background Markov chain are ordered according to the sign of the unit fluid rate and the generator matrix is partitioned into the following blocks $\mathbf{Q} = \begin{array}{|c|c|}\hline \mathbf{Q}_{++} & \mathbf{Q}_{+-} \\ \hline \mathbf{Q}_{-+} & \mathbf{Q}_{--} \\ \hline \end{array}$. The stationary distribution of the background Markov chain is denoted by $\gamma = [\gamma_+ \mid \gamma_-]$, that is $\gamma\mathbf{Q} = \mathbf{0}$, $\gamma\mathbb{1} = 1$. The drift of the Markov fluid model with unit fluid rate is $d = \gamma_+\mathbb{1}_+ - \gamma_-\mathbb{1}_-$ and the partitioned form of (11) is

$$\frac{d}{dx}[\pi_+(x)|\pi_-(x)] \begin{array}{|c|c|}\hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{I} \\ \hline \end{array} = [\pi_+(x)|\pi_-(x)] \begin{array}{|c|c|}\hline \mathbf{Q}_{++} & \mathbf{Q}_{+-} \\ \hline \mathbf{Q}_{-+} & \mathbf{Q}_{--} \\ \hline \end{array} \ . \tag{12}$$

Using the vector and matrix blocks associated with the partitioning of the states according to the sign of the fluid rate the boundary conditions at fluid level 0 and b are [7], [10]:

$$\mathbf{p}_+(0) = \mathbf{0}, \ \mathbf{p}_-(b) = \mathbf{0} \ , \tag{13}$$

$$\pi_-(0) + \mathbf{p}_-(0)\mathbf{Q}_{--} = \mathbf{0}, \ \pi_+(b) + \mathbf{p}_+(b)\mathbf{Q}_{++} = \mathbf{0}, \tag{14}$$

$$\pi_+(0) = \mathbf{p}_-(0)\mathbf{Q}_{-+}, \ \pi_-(b) = \mathbf{p}_+(b)\mathbf{Q}_{+-}. \tag{15}$$

(13) states that the fluid level cannot be 0 when the fluid rate is $+1$ and it cannot be b when the fluid rate is -1. Due to (13) we use the short notation $\mathbf{p}_+ = \mathbf{p}_+(b)$, $\mathbf{p}_- = \mathbf{p}_-(0)$. The first part of (14) means that the fluid level can be 0 due to a state transition of the environment from an other state with negative fluid rate or due to the fact that the fluid level reduced to 0 in a state with negative fluid rate. Finally, the first part of (15) represents that the fluid level can start increasing from 0 due to the fact that the process stayed in a state with negative fluid rate at level 0 and a state transition occurred to a state with positive fluid rate. The second parts of (14) and (15) are the counterpart statements for buffer level b.

The are several solution methods for the stationary analysis of finite buffer Markov fluid models. It is possible to look for a purely analytic solution based on the differential equation (11) and the set of boundary equations (13)-(15) [7] and it is also possible to evaluate the model behavior based on a deeper stochastic interpretation of the model behavior [14]. In this paper we focus on the solution method of [9] which is based on the matrix analytic approach. It analyzes the Markov fluid model based on some similarities with the QBD processes.

Later on we propose a modification of this method such that it exploits the similarities between the QBD and the MFQ models also at the limit of saturation.

B. Computational method for non-zero drift

Soares and Latouche [9] proposed the following procedure to compute the stationary distribution of this finite Markov fluid model:

1) Compute the characterizing matrix $\mathbf{\Psi} = \mathbf{\Psi}_{+-}$ and $\hat{\mathbf{\Psi}} = \hat{\mathbf{\Psi}}_{-+}$ (where the interchangeably used verbose notations explicitly refer to the dimension of the matrices) for the forward and the inverse level process based on the Ricatti matrix equations

$$\mathbf{Q}_{+-} + \mathbf{Q}_{++}\mathbf{\Psi}_{+-} + \mathbf{\Psi}_{+-}\mathbf{Q}_{-+}\mathbf{\Psi}_{+-} + \mathbf{\Psi}_{+-}\mathbf{Q}_{--} = \mathbf{0},$$
$$\mathbf{Q}_{-+} + \mathbf{Q}_{--}\hat{\mathbf{\Psi}}_{-+} + \hat{\mathbf{\Psi}}_{-+}\mathbf{Q}_{+-}\hat{\mathbf{\Psi}}_{-+} + \hat{\mathbf{\Psi}}_{-+}\mathbf{Q}_{++} = \mathbf{0}.$$

2) Compute

$$\mathbf{K} = \mathbf{K}_{++} = \mathbf{Q}_{++} + \mathbf{\Psi}_{+-}\mathbf{Q}_{-+}, \tag{16}$$
$$\mathbf{U} = \mathbf{U}_{--} = \mathbf{Q}_{--} + \mathbf{Q}_{-+}\mathbf{\Psi}_{+-}, \tag{17}$$
$$\hat{\mathbf{K}} = \hat{\mathbf{K}}_{--} = \mathbf{Q}_{--} + \hat{\mathbf{\Psi}}_{-+}\mathbf{Q}_{+-}, \tag{18}$$
$$\hat{\mathbf{U}} = \hat{\mathbf{U}}_{++} = \mathbf{Q}_{++} + \mathbf{Q}_{+-}\hat{\mathbf{\Psi}}_{-+}. \tag{19}$$

3) Compute the probability mass at the boundaries based on

$$[\mathbf{p}_+ \mid \mathbf{p}_-] \;\; \mathbf{V} = [\mathbf{0} \mid \mathbf{0}] \tag{20}$$

where

$$\mathbf{V} = \begin{array}{|c|c|} \hline \mathbf{Q}_{++} + \mathbf{Q}_{+-}\hat{\mathbf{\Psi}}_{-+} & (\mathbf{Q}_{+-} + \mathbf{Q}_{++}\mathbf{\Psi}_{+-})\, e^{\mathbf{U}b} \\ \hline \left(\mathbf{Q}_{-+} + \mathbf{Q}_{--}\hat{\mathbf{\Psi}}_{-+}\right) e^{\hat{\mathbf{U}}b} & \mathbf{Q}_{--} + \mathbf{Q}_{-+}\mathbf{\Psi}_{+-} \\ \hline \end{array}$$

with normalizing condition $\mathbf{p}_+\mathbb{1}_+ - \mathbf{p}_-\mathbb{1}_- = \gamma_+\mathbb{1}_+ - \gamma_-\mathbb{1}_-$.

