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The two-matrix problem

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1 Introduction

There are two real-valued square matrices $\mathbf{G}_0$ *and* $\mathbf{G}_1$. *Do they describe a valid rational arrival process (RAP) with row vector* ν*?*

That is,

$$\nu e^{\mathbf{G}_0 t_1} \mathbf{G}_1 e^{\mathbf{G}_0 t_2} \mathbf{G}_1 \dots e^{\mathbf{G}_0 t_k} \mathbf{G}_1 \mathbb{1} \geq 0 \tag{1}$$

for all $k \geq 1$ *and all* $t_1, t_2, \dots, t_k \in \mathbb{R}^+$*, where* $\mathbb{1}$ *is the column vector of ones.*

Continuous-time Markov arrival processes (MAPs) are efficiently used to model point processes with dependent interarrival times [11]. Any interarrival time of a MAP is phase-type (PH) distributed [10]. A non-Markovian generalization of MAPs is the rational arrival processes (RAPs) [2], whose interarrival time is matrix exponentially (ME) distributed [1].

The advantage of using Markovian stochastic models, like PH distribution and MAP, is in their simple stochastic interpretation via an underlying continuous-time Markov chain (CTMC), which modulates the terminating event of the PH distribution and the arrival event of the MAP. The advantage of using non-Markovian models, like RAP and ME distribution, is that they describe a broader class of stochastic models [4, 6, 3, 8].

The relation of PH and ME distributions has been investigated for a long time [12, 7, 9, 13], while the relation of MAPs and RAPs is much less explored. The most important open problems are

- the validity of RAP models and
- the minimal Markovian representation of valid RAP models.

This note is devoted to the first problem.

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2 Problem statement

Definition 1 A *ME distribution* [1] is a distribution on $\mathbb{R}^+$ such that its density function is a matrix exponential function of the parameter

$$f_{(\nu,\mathbf{G}_0)}(t) = -\nu\, e^{\mathbf{G}_0 t}\, \mathbf{G}_0 \mathbb{1}, \tag{2}$$

where ν is a row vector and $\mathbf{G}_0$ is a square matrix of size $m < \infty$.

The $(\nu, \mathbf{G}_0)$ pair defines a ME exponential distribution, if and only if $f_{(\nu,\mathbf{G}_0)}(t) \geq 0$ for all $t \in \mathbb{R}^+$, which we refer to as ME non-negativity criteria, and $\int_0^\infty f_{(\nu,\mathbf{G}_0)}(t)dt \leq 1$.

Definition 2 A $(\nu, \mathbf{G}_0)$ pair is *Markovian*, if $\nu \geq \mathbf{0}$, the diagonal elements of $\mathbf{G}_0$ are negative, the rest of its elements are non-negative and $\mathbf{G}_0 \mathbb{1} \leq \mathbf{0}$.

Any Markovian $(\nu, \mathbf{G}_0)$ pair satisfies the ME non-negativity criteria.

Definition 3 A *PH distribution* is a ME distribution which has a finite Markovian representation.

Definition 4 A *RAP* [2] is a point process whose joint density of consecutive interarrival times is a matrix exponential function of the variables

$$f_{(\nu,\mathbf{G}_0,\mathbf{G}_1)}(t_1,\ldots,t_k) = \nu e^{\mathbf{G}_0 t_1} \mathbf{G}_1 e^{\mathbf{G}_0 t_2} \mathbf{G}_1 \ldots e^{\mathbf{G}_0 t_k} \mathbf{G}_1 \mathbb{1}, \tag{3}$$

where ν is a row vector and $\mathbf{G}_0$ and $\mathbf{G}_1$ are square matrices of size $m < \infty$.

The triple $(\nu, \mathbf{G}_0, \mathbf{G}_1)$ represents a RAP if and only if $f_{(\nu,\mathbf{G}_0,\mathbf{G}_1)}(t_1,\ldots,t_k) \geq 0$ for all $k \geq 1$ and $t_1, t_2, \ldots, t_k \in \mathbb{R}^+$, which we refer to as RAP non-negativity criteria, and $\int_0^\infty \cdots \int_0^\infty f_{(\nu,\mathbf{G}_0,\mathbf{G}_1)}(t_1,\ldots,t_k)dt_k \ldots dt_1 \leq 1$.

Definition 5 The $(\nu, \mathbf{G}_0, \mathbf{G}_1)$ triple is *Markovian*, if $\nu \geq \mathbf{0}$, $\mathbf{G}_1 \geq \mathbf{0}$, the diagonal elements of $\mathbf{G}_0$ are negative, the rest of its elements are non-negative and $\mathbf{G}_0 \mathbb{1} \leq \mathbf{0}$.

