Performance Analysis of Markov Regenerative Reward Models ⋆

Miklós Telek, András Pfening

Department of Telecommunications, Technical University of Budapest, 1521 Budapest, Hungary; {telek,pfening}@hit.bme.hu

Abstract

The problem considered in the paper can not be solved by the traditional technique available for the analysis of Markov Regenerative Processes (*MRGP*). The widely used description of *MRGP*s, i.e. by the local and the global kernels, do not contain sufficient information on the process to evaluate the distribution of reward measures. A new analytical approach is proposed and studied to utilize better the Markov regenerative property for the analysis of Markov Regenerative Reward Models (*MRRM*). The distribution of the accumulated reward and the completion time of *MRRM*s is provided in transform domain, as well as considerations about the time domain numerical evaluation of these measures. As a simple application example the performance of a finite queueing system is analyzed.

Key words: Stochastic reward models, Markov regenerative processes, accumulated reward, task completion time.

1 Introduction

The modeling framework applied in the analysis of complex computer/communication systems depends on the behaviour of the analyzed system and the aim of the analysis. The most frequently applied stochastic modeling technique is the Markovian approach, which is based on the memoryless (Markov) property of the system behaviour. Nevertheless, this property and its consequence, the exponentially distributed event times in case of time homogeneous system behaviour, have been recognized as one of the main restrictions in the application of Markovian models [6]. An alternative non-Markovian modeling

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approach is based on the Markov renewal theory [5] and therefore the application of *MRGPs* in stochastic modeling received an increasing attention recently [15,7]. The automated generation of such models by non-Markovian Stochastic Petri Nets [4] as well as new results on their transient [2,8] and steady state [1,20] analysis increase the applicability of this modeling framework.

A more detailed insight of stochastic models can be obtained by associating a so called reward variable to the analyzed stochastic process which changes its value according to the actual state of the stochastic process (rate reward) and/or the state transition takes place (impulse reward or reward loss). A wide range of practically important performance measures, such as the computational performance of multimode computers [18], throughput [17] etc., can be evaluated based on the associated reward variable. Another well known application of stochastic reward models is termed as *performability analysis* [16], which aims to capture the performance delivered by the system in the presence of failures.

The former studied stochastic reward models are based on underlying Continuous Time Markov Chains (*CTMC*) or Semi-Markov Processes (*SMP*), and various techniques have been published for the evaluation of the accumulated reward, the completion time and other related reward measures of these models [17,12,13].

The terminology *MRRM* comes from Logothetis, who was the first to consider stochastic reward models with underlying *MRGP*s [14]. He provided the analysis of the mean accumulated reward up to time t, the mean instantaneous reward and its limiting behaviour [14,15]. The main limitation of these works comes from the fact that they are based on the widely used kernels representation of *MRGP*s.

The global (usually denoted as $\mathbf{K}(t)$) and the local ($\mathbf{E}(t)$) kernels of *MRGP*s do not define the stochastic process in the sense that they do not define all the finite dimensional joint distributions of the process. They give a limited description of *MRGP*s that can be used

- to define the embedded Markov renewal sequence of the regeneration epochs (in the above mentioned "complete" sense) and
- to evaluate the transient state probabilities.

All the reward measures considered in [14,15] are based on the transient state probabilities of the *MRGP*. Reward measures which require a more detailed knowledge on the process, such as the higher moments of the accumulated reward or the mean of the completion time, can not be evaluated based on the global and the local kernels of *MRGP*s.

The majority of the former literature on *MRGP*s follows the approach summarized in [6]:

".. solving problems using Markov renewal theory is a two step process:

- *First, we need to construct both kernel matrices* $\mathbf{K}(t)$ *and* $\mathbf{E}(t)$.
- *We then solve one set of Volterra integral equations for the conditional transition probabilities or for some measures of interest."*

The first step, of course, should be based on the "complete" knowledge of the evolution role of the process and it results in a dense description which can be used for the transient analysis, but which does not contain the "complete" description of the process any more. This way we loose information about the process at the first step. The approach we propose in this paper is similar, but instead of the local and the global kernels we introduce a more proper pair of "kernels" (referred to as *reward kernels*), that contain all the required information for the purposes of the analysis of reward measures.

A different approach appears in [9], where instead of the Markov renewal theory the analysis of the considered class of *MRRM*s is based on the application of supplementary variables.