4) Compute the fluid density based on

$$[\pi_+(x)|\pi_-(x)] = [\mathbf{v}_+ \mid \mathbf{v}_-] \begin{array}{|c|c|} \hline e^{\mathbf{K}x} & e^{\mathbf{K}x}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}(b-x)}\hat{\mathbf{\Psi}} & e^{\hat{\mathbf{K}}(b-x)} \\ \hline \end{array} \tag{21}$$

where

$$[\mathbf{v}_+ \mid \mathbf{v}_-] = [\mathbf{p}_+ \mid \mathbf{p}_-] \begin{array}{|c|c|} \hline \mathbf{0} & \mathbf{Q}_{+-} \\ \hline \mathbf{Q}_{-+} & \mathbf{0} \\ \hline \end{array} \begin{array}{|c|c|} \hline \mathbf{I} & e^{\mathbf{K}b}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}b}\hat{\mathbf{\Psi}} & \mathbf{I} \\ \hline \end{array}^{-1}. \tag{22}$$

In step 1), a possible solution method of the Ricatti matrix equations is the following:

	$d<0$	$d=0$	$d>0$
$\mathbf{\Psi}$	$\mathbf{\Psi}_{+-}\mathbb{1}_- = \mathbb{1}_+$	$\mathbf{\Psi}_{+-}\mathbb{1}_- = \mathbb{1}_+$	$\mathbf{\Psi}_{+-}\mathbb{1}_- < \mathbb{1}_+$
$\hat{\mathbf{\Psi}}$	$\hat{\mathbf{\Psi}}_{-+}\mathbb{1}_+ \leq \mathbb{1}_-$	$\hat{\mathbf{\Psi}}_{-+}\mathbb{1}_+ = \mathbb{1}_-$	$\hat{\mathbf{\Psi}}_{-+}\mathbb{1}_+ = \mathbb{1}_-$
$\mathbf{\Psi}$	$\gamma_+\mathbf{\Psi}_{+-} \leq \gamma_-$	$\gamma_+\mathbf{\Psi}_{+-} = \gamma_-$	$\gamma_+\mathbf{\Psi}_{+-} = \gamma_-$
$\hat{\mathbf{\Psi}}$	$\gamma_-\hat{\mathbf{\Psi}}_{-+} = \gamma_+$	$\gamma_-\hat{\mathbf{\Psi}}_{-+} = \gamma_+$	$\gamma_-\hat{\mathbf{\Psi}}_{-+} \leq \gamma_+$
$\mathbf{K}$	$Mr(\mathbf{K}) < 0$	$Mr(\mathbf{K}) = 0$	$Mr(\mathbf{K}) = 0$
$\hat{\mathbf{K}}$	$Mr(\hat{\mathbf{K}}) = 0$	$Mr(\hat{\mathbf{K}}) = 0$	$Mr(\hat{\mathbf{K}}) < 0$
$\mathbf{U}$	$Mr(\mathbf{U}) = 0$	$Mr(\mathbf{U}) = 0$	$Mr(\mathbf{U}) < 0$
$\hat{\mathbf{U}}$	$Mr(\hat{\mathbf{U}}) < 0$	$Mr(\hat{\mathbf{U}}) = 0$	$Mr(\hat{\mathbf{U}}) = 0$

TABLE II
DRIFT RELATED PROPERTIES OF FINITE MFQS

Let $c = \max_{i\in S}|\mathbf{Q}_{ii}|$ and define matrix $\mathbf{P} = \mathbf{I} + \mathbf{Q}/c$ which is identically partitioned as $\mathbf{Q}$. Let

$$\mathbf{F} = \begin{array}{|c|c|} \hline \frac{1}{2}\mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \\ \hline \end{array}, \quad \mathbf{L} = \begin{array}{|c|c|} \hline \frac{1}{2}\mathbf{P}_{++} - \mathbf{I} & \mathbf{0} \\ \hline \mathbf{P}_{-+} & -\mathbf{I} \\ \hline \end{array}, \quad \mathbf{B} = \begin{array}{|c|c|} \hline \mathbf{0} & \frac{1}{2}\mathbf{P}_{+-} \\ \hline \mathbf{0} & \mathbf{P}_{--} \\ \hline \end{array}.$$

Finally, $\mathbf{\Psi} = \mathbf{G}_{+-}$ obtained from the minimal non-negative solution of $\mathbf{B} + \mathbf{L}\mathbf{G} + \mathbf{F}\mathbf{G}^2 = \mathbf{0}$.

C. Matrix analytic method for finite buffer Markov fluid model

In this paper we restrict our attention to this method and recommend its modification. To be self explaining, we summarize some of the results from [9] below. These results are used in the sequel to prove the validity of the proposed modifications. We apply similar notations as the ones in [9], except that the generator matrix of the background Markov chain is denoted by $\mathbf{T}$ in [9] and by $\mathbf{Q}$ here.

We start with the stochastic interpretation of the characterizing matrices. Assuming that the buffer is infinite the evolution of the buffer level process can be divided into idle buffer and busy buffer (when the fluid level is positive) periods. The busy buffer period starts when the buffer is idle and the background process moves from a state with negative rate to another with positive rate and the busy buffer period (if finite) completes in a state with negative rate. The i, j ($i \in S^+, j \in S^-$) element of $\mathbf{\Psi}$ is the probability that the busy buffer period is finite and it completes in state j given that it starts in state i. The drift indicates the long run tendency of the fluid level. If the drift is negative or zero then the busy buffer period is finite with probability 1, that is $\mathbf{\Psi}\mathbb{1}_- = \mathbb{1}_+$. If the drift is positive then the busy buffer period might be infinite with positive probability, that is $\mathbf{\Psi}\mathbb{1}_- \leq \mathbb{1}_+$.