Any Markovian $(\nu, \mathbf{G}_0, \mathbf{G}_1)$ triple satisfies the RAP non-negativity criteria.

Definition 6 A *MAP* is a RAP which has a Markovian representation.

There are infinitely many different vector-matrix pairs representing a ME distribution and infinitely many different vector-matrix-matrix triples representing a RAP [5]. If $(\nu, \mathbf{G}_0, \mathbf{G}_1)$ of size m, $(\phi, \mathbf{C}_0, \mathbf{C}_1)$ of size n and matrix $\mathbf{W}$ of size $n \times m$ are such that $\nu = \phi \mathbf{W}$, $\mathbf{C}_0 \mathbf{W} = \mathbf{W} \mathbf{G}_0$, $\mathbf{C}_1 \mathbf{W} = \mathbf{W} \mathbf{G}_1$, $\mathbb{1}_n = \mathbf{W} \mathbb{1}_m$, then $(\nu, \mathbf{G}_0, \mathbf{G}_1)$ and $(\phi, \mathbf{C}_0, \mathbf{C}_1)$ represent the same RAP, because

$$\begin{aligned}
f_{(\nu,\mathbf{G}_0,\mathbf{G}_1)}(t_1,\ldots,t_k) &= \nu e^{\mathbf{G}_0 t_1} \mathbf{G}_1 e^{\mathbf{G}_0 t_2} \mathbf{G}_1 \ldots e^{\mathbf{G}_0 t_k} \mathbf{G}_1 \mathbb{1} = \phi \mathbf{W} e^{\mathbf{G}_0 t_1} \mathbf{G}_1 e^{\mathbf{G}_0 t_2} \mathbf{G}_1 \ldots e^{\mathbf{G}_0 t_k} \mathbf{G}_1 \mathbb{1} \\
&= \phi e^{\mathbf{C}_0 t_1} \mathbf{W} \mathbf{G}_1 e^{\mathbf{G}_0 t_2} \mathbf{G}_1 \ldots e^{\mathbf{G}_0 t_k} \mathbf{G}_1 \mathbb{1} = \phi e^{\mathbf{C}_0 t_1} \mathbf{C}_1 \mathbf{W} e^{\mathbf{G}_0 t_2} \mathbf{G}_1 \ldots e^{\mathbf{G}_0 t_k} \mathbf{G}_1 \mathbb{1} \\
&= \ldots = \phi e^{\mathbf{C}_0 t_1} \mathbf{C}_1 e^{\mathbf{C}_0 t_2} \mathbf{C}_1 \ldots e^{\mathbf{C}_0 t_k} \mathbf{C}_1 \mathbb{1} = f_{(\phi,\mathbf{C}_0,\mathbf{C}_1)}(t_1,\ldots,t_k).
\end{aligned}$$

This transformation can be used to obtain an equivalent Markovian representation $(\phi, \mathbf{C}_0, \mathbf{C}_1)$ of the RAP, which is defined by a non-Markovian representation $(\nu, \mathbf{G}_0, \mathbf{G}_1)$.

3 Discussion

To check the ME non-negativity criteria is a difficult task already. There are many necessary conditions for $(\nu, \mathbf{G}_0)$ (e.g., the eigenvalues of $\mathbf{G}_0$ have negative real part, there is a real eigenvalue among the eigenvalues with maximal real part, etc.), but sufficient conditions are difficult to find. For a subset of ME distributions, the characterization theorem of O'Cinneide [12] provides a sufficient condition, which proves that any ME distribution with strictly positive density function in $(0,\infty)$ and with unique real eigenvalue with maximal real part has a finite-dimensional Markovian representation. Additionally, [9] recommends a method for constructing such Markovian representation for ME distributions satisfying the conditions of O'Cinneide's characterization theorem.

Consequently, to check if $(\nu, \mathbf{G}_0)$ defines a ME distribution one needs to apply the numerical procedure to transform $(\nu, \mathbf{G}_0)$ into a Markovian representation according to [13] and if the procedure succeeds then the answer is positive.

In spite of the related efforts, the counterparts of these results, which we summarize as a conjecture and a challenge, are not available for RAPs.

Conjecture 1 *Any RAP with strictly positive joint density function for* $t_1, \ldots, t_k \in (0,\infty)$ *and* $\mathbf{G}_0$ *with unique real eigenvalue with maximal real part has a finite-dimensional Markovian representation.*

Challenge 1 *Develop a procedure for transforming* $(\nu, \mathbf{G}_0, \mathbf{G}_1)$ *into a Markovian representation if the conditions of Conjecture 1 hold.*

A possible way to transform ν and $\mathbf{G}_0$ into a potentially larger Markovian representation could be the same as in [9]. Following this approach, our efforts in Challenge 1 failed because we could not find the associated Markovian representation of $\mathbf{G}_1$.

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