The rest of the paper is organized as follows. Section 2 introduces the traditional theory of *MRGP*s. In Section 3 the reward kernels are defined and the main result of the paper is provided. Some ways of evaluation of reward measures by the proposed approach are discussed in Section 4. In Section 5 the analysis of the reward kernels are provided for some special subordinated processes. A simple example is evaluated in Section 6. We encourage the reader to have a look to the studied example ahead for better understanding the sections of formal analysis. The paper is concluded in Section 7.

2 Introduction of *MRGP*s

For a detailed study of *MRGP*s we recommend [11], here we only briefly summarize the main results.

A continuous time stochastic process $(Z(t))$ enjoys the *Markov property* (or Markov regenerative property) at time T if for any $0 < t_1 < t_2 < \ldots < t_k$ and $x_1, x_2, \ldots, x_k$:

$$
\begin{aligned}
&P\{Z(T+t_1) \le x_1, Z(T+t_2) \le x_2, \ldots, Z(T+t_k) \le x_k \mid \\
&\qquad Z(T), Z(\tau), 0 \le \tau < T\} = \\
&P\{Z(T+t_1) \le x_1, Z(T+t_2) \le x_2, \ldots, Z(T+t_k) \le x_k \mid Z(T)\}
\end{aligned}
$$

The Markov regenerative process $\{Z(t) \in \Omega,\ t \geq 0\}$ does not have the Markov property in general, but there is a sequence of embedded (random) time points $(T_0 = 0, T_1, ..., T_n, ...)$ such that the process at these time points satisfies the Markov property. These time points are the Markov regeneration epochs.

The transient state probability analysis of time homogeneous *MRGP*s is usually based on the following conditional probabilities:

$$\begin{aligned} K_{ij}(t) &= \Pr\{Z(T_1) = j,\ T_1 \leq t \mid Z(0) = i\}, \\ E_{ij}(t) &= \Pr\{Z(t) = j, T_1 > t \mid Z(0) = i\}. \end{aligned} \tag{1}$$

The matrix $\mathbf{K}(t) = [K_{ij}(t)]$ is termed the **global kernel** [4] and is the joint conditional probability of the time to the next Markov regeneration and the state right after the next Markov regeneration given the state at the current Markov regeneration. The matrix $\mathbf{E}(t) = [E_{ij}(t)]$ (called the **local kernel**) describes the state transition probabilities of the *MRGP* between two consecutive Markov regeneration epochs. The matrices $\mathbf{K}(t)$ and $\mathbf{E}(t)$ can be used in computing the transient probability: $V_{ij}(t) = P\{Z(t) = j \mid Z(0) = i\}$.

Let $\mathbf{K}*\mathbf{V}(t)$ denote a matrix whose (i,j)th element is

$$[\mathbf{K}*\mathbf{V}(t)]_{ij} = \sum_u \int_0^t dK_{iu}(x) V_{uj}(t-x).$$

Then the matrix of transient probabilities $\mathbf{V}(t) = [V_{ij}(t)]$ satisfies the Markov renewal equation [5]: $\mathbf{V}(t) = \mathbf{E}(t) + \mathbf{K}*\mathbf{V}(t)$. $\mathbf{V}(t)$ can be expressed in closed form in Laplace-Stieltjes (LST) transform domain:
$\mathbf{V}^{\sim}(s) = [\mathbf{I} - \mathbf{K}^{\sim}(s)]^{-1}\, \mathbf{E}^{\sim}(s)$.

Note that several other simple measures of *MRGP*s, such as the sojourn time in a state or in a group of states etc., can not be evaluated based on $\mathbf{E}(t)$ and $\mathbf{K}(t)$, since the evolution of the process between the consecutive Markov regeneration epochs is not "completely" defined, only the transient state probabilities are described by $\mathbf{E}(t)$. The Markov regenerative property of the state transition probabilities is utilized in the above mentioned results. In the following section we propose an analysis approach based on the Markov regenerative property of the studied reward measures.