The main drift related properties of finite MFQs are summarized in Table II, where $Mr(\mathbf{M})$ is the real part of the eigenvalue of $\mathbf{M}$ with the maximal real part. In all of the cases when the eigenvalue with the maximal real part is zero, zero is an eigenvalue of the matrix with multiplicity one and the associated eigenvectors play important roles. When $Mr(\mathbf{U}) = 0$ we have $\mathbf{\Psi}_{+-}\mathbb{1}_- = \mathbb{1}_+$ and from (17)

$$\mathbf{U}\mathbb{1}_- = \mathbf{Q}_{--}\mathbb{1}_- + \mathbf{Q}_{-+}\mathbf{\Psi}_{+-}\mathbb{1}_- = \mathbf{Q}_{--}\mathbb{1}_- + \mathbf{Q}_{-+}\mathbb{1}_+ = \mathbf{0}.$$

Similarly, when $Mr(\hat{\mathbf{U}}) = 0$ we have $\hat{\mathbf{U}}\mathbb{1}_+ = \mathbf{0}$. With respect to matrix $\mathbf{K}$, when $Mr(\mathbf{K}) = 0$ we have $\gamma_+\mathbf{\Psi}_{+-} = \gamma_-$ and

from (16)

$$\gamma_+\mathbf{K} = \gamma_+\mathbf{Q}_{++} + \gamma_+\mathbf{\Psi}_{+-}\mathbf{Q}_{-+} = \gamma_+\mathbf{Q}_{++} + \gamma_-\mathbf{Q}_{-+} = \mathbf{0},$$

where the last equation is the partitioned form of stationary equation, $\gamma\mathbf{Q} = \mathbf{0}$. Similarly, when $Mr(\hat{\mathbf{K}}) = 0$ we have $\gamma_-\hat{\mathbf{K}} = \mathbf{0}$.

D. Finite buffer Markov fluid model at saturation

The above described procedure is proposed for the case when the drift is non-zero. In this section we focus on the case when the drift is zero. First we discuss the reasons why the procedure is not applicable when the drift is zero and after that we propose a modified procedure which is applicable both when the drift is non-zero and zero. The discussion on the applicability of the above procedure at $d = 0$ follows the steps of the procedure.

1) When $d = 0$ both the forward and the inverse fluid level process are recurrent and the probability of returning to a given initial fluid level is one. Consequently

$$\mathbf{\Psi}_{+-}\mathbb{1}_- = \mathbb{1}_+ \quad \text{and} \quad \hat{\mathbf{\Psi}}_{-+}\mathbb{1}_+ = \mathbb{1}_-.$$

2) When $d = 0$ the results of the second step have the following properties:

$$\mathbf{U}_{--}\mathbb{1}_- = \mathbf{0}, \quad \hat{\mathbf{U}}_{++}\mathbb{1}_+ = \mathbf{0},$$
$$\gamma_+\mathbf{K} = \mathbf{0}, \quad \gamma_-\hat{\mathbf{K}} = \mathbf{0}.$$

3) $[\mathbf{p}_+ \mid \mathbf{p}_-]$ cannot be computed based on (20), because of the rank deficit of $\mathbf{V}$ according to Theorem 5. Additionally, if the drift is zero (i.e., $\gamma_+\mathbb{1}_+ - \gamma_-\mathbb{1}_- = 0$) then $\mathbf{p}_+^{(b)}\mathbb{1}_+ - \mathbf{p}_-^{(b)}\mathbb{1}_- = \gamma_+\mathbb{1}_+ - \gamma_-\mathbb{1}_-$ cannot be used for normalization.

4) (22) is not applicable, because $\begin{array}{|c|c|}\hline \mathbf{I} & e^{\mathbf{K}b}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}b}\hat{\mathbf{\Psi}} & \mathbf{I} \\ \hline \end{array}$ is singular, since

$$[\ \gamma_+ \mid -\gamma_-\ ] \begin{array}{|c|c|}\hline \mathbf{I} & e^{\mathbf{K}b}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}b}\hat{\mathbf{\Psi}} & \mathbf{I} \\ \hline \end{array} = [\ \mathbf{0} \mid \mathbf{0}\ ],$$

due to $\gamma_+e^{\mathbf{K}b} = \gamma_+$ and $\gamma_+\mathbf{\Psi} = \gamma_-$ and the related properties of the reverse level process. $\gamma_+e^{\mathbf{K}b} = \gamma_+$ comes from $\gamma_+\mathbf{K} = \mathbf{0}$.

We can summarize the main message of this detailed list as the procedure of Section III-B is not applicable for the case of $d = 0$ for several reasons.

Theorem 5. *When the drift is zero the rank of the null space of* $\mathbf{V}$ *is at least* 2*.*

Proof: Using $e^{\mathbf{U}b}\mathbb{1}_- = \mathbb{1}_-$ (due to $\mathbf{U}\mathbb{1}_- = \mathbf{0}$), $\mathbf{\Psi}_{+-}\mathbb{1}_- = \mathbb{1}_+$, $\mathbf{Q}_{++}\mathbb{1}_+ + \mathbf{Q}_{+-}\mathbb{1}_- = \mathbf{0}$ and the related properties of the reverse level process we have $\mathbf{V}\begin{bmatrix}\mathbb{1}_+ \\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{0} \\ \mathbf{0}\end{bmatrix}$ and $\mathbf{V}\begin{bmatrix}\mathbf{0} \\ \mathbb{1}_-\end{bmatrix} = \begin{bmatrix}\mathbf{0} \\ \mathbf{0}\end{bmatrix}$. ∎

E. Proposed stationary analysis method

The steps of the proposed analysis method are as follows.

