# Markov fluid models for energy and performance analysis *

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Traditional performance analysis methods are often based on discrete state continuous time Markov chains (CTMC). One of the main reasons of this choice is the relative simplicity of the analysis of CTMCs. Discrete state models are applicable when the considered system behavior can be represented with a finite or countable set.

More recently performance models with hybrid (discrete and continuous) state space gain attention in performance analysis. In these models the system parameters with continuous values can be considered as well.

We focus on the available analysis methods of Markov models with hybrid (discrete and continuous) state space and present application examples.

Keywords: Markov fluid model, stationary behaviour, differential equation.

## 1 Introduction

Markov fluid model [11] is an efficient tool to describe stochastic system behaviour in a wide range of application fields. Examples of its application can be found in risk process modeling [3], in operation and maintenance modeling [4] and in modeling various telecommunication systems including e. g. congestion control of high-speed networks [6], traffic shapers for an on-off source [1] and single-wavelength optical buffers [10].

In the basic version of Markov fluid models [11] the fluid rates are independent of the fluid level. The numerically stable solution of these models requires a kind of eigenvalue separation. Two main ways are proposed for doing that. A purely algebraic method is proposed in [15] and a different thread of papers propose methods based on the stochastic interpretation of the fluid level process [2], [13].

In a more general set of Markov fluid models the fluid rates, as well as the transition rates of the governing Markov chain, might depend on the fluid level but in a simple, piecewise constant way. The above mentioned solution methods are extended to the piecewise constant case in [9] and [14]. When the fluid rates and the transition rates are continuous, non-constant functions of the fluid level [5, 12] then the numerically stable methods based on eigenvalue separation are not applicable any more. A numerical solution method is proposed in [7] for the analysis of this case with the use of flux transition functions assuming that the fluid rate functions do not approach a predefined environment of zero.

The rest of the paper is organized as follows. Section 2 briefly summarizes the background. A demonstrative example is presented in Section 3 and the paper is concluded in Section 4.

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## 2 Description of the system

The $Z(t) = \{M(t), X(t); t \geq 0\}$ process represents the state of a fluid model with single fluid buffer, where $M(t) \in \mathscr{S}$ is the (discrete) state of the environment process and $X(t) \in [0, B]$ is the fluid level of the fluid buffer at time $t$. $\mathscr{S}$ denotes the finite set of states of the environment and $B$ denotes the maximum fluid level. The fluid level distribution might have probability masses at particular fluid levels and it is continuous between these levels. We define $\hat{\pi}_j(t,x)$ and $\hat{c}_j(t,x)$ to describe the transient fluid densities and the transient probability masses of the fluid distribution as follows

$$\hat{\pi}_j(t,x) = \lim_{\Delta \to 0} \frac{Pr(M(t) = j, x \leq X(t) < x + \Delta)}{\Delta} \ , \ \ \hat{c}_j(t,x) = Pr(M(t) = j, X(t) = x) \ .$$

On the continuous intervals of the fluid level distribution, row vector $\hat{\pi}(t,x) = \{\hat{\pi}_j(t,x)\}$ satisfies (see [8])

$$\frac{\partial}{\partial t}\hat{\pi}(t,x) + \frac{\partial}{\partial x}\left(\hat{\pi}(t,x)\mathbf{R}(x)\right) = \hat{\pi}(t,x)\mathbf{Q}(x) \ , \tag{1}$$

with initial condition $\hat{\pi}(0,x)$. At the boundaries the row vector $\hat{c}(t,x) = \{\hat{c}_j(t,x)\}$ satisfies

$$\frac{d}{dt}\hat{c}_j(t,0) = -\hat{\pi}_j(t,0)R_j(0) + \sum_k \hat{c}_k(t,0)Q_{kj}(0) \ ,$$

and

$$\frac{d}{dt}\hat{c}_j(t,B) = \hat{\pi}_j(t,B)R_j(B) + \sum_k \hat{c}_k(t,B)Q_{kj}(B) \ ,$$

with initial condition $\hat{c}(0,0)$ and $\hat{c}(0,B)$. Based on the stochastic interpretation of $\hat{c}_j(0,0)$ and $\hat{c}_j(0,B)$, we have the following important properties

$$\hat{c}_j(t,0) = 0 \ , \text{ if } R_j(0) > 0 \ , \quad \hat{c}_j(t,B) = 0 \ , \text{ if } R_j(B) < 0$$

for $t > 0$. When the fluid rate increases ($R_j(0) > 0$) the probability that the fluid buffer is empty is zero and similarly when the fluid rate decreases ($R_j(B) < 0$) the probability that the fluid buffer is full is zero.