3 Analysis of *MRRM*s

A reward rate (r_i) is assigned to each state and an impulse reward (ϱ_{ij}) to each pair of states of an *MRGP* ($Z(t) \in \Omega$). The r.v. of the accumulated reward up to time t is defined as

$$B(t) = \int_{\tau=0}^{t} r_{Z(\tau)} d\tau + \sum_i \sum_j N_{ij}(t) \varrho_{ij} .$$

where $N_{ij}(t)$ is the number of state transitions from state i to state j up to time t [1].

To utilize the Markov regenerative property of the analyzed reward measures we define the following random variables:

$$\begin{aligned} \mathcal{R}_{ij}(t) &= \{B(t) \mid Z(t) = j,\ Z(0) = i\}, \\ \mathcal{G}_{ij}(t) &= \{B(t) \mid Z(t) = j,\ Z(0) = i,\ T_1 > t\}, \\ \mathcal{S}_{ij}(t) &= \{B(t) \mid Z(t) = j,\ Z(0) = i,\ T_1 = t\}. \end{aligned}$$

- $\mathcal{R}_{ij}(t)$ is the accumulated reward given that the process started in state i and it stays in state j at time t.
- $\mathcal{G}_{ij}(t)$ is the accumulated reward supposed that the process started in state i, it stays in state j at time t and t is inside the first regeneration period.
- $\mathcal{S}_{ij}(t)$ is the accumulated reward supposed that the process started in state i, it stays in state j at time t and t is the first regeneration instance.

Furthermore we define the local and the global *reward kernels*, respectively:

$$\begin{aligned} G_{ij}(t, w) &= \Pr\{B(t) \le w,\ Z(t) = j,\ T_1 > t \mid Z(0) = i\}, \\ S_{ij}(t, w) &= \Pr\{B(T_1) \le w,\ Z(T_1) = j,\ T_1 \le t \mid Z(0) = i\}, \end{aligned}$$

and the state dependent distribution of the accumulated reward:

$$R_{ij}(t, w) = \Pr\{B(t) \le w,\ Z(t) = j \mid Z(0) = i\} .$$

The matrices composed by these elements are denoted as $\mathbf{R}(t, w) = [R_{ij}(t, w)]$, $\mathbf{G}(t, w) = [G_{ij}(t, w)]$ and $\mathbf{S}(t, w) = [S_{ij}(t, w)]$. The main result of this paper, the relation of these quantities, is provided by the following theorem.

[1] The framework proposed in this section is general enough to evaluate models with rate and impulse rewards as well, but in the subsequent analysis of the subordinated processes we focus on rate reward models.

Theorem 1 *The distribution of the accumulated reward of an* MRGP *is characterized by the following double LST domain equation:*

$$\mathbf{R}^{\sim\sim}(s,v) = [\mathbf{I} - \mathbf{S}^{\sim\sim}(s,v)]^{-1} \ \mathbf{G}^{\sim\sim}(s,v) \tag{2}$$

Proof: Conditioning on the occurrence of the first regeneration instance (T_1) and unconditioning based on its distribution ($K_i(t) = \sum_{j\in\Omega} K_{ij}(t)$) we have [2]:

$$\mathcal{R}_{ij}(t) = E_{ij}(t)\ \mathcal{G}_{ij}(t) \ + \ \sum_k \int_{\tau=0}^{t} \mathcal{S}_{ik}(\tau) + \mathcal{R}_{kj}(t-\tau)\ dK_{ik}(\tau),$$

from which

$$R_{ij}(t,w) = G_{ij}(t,w) \ + \ \sum_k \int_{\tau=0}^{t} \int_{\alpha=0}^{w} R_{kj}(t-\tau, w-\alpha)\ dS1_{ik}(\tau,\alpha)\ dK_{ik}(\tau),$$

where $S1_{ij}(t,w) = \Pr\{B(T_1) \le w \mid Z(T_1) = j,\ T_1 = t,\ Z(0) = i\}$.