1) Compute $\mathbf{\Psi}$ and $\hat{\mathbf{\Psi}}$ in the same way as in the standard method.
2) Compute $\mathbf{K}$, $\mathbf{U}$ and $\hat{\mathbf{K}}$, $\hat{\mathbf{U}}$ in the same way as in the standard method. Additionally, compute the right eigenvectors and the diads associated with the zero eigenvalue of $\mathbf{K}$ and $\hat{\mathbf{K}}$. Let ϱ and $\hat{\varrho}$ be the nonzero solutions of $\mathbf{K}\varrho = \mathbf{0}$ and $\hat{\mathbf{K}}\hat{\varrho} = \mathbf{0}$, respectively, and define $\mathbf{\Pi} = \dfrac{\varrho\gamma_+}{\gamma_+\varrho}$ and $\hat{\mathbf{\Pi}} = \dfrac{\hat{\varrho}\gamma_-}{\gamma_-\hat{\varrho}}$.
3) Compute the spanning vectors of the null space of $\mathbf{W}$, denoted by $\mathbf{z}_i$, where

$$\mathbf{W} = \begin{array}{|c|c|}\hline \mathbf{I} & e^{\mathbf{K}b}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}b}\hat{\mathbf{\Psi}} & \mathbf{I} \\ \hline \end{array} \begin{array}{|c|c|}\hline e^{\hat{\mathbf{U}}b} & \mathbf{\Psi} \\ \hline \hat{\mathbf{\Psi}} & e^{\mathbf{U}b} \\ \hline \end{array} - \begin{array}{|c|c|}\hline e^{\mathbf{K}b} & \mathbf{\Psi} \\ \hline \hat{\mathbf{\Psi}} & e^{\hat{\mathbf{K}}b} \\ \hline \end{array} \begin{array}{|c|c|}\hline \mathbf{I} & \mathbf{\Psi}e^{\mathbf{U}b} \\ \hline \hat{\mathbf{\Psi}}e^{\hat{\mathbf{U}}b} & \mathbf{I} \\ \hline \end{array}.$$

- If the null space is rank one then $[\mathbf{v}_+ \mid \mathbf{v}_-] = \mathbf{z}_1$.
- If the null space is rank two then compute $\mathbf{w} = \mathbf{z}_1\begin{array}{|c|c|}\hline \mathbf{I} & e^{\mathbf{K}b}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}b}\hat{\mathbf{\Psi}} & \mathbf{I} \\ \hline \end{array}$.
- If $\mathbf{w} \neq \mathbf{0}$ then $[\mathbf{v}_+ \mid \mathbf{v}_-] = \mathbf{z}_1$, otherwise $[\mathbf{v}_+ \mid \mathbf{v}_-] = \mathbf{z}_2$.

Now $[\mathbf{v}_+ \mid \mathbf{v}_-]$ is the non-normalized coefficient vector of the combined matrix exponential solution of the fluid density.

4) Compute the non-normalized fluid density and probability mass based on

$$[\pi_+(x)|\pi_-(x)] = [\mathbf{v}_+ \mid \mathbf{v}_-] \begin{array}{|c|c|}\hline e^{\mathbf{K}x} & e^{\mathbf{K}x}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}(b-x)}\hat{\mathbf{\Psi}} & e^{\hat{\mathbf{K}}(b-x)} \\ \hline \end{array}$$

and

$$[\mathbf{p}_+(b)|\mathbf{p}_-(0)] = -[\mathbf{v}_+ \mid \mathbf{v}_-] \begin{array}{|c|c|}\hline e^{\mathbf{K}b} & \mathbf{\Psi} \\ \hline \hat{\mathbf{\Psi}} & e^{\hat{\mathbf{K}}b} \\ \hline \end{array} \begin{array}{|c|c|}\hline \mathbf{Q}_{++}^{-1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{Q}_{--}^{-1} \\ \hline \end{array} \tag{23}$$

5) Compute the normalizing constant

$$c = [\mathbf{v}_+ \mid \mathbf{v}_-]\left(\begin{bmatrix}f_+ \\ f_-\end{bmatrix} - \begin{bmatrix}e_+ \\ e_-\end{bmatrix}\right)$$

where

$$\begin{bmatrix} e_+ \\ e_- \end{bmatrix} = \begin{array}{|c|c|}\hline e^{\mathbf{K}b} & \mathbf{\Psi} \\ \hline \hat{\mathbf{\Psi}} & e^{\hat{\mathbf{K}}b} \\ \hline \end{array} \begin{array}{|c|c|}\hline \mathbf{Q}_{++}^{-1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{Q}_{--}^{-1} \\ \hline \end{array} \begin{bmatrix} \mathbb{1}_+ \\ \mathbb{1}_- \end{bmatrix}$$

and

$$\begin{bmatrix} f_+ \\ f_- \end{bmatrix} = \begin{array}{|c|c|}\hline \left(e^{\mathbf{K}b}\text{–}\mathbf{I}\text{–}b\mathbf{\Pi}\right) & \mathbf{0} \\ \hline \mathbf{0} & \left(e^{\hat{\mathbf{K}}b}\text{–}\mathbf{I}\text{–}b\hat{\mathbf{\Pi}}\right) \\ \hline \end{array} \times$$

$$\times \begin{array}{|c|c|}\hline (\mathbf{K}\text{–}\mathbf{\Pi})^{-1} & \mathbf{0} \\ \hline \mathbf{0} & (\hat{\mathbf{K}}\text{–}\hat{\mathbf{\Pi}})^{-1} \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \mathbf{I} & \mathbf{\Psi} \\ \hline \hat{\mathbf{\Psi}} & \mathbf{I} \\ \hline \end{array} \begin{bmatrix} \mathbb{1}_+ \\ \mathbb{1}_- \end{bmatrix}$$

Finally obtain the normalized measures by dividing the non-normalized ones with c.

This computational method contains several modifications of that in Section III-B. These modifications are motivated with the similarities of the QBD and the MFQ models. The QBD solution method computes the coefficient vectors of the matrix geometric series (α, β). We use this idea also in case of MFQs. It means that instead of $(\mathbf{p}_+, \mathbf{p}_-)$ the coefficient vector of the matrix exponential solution $(\mathbf{v}_+, \mathbf{v}_-)$ is computed first. There are several important consequences of this modification:

- a modified system of linear equation needs to be solved,
- the modified system of linear equations exhibits properties which are very similar to those of the QBD model,
- the required performance measures of the model, $(\mathbf{p}_+, \mathbf{p}_-)$ and $(\pi_+(x), \pi_-(x))$ are computed from $(\mathbf{v}_+, \mathbf{v}_-)$ in an explicit way (without matrix inversion).