Assuming that the system converges to a unique stationary solution, the stationary fluid density function and fluid mass function are $\pi_j(x) = \lim_{t \to \infty} \hat{\pi}_j(t,x)$ and $c_j(x) = \lim_{t \to \infty} \hat{c}_j(t,x)$. On the continuous intervals of the fluid level distribution, row vector $\pi(x) = \{\pi_j(x)\}$, satisfies (see [8])

$$\frac{d}{dx}\left(\pi(x)\mathbf{R}(x)\right) = \pi(x)\mathbf{Q}(x) \ , \tag{2}$$

where matrix $\mathbf{Q}(x) = \{Q_{ij}(x)\}$ is the transition rate matrix of the environment process when the fluid level is $x$, and the diagonal matrix $\mathbf{R}(x) = \text{diag}\langle R_j(x) \rangle$ is composed by the fluid rates $R_j(x)$, $j \in \mathscr{S}$. The fluid rate determines the rate at which the fluid level changes when the environment is in state $j$ and the fluid level is $x$, i.e., $\frac{d}{dt}X(t) = R_j(x)$ when $X(t) = x$ and $M(t) = j$ and the transition rate matrix determines the rate at which discrete state transitions occur, i.e., $Q_{ij}(x) = \lim_{\Delta \searrow 0} Pr(M(t+\Delta) = j | M(t) = i, X(t) = x)/\Delta$ for $i \neq j$ and $Q_{ii}(x) = -\sum_{j \in \mathscr{S}, j \neq i} Q_{ij}(x)$, where $\searrow$ indicates that $\Delta$ converges to 0 from the right.

The stationary solution of the fluid model is characterized by the ordinary differential equation (ODE) (2). The main difficulty of finding the stationary solution is to find an appropriate set of boundary conditions for the ODE based on the stochastic behaviour of the fluid model. The boundary conditions at fluid level 0 and $B$ are [11, 7]:

$$-\pi_j(0)R_j(0)+\sum_k c_k(0)Q_{kj}(0)=0\ ,$$

$$\pi_j(B)R_j(B)+\sum_k c_k(B)Q_{kj}(B)=0\ ,$$

$$c_j(0)=0\ ,\ \text{if}\ R_j(0)>0\ ,\quad c_j(B)=0\ ,\ \text{if}\ R_j(B)<0\ .$$

The transient and stationary solution of this model is based on the solution of the set of partial differential equations (PDEs) and ordinary differential equations (ODEs), respectively.

## 3 Demonstrative example

We consider a battery supplied network element, which communicates with other elements through radio channels. The power consumptions of the network element is a function of its communication activity, more parallel radio communications results in higher power consumptions, but the relation is sub-linear. One of the simples load model of this network element is when communication request arrive according to a Poisson process and the length of the communications is exponentially distributed. The network element can serve at most $m$ communication requests at a time and new request are dropped when there are $m$ ongoing communications. Borrowing the standard queueing notation, the load of the network element is according to an M/M/m/m queue.

Having this simple model of the network element several interesting energy and performance parameters can be analyzed in an accurate qualitative level. First, we assume that the network element is set up for operation with a fully charged battery and it is not recharged during the operation. In this case a primary measure of interest is the distribution of the operation time, but several related design questions rise after this. Obviously, lower communication load of the network element results in longer operational time, but due to the sub-linear power consumption the energy utilization (performed radio communication per consumed energy) is low when the load is low. A potential related measure of interest is the optimal load for energy utilization. It is still straight forward from the sub-linear power consumption that larger load results in better specific energy utilization, but when the network level overall transmitted data is considered during the operation period then the combined effect of power consumption and request dropping has to evaluated.

Several similar performance and energy related parameters can be considered then the battery is occasionally recharged and we look for long run behavior.

## 4 Conclusions

Markov fluid models allow to describe a wide range of practically interesting system behavior an associated performance parameters. We briefly summarized the characterizing equations of these models. There are well established solution methods which are not considered in this extended abstract. The presented demonstrative example intends to shed light on the potential practical applications.

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