The third consequence is visible from the modified procedure. Here we show the other two consequences. First we derive $(\mathbf{v}_+, \mathbf{v}_-)\mathbf{W} = (\mathbf{0}, \mathbf{0})$, then we study its properties.

The left hand side of (20) can be rewritten in the following form (see also [9, page 307])

$$\underbrace{[\mathbf{p}_+ \mid \mathbf{p}_-] \begin{array}{|c|c|}\hline \mathbf{Q}_{++} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{Q}_{--} \\ \hline \end{array} \begin{array}{|c|c|}\hline \mathbf{I} & \mathbf{\Psi} e^{\mathbf{U}b} \\ \hline \hat{\mathbf{\Psi}} e^{\hat{\mathbf{U}}b} & \mathbf{I} \\ \hline \end{array}}_{-[\pi_+(b)|\pi_-(0)]}$$

$$+ \underbrace{[\mathbf{p}_+ \mid \mathbf{p}_-] \begin{array}{|c|c|}\hline \mathbf{0} & \mathbf{Q}_{+-} \\ \hline \mathbf{Q}_{-+} & \mathbf{0} \\ \hline \end{array}}_{[\pi_+(0)|\pi_-(b)]} \begin{array}{|c|c|}\hline e^{\hat{U}b} & \hat{\mathbf{\Psi}} \\ \hline \hat{\mathbf{\Psi}} & e^{\hat{\mathbf{U}}b} \\ \hline \end{array}, \tag{24}$$

where the under braced expressions are from (14) and (15). From (21) we have

$$[\pi_+(b)|\pi_-(0)] = [\mathbf{v}_+ \mid \mathbf{v}_-] \begin{array}{|c|c|}\hline e^{\mathbf{K}b} & \mathbf{\Psi} \\ \hline \hat{\mathbf{\Psi}} & e^{\hat{\mathbf{K}}b} \\ \hline \end{array}, \tag{25}$$

$$[\pi_+(0)|\pi_-(b)] = [\mathbf{v}_+ \mid \mathbf{v}_-] \begin{array}{|c|c|}\hline \mathbf{I} & e^{\mathbf{K}b}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}b}\hat{\mathbf{\Psi}} & \mathbf{I} \\ \hline \end{array}. \tag{26}$$

Substituting the last two expressions into the under braced ones results in the modified linear system $(\mathbf{v}_+, \mathbf{v}_-)\mathbf{W} = (\mathbf{0}, \mathbf{0})$.

The following theorems summarize the properties of the non-zero solutions of this modified linear system.

Theorem 6. *When the drift is zero the rank of the null space of* $\mathbf{W}$ *is at least* 2*.*

Proof: $\mathbf{W} \begin{bmatrix} \mathbb{1}_+ \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$ and $\mathbf{W} \begin{bmatrix} \mathbf{0} \\ \mathbb{1}_- \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$. ■

Theorem 7. *If the drift is 0 then* $(\gamma_+, -\gamma_-)\mathbf{W} = (\mathbf{0}, \mathbf{0})$.

Proof: Matrix $\mathbf{W}$ is composed by two terms where both terms are products of two matrices. We consider the product of $(\gamma_+, -\gamma_-)$ and the left matrices of the products composing $\mathbf{W}$ using $\gamma_+ e^{\mathbf{K}b} = \gamma_+$ and $\gamma_+ \mathbf{\Psi} = \gamma_-$ and the related properties of the reverse level process. We have

$$(\gamma_+, -\gamma_-) \begin{array}{|c|c|}\hline \mathbf{I} & e^{\mathbf{K}b}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}b}\hat{\mathbf{\Psi}} & \mathbf{I} \\ \hline \end{array} = (\mathbf{0}, \mathbf{0}) \ ,$$

$$(\gamma_+, -\gamma_-) \begin{array}{|c|c|}\hline e^{\mathbf{K}b} & \mathbf{\Psi} \\ \hline \hat{\mathbf{\Psi}} & e^{\hat{\mathbf{K}}b} \\ \hline \end{array} = (\mathbf{0}, \mathbf{0}) \ .$$

■

Corollary 2. *If* $[\nu_+ \mid \nu_-]$ *is a solution of* $(\mathbf{v}_+, \mathbf{v}_-)\mathbf{W} = (\mathbf{0}, \mathbf{0})$ *and* c *is a constant then* $[\nu_+ + c\gamma_+ \mid \nu_- - c\gamma_-]$ *is also a solution of* $(\mathbf{v}_+, \mathbf{v}_-)\mathbf{W} = (\mathbf{0}, \mathbf{0})$.

The corollary is a straightforward consequence of Theorem 7.

Theorem 8. *If* $[\nu_+ \mid \nu_-]$ *is a solution of* $(\mathbf{v}_+, \mathbf{v}_-)\mathbf{W} = (\mathbf{0}, \mathbf{0})$ *and* $[\pi_+(x)|\pi_-(x)]$ *is computed from* (21) *is non-zero, then* $[\nu'_+ \mid \nu'_-] = [\nu_+ + c\gamma_+ \mid \nu_- - c\gamma_-]$ *and* (21) *results in the same stationary fluid density.*

Proof:

$$[\nu_+ + c\gamma_+ \mid \nu_- - c\gamma_-] \begin{array}{|c|c|}\hline e^{\mathbf{K}x} & e^{\mathbf{K}x}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}(b-x)}\hat{\mathbf{\Psi}} & e^{\hat{\mathbf{K}}(b-x)} \\ \hline \end{array} =$$

$$= [\nu_+ \mid \nu_-] \begin{array}{|c|c|}\hline e^{\mathbf{K}x} & e^{\mathbf{K}x}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}(b-x)}\hat{\mathbf{\Psi}} & e^{\hat{\mathbf{K}}(b-x)} \\ \hline \end{array} +$$

$$+ c\,[\gamma_+ \mid -\gamma_-] \begin{array}{|c|c|}\hline e^{\mathbf{K}x} & e^{\mathbf{K}x}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}(b-x)}\hat{\mathbf{\Psi}} & e^{\hat{\mathbf{K}}(b-x)} \\ \hline \end{array} =$$

$$= [\pi_+(x)|\pi_-(x)] + [\mathbf{0}|\mathbf{0}]$$

where $\gamma_+ e^{\mathbf{K}x} = \gamma_+$ and $\gamma_+\mathbf{\Psi} = \gamma_-$ and the related properties of the reverse level process are used in the last step. ∎

Theorem 9. *If the drift is 0, the rank of the null space of matrix* $\mathbf{W}$ *is* 2*.*

Proof: The proof follows the line of the proof of Theorem 4 and we discuss the MFQ specific step only. In case of the MFQ, (10) needs to be replaced with (26). Following the same line of derivation the null space of the matrix in (26) is associated with the solution of $\eta = \eta e^{\mathbf{K}b}\mathbf{\Psi} e^{\hat{\mathbf{K}}b}\hat{\mathbf{\Psi}}$. According to the drift related properties summarized in Table II the only non-zero solution is $\eta = \gamma_+$ and if the drift is not 0 the matrix in (26) is non-singular. ∎

It is worth noting that our proposed solution also contains a matrix inversion to compute $[\mathbf{p}_+|\mathbf{p}_-]$ based on $[\mathbf{v}_+|\mathbf{v}_-]$ in (23). That expression is based on the first under braced expression in (24), where matrix $\begin{array}{|c|c|}\hline \mathbf{Q}_{++} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{Q}_{--} \\ \hline \end{array}$ is always invertible independent of the drift.

F. Computation of the normalizing constant

Based on the the non-normalized fluid density and probability mass we compute the normalizing constant

$$c = \mathbf{p}_+^{(b)}\mathbb{1}_+ + \int_0^b \pi_+(x)dx\mathbb{1}_+ + \mathbf{p}_-^{(b)}\mathbb{1}_- + \int_0^b \pi_-(x)dx\mathbb{1}_-$$

From (23) and (21) we have

$$\mathbf{p}_+^{(b)}\mathbb{1}_+ + \mathbf{p}_-^{(b)}\mathbb{1}_- =$$

$$= -[\mathbf{v}_+ \mid \mathbf{v}_-] \begin{array}{|c|c|}\hline e^{\mathbf{K}b} & \mathbf{\Psi} \\ \hline \hat{\mathbf{\Psi}} & e^{\hat{\mathbf{K}}b} \\ \hline \end{array} \begin{array}{|c|c|}\hline \mathbf{Q}_{++}^{-1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{Q}_{--}^{-1} \\ \hline \end{array} \begin{bmatrix}\mathbb{1}_+ \\ \mathbb{1}_-\end{bmatrix}$$

and

$$\int_0^b \pi_+(x)dx\mathbb{1}_+ + \int_0^b \pi_-(x)dx\mathbb{1}_- =$$

$$= [\mathbf{v}_+ \mid \mathbf{v}_-] \int_0^b \begin{array}{|c|c|}\hline e^{\mathbf{K}x} & \mathbf{0} \\ \hline \mathbf{0} & e^{\hat{\mathbf{K}}(b-x)} \\ \hline \end{array} dx \begin{array}{|c|c|}\hline \mathbf{I} & \mathbf{\Psi} \\ \hline \hat{\mathbf{\Psi}} & \mathbf{I} \\ \hline \end{array} \begin{bmatrix}\mathbb{1}_+ \\ \mathbb{1}_-\end{bmatrix},$$

where the integral of the block diagonal matrix can be computed by blocks. For a closed form solution of $\int_0^b e^{\mathbf{K}x}dx$ we need to separate the unique zero eigenvalue of $\mathbf{K}$ as it is proposed in [12, page 64]. It is similar to the computation used in the case of the finite QBD process with zero drift.

$$\int_0^b e^{\mathbf{K}x}dx = \sum_{i=0}^{\infty}\int_0^b \frac{x^i}{i!}\mathbf{K}^i dx =$$
$$= \sum_{i=0}^{\infty}\frac{b^{i+1}}{(i+1)!}\mathbf{K}^i(\mathbf{K}-\mathbf{\Pi})(\mathbf{K}-\mathbf{\Pi})^{-1} =$$
$$= \sum_{i=0}^{\infty}\frac{b^{i+1}}{(i+1)!}(\mathbf{K}^{i+1}-\mathbf{K}^i\mathbf{\Pi})(\mathbf{K}-\mathbf{\Pi})^{-1} =$$
$$= \left(e^{\mathbf{K}b} - \mathbf{I} - b\mathbf{\Pi}\right)(\mathbf{K}-\mathbf{\Pi})^{-1}$$

Where we first observed that $\mathbf{K}-\mathbf{\Pi}$ is non-singular, because the multiplicity of the zero eigenvalue is one, than the fact that for $i \geq 1$, $\mathbf{K}^i\mathbf{\Pi} = \mathbf{0}$. The computation of $\int_0^b e^{\hat{\mathbf{K}}(b-x)}dx = \int_0^b e^{\hat{\mathbf{K}}x}dx$ follows a similar pattern. Using this closed form of the integral we have

$$\int_0^b \pi_+(x)dx\mathbb{1}_+ + \int_0^b \pi_-(x)dx\mathbb{1}_- =$$
$$= [\mathbf{v}_+ \mid \mathbf{v}_-]\times$$
$$\times \begin{array}{|c|c|}\hline \left(e^{\mathbf{K}b}\text{–}\mathbf{I}\text{–}b\mathbf{\Pi}\right)(\mathbf{K}\text{–}\mathbf{\Pi})^{-1} & \mathbf{0} \\ \hline \mathbf{0} & \left(e^{\hat{\mathbf{K}}b}\text{–}\mathbf{I}\text{–}b\hat{\mathbf{\Pi}}\right)(\hat{\mathbf{K}}\text{–}\hat{\mathbf{\Pi}})^{-1} \\ \hline \end{array} \times$$
$$\times \begin{array}{|c|c|}\hline \mathbf{I} & \mathbf{\Psi} \\ \hline \hat{\mathbf{\Psi}} & \mathbf{I} \\ \hline \end{array} \begin{bmatrix}\mathbb{1}_+ \\ \mathbb{1}_-\end{bmatrix}.$$

In the special case when $d = 0$ the normalizing equation simplifies a bit due to

$$\begin{array}{|c|c|}\hline \mathbf{I} & \mathbf{\Psi} \\ \hline \hat{\mathbf{\Psi}} & \mathbf{I} \\ \hline \end{array} \begin{bmatrix}\mathbb{1}_+ \\ \mathbb{1}_-\end{bmatrix} = 2\begin{bmatrix}\mathbb{1}_+ \\ \mathbb{1}_-\end{bmatrix}.$$

G. Enhanced algorithm

Both, the numerical method proposed in [9] (summarized in Section III-B) and the modified numerical method presented in Section III-E require the computation of $e^{\mathbf{K}b}$, $e^{\hat{\mathbf{K}}b}$ and $e^{\mathbf{U}b}$, $e^{\hat{\mathbf{U}}b}$, such that $e^{\mathbf{U}b}$ and $e^{\hat{\mathbf{U}}b}$ occur only in the linear system.

It is possible to rearrange the linear system of the modified algorithm, $(\mathbf{v}_+, \mathbf{v}_-)\mathbf{W} = (\mathbf{0}, \mathbf{0})$, such that $e^{\mathbf{U}b}$ and $e^{\hat{\mathbf{U}}b}$ disappears. The modified linear system is $(\mathbf{v}_+, \mathbf{v}_-)\mathbf{W}' = (\mathbf{0}, \mathbf{0})$ where

$$\mathbf{W}' = \begin{array}{|c|c|}\hline \mathbf{I} & e^{\mathbf{K}b}\mathbf{\Psi} \\ \hline e^{\hat{\mathbf{K}}b}\hat{\mathbf{\Psi}} & \mathbf{I} \\ \hline \end{array} + \begin{array}{|c|c|}\hline e^{\mathbf{K}b} & \mathbf{\Psi} \\ \hline \hat{\mathbf{\Psi}} & e^{\hat{\mathbf{K}}b} \\ \hline \end{array} + \begin{array}{|c|c|}\hline \mathbf{0} & \mathbf{Q}_{++}^{-1}\mathbf{Q}_{+-} \\ \hline \mathbf{Q}_{--}^{-1}\mathbf{Q}_{-+} & \mathbf{0} \\ \hline \end{array} .$$

This linear system cames from the relations of the fluid densities at the bottom and at the top of the fluid buffer, which can be obtained from (14) and (15). These relations are $\pi_+(0) = \pi_-(0)(-\mathbf{Q}_{--})^{-1}\mathbf{Q}_{-+}$ and $\pi_-(b) = \pi_+(b)(-\mathbf{Q}_{++})^{-1}\mathbf{Q}_{+-}$, which can be organized in the following matrix form

$$[\pi_+(0)|\pi_-(b)] = \\ = [\pi_+(b)|\pi_-(0)] \begin{array}{|c|c|}\hline \mathbf{0} & (-\mathbf{Q}_{++})^{-1}\mathbf{Q}_{+-} \\ \hline (-\mathbf{Q}_{--})^{-1}\mathbf{Q}_{-+} & \mathbf{0} \\ \hline \end{array} .$$

Substituting (25) and (26) into this matrix form results in the linear system with $\mathbf{W}'$. The enhanced modified method is the same as the one in Section III-E except that $\mathbf{W}'$ is used instead of $\mathbf{W}$. The numerical results reported in the next section are computed via the linear system $\mathbf{W}'$.

IV. Numerical experiment

We analyzed the numerical properties of the algorithms for finite buffer MFQs using our MATLAB implementations, which are parts of the BuTools package (available at http://webspn.hit.bme.hu/∼butools/). We compared the proposed procedure (Section III-E with the linear system from Section III-G), with the matrix analytic algorithm proposed in [9] (Section III-B) at two different drift values, one far from zero and one close to zero. The "grey" zone around the limit of saturation is handled with built in MATLAB functions like `null` using the standard floating point precision of MATLAB.

A. Comparison of methods when the drift is far from zero

First we evaluated the MFQ with buffer size $b = 30$, generator matrix

$$\mathbf{Q} = \begin{array}{|cc|ccc|}\hline -4 & 0 & 2 & 1 & 1 \\ 3 & -6 & 0 & 2 & 1 \\ \hline 1 & 3 & -5 & 1 & 0 \\ 3 & 1 & 1 & -7 & 2 \\ 1 & 1 & 0 & 1 & -3 \\ \hline \end{array}$$

such that the fluid rate is $+1$ in states $1, 2$ and -1 in states $3, 4, 5$. The stationary distribution of this background CTMC is $\gamma = (0.314, 0.142, 0.154, 0.143, 0.247)$ and the drift is $d = -0.00933$. To quantify the difference between the results of the methods we used the following error measure:

$$\Delta = \sum_{i\in S}\int_0^b |\pi_i^{orig}(x) - \pi_i^{new}(x)|dx + \\ + \sum_{i\in S} |p_i^{orig}(0) - p_i^{new}(0)| + \\ + \sum_{i\in S} |p_i^{orig}(b) - p_i^{new}(b)| \ ,$$

where $\pi_i^{orig}(x)$ and $\pi_i^{new}(x)$ correspond to the fluid density for state i at level x for the new and the original algorithm. $p_i^{orig}(0)$ and $p_i^{new}(0)$ are the probabilities for the empty buffer and $p_i^{orig}(b)$ and $p_i^{new}(b)$ are for the full buffer. For this MFQ we obtained

$$\mathbf{K} = \begin{array}{|cc|}\hline -2.514 & 1.687 \\ 4.656 & -4.598 \\ \hline \end{array} , \quad \hat{\mathbf{K}} = \begin{array}{|ccc|}\hline -3.982 & 2.244 & 0.876 \\ 2.298 & -5.934 & 2.857 \\ 1.156 & 2.029 & -2.197 \\ \hline \end{array}$$

for which $Mr(\mathbf{K}) = -0.567$, $Mr(\hat{\mathbf{K}}) = 8.8 \cdot 10^{-16}$ and $\gamma_+\mathbf{K} = (-0.128, -0.124)$, $\gamma_-\hat{\mathbf{K}} = (-5.6, 5.6, 0) \cdot 10^{-16}$. The fluid density curves computed by the two methods are depicted in Figure 1.

We also calculated the difference between the methods for systems with state space cardinalities of 20 and 50. The results were similar. The average error was $3.798 \cdot 10^{-5}$.

1) Comparison of the methods when the drift is close to zero: In our second example the buffer size is $b = 30$ the generator matrix is

$$\mathbf{Q} = \begin{array}{|cc|ccc|}\hline -5 & 3 & 1 & 0 & 1 \\ 5 & -8 & 0 & 2 & 1 \\ \hline 1 & 0 & -4 & 2 & 1 \\ 4 & 1 & 0 & -6 & 1 \\ 1 & 0 & 0 & 2 & -3 \\ \hline \end{array}$$

such that the fluid rate is $+1$ in states $1, 2$ and -1 in states $3, 4, 5$. The stationary distribution for this CTMC process is $\gamma = (0.349, 0.151, 0.087, 0.163, 0.250)$ and the drift is $d = -1.11 \cdot 10^{-16}$. The corresponding K an $\hat{K}$ matrices are the following:

$$\mathbf{K} = \begin{array}{|cc|}\hline -3.111 & 3.297 \\ 7.718 & -7.607 \\ \hline \end{array} , \quad \hat{\mathbf{K}} = \begin{array}{|ccc|}\hline -3.306 & 2.612 & 2.000 \\ 7.058 & -5.412 & 2.000 \\ 6.938 & 2.613 & -2.000 \\ \hline \end{array} .$$

The related eigenvalues and vectors are $Mr(\mathbf{K}) = 0$, $Mr(\hat{\mathbf{K}}) = -2.66 \cdot 10^{-15}$ and $\gamma_+\mathbf{K} = (0, 4.44) \cdot 10^{-16}$, $\gamma_-\hat{\mathbf{K}} = (-5.55, -2.22, -7.22) \cdot 10^{-16}$.

The fluid density curves are depicted in Figure 2. When the drift is close to zero the original procedure gets numerically instable as it is clearly visible on the figure.

V. Conclusions

The standard solution methods for Markovian queueing models with finite buffer are designed for the analysis of the cases when the drift is different from 0. Consequently, it is rather dangerous to apply them in general purpose tools where the drift can get very close to 0 where the results

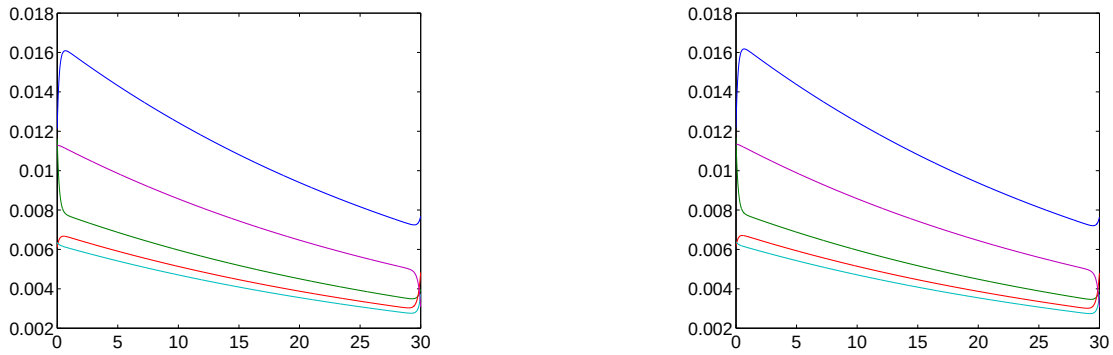

Fig. 1. The fluid density functions ($\pi_i(x)$ versus fluid level x) of the example with non-zero drift ($b = 30$, $d = -0.00933$, $\Delta = 5.3214e-007$). The left graph corresponds to the method proposed in [9], the right graph corresponds to the method proposed in this paper.

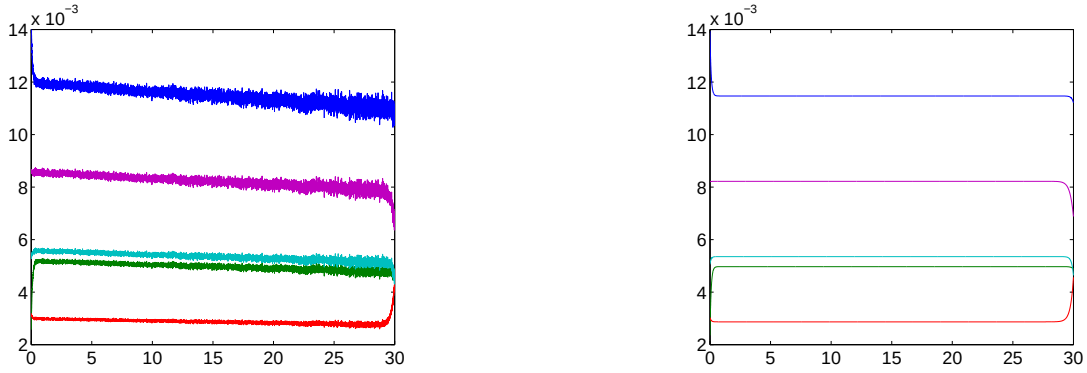

Fig. 2. The fluid density functions ($\pi_i(x)$ versus fluid level x) for a queue with zero drift. The left graph corresponds to the original method, the right to the new one. ($d = -1.11 \cdot 10^{-16}$, $b = 30$, $\Delta = 2.3 \cdot 10^{-2}$)

became numerically sensitive. In case of finite QBD models it is enough to replace the explicit solver of the characterizing linear system for the procedure which computes the spanning vectors of the zero space. In case of finite buffer MFQs a modified characterizing linear system needs to be solved using the procedure which computes the spanning vectors of the zero space. With these simple modifications of the computational methods the procedures become numerically stable at all drift values.

ACKNOWLEDGEMENTS

This work was supported by OTKA grant no. K-101150.

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