An LST with respect to w, denoting the transform variable by v, results in:

$$R_{ij}^{\sim}(t,v) = G_{ij}^{\sim}(t,v) + \sum_k \int_{\tau=0}^{t} R_{kj}^{\sim}(t-\tau,v)\ S1_{ik}^{\sim}(\tau,v)\ dK_{ik}(\tau) =$$
$$G_{ij}^{\sim}(t,v) + \sum_k \int_{\tau=0}^{t} R_{kj}^{\sim}(t-\tau,v)\ dS_{ik}^{\sim}(\tau,v).$$

A second LST with respect to t, denoting the transform variable by s, results in

$$R_{ij}^{\sim\sim}(s,v) = G_{ij}^{\sim\sim}(s,v) + \sum_k S_{ik}^{\sim\sim}(s,v)\ R_{kj}^{\sim\sim}(s,v). \tag{3}$$

In matrix form $\mathbf{R}^{\sim\sim}(s,v) = \mathbf{G}^{\sim\sim}(s,v) + \mathbf{S}^{\sim\sim}(s,v)\ \mathbf{R}^{\sim\sim}(s,v)$ from which the theorem follows. □

[2] The proof is based on the Markov Renewal Theory, i.e. it is similar to the one applied for reward analysis of *CTMC*s and *SMP*s (see for example [17,12,13]), but in this case the stochastic process can experience state transitions up to T_1 which makes our analysis problem rather complex.

4 Evaluation of reward measures based on $\mathbf{R}^{\sim\sim}(s,v)$

4.1 Accumulated reward

The distribution of the accumulated reward is given by:

$$\begin{aligned} B(t,w) &= \Pr\{B(t) \le w\} = \textstyle\sum_i \sum_j P_i(0) R_{ij}(t,w) = \underline{P}^T(0)\ \mathbf{R}(t,w)\ \underline{h} \\ &= LST^{-1}_{s\to t}\ LST^{-1}_{v\to w}\ \underline{P}^T(0)\ \mathbf{R}^{\sim\sim}(s,v)\ \underline{h} \end{aligned} \tag{4}$$

where $\underline{P}^T(0) = \{P_i(0)\}$ is the row vector of the initial state probabilities and $\underline{h} = \{1\}$ is the column vector with all the entries equal to 1.

For the numerical evaluation of the distribution of the accumulated reward based on (4) two inverse transformations are necessary according to the time $(s \to t)$ and the reward variables $(v \to w)$. As it can be seen in the subsequent numerical example a symbolic inverse transformation can be very hard even for a simple model.

Instead, the evaluation of the moments of the accumulated reward at time t is based on a single inverse transformation according to the time variable $(s \to t)$ by applying the following equation:

$$\mathbf{E}\Big[B(t)^k\Big] = LST^{-1}_{s\to t}\left\{ (-1)^k\ \frac{\partial^k}{\partial\ v^k}\ \underline{P}^T(0)\ \mathbf{R}^{\sim\sim}(s,v)\ \underline{h}\bigg|_{v\to 0} \right\} \tag{5}$$

A symbolic evaluation of the k-th derivative of $\underline{P}^T(0)\ \mathbf{R}^{\sim\sim}(s,v)\ \underline{h}$ and a numerical inverse transformation of the result can be performed in a reasonable respond time.

4.2 Completion time

The completion time C is a random variable representing the time to accumulate a reward requirement equal to a random variable W [13]:

$$C\ =\ \min\ [t \ge 0\ :\ B(t)\ =\ W]\ .$$

C is the time at which the work accumulated by the system reaches the value W for the first time. We assume, in general, that W is a random variable with distribution $W(w)$ with support on $(0,\ \infty)$. The degenerate case, in which

W is deterministic and the distribution $W(w)$ becomes the unit step function $U(w - w_d)$, can be considered as well.

For a given sample of the reward requirement $W = w$, the completion time $C(w)$ is defined as: $C(w) = \min\left[t \geq 0 : B(t) = w\right]$. Let $C(t,w)$ be the *cdf* of the completion time when the reward requirement is w: $C(t,w) = Pr\{C(w) \leq t\}$. The completion time ($C$) of the random reward requirement W is characterized by the following distribution:

$$\widehat{C}(t) = Pr\{C \leq t\} = \int_0^\infty C(t,w)\, dW(w) \tag{6}$$

The distribution of the completion time is closely related to the distribution of the accumulated reward by the following relation:

$$B(t,w) = Pr\{B(t) \leq w\} = Pr\{C(w) \geq t\} = 1 - C(t,w) \tag{7}$$

from which

$$C^{\sim\sim}(s,v) = 1 - B^{\sim\sim}(s,v) = 1 - \underline{P}^T(0)\, \mathbf{R}^{\sim\sim}(s,v)\, \underline{h} \tag{8}$$

The kth moments of the completion time of the reward requirement w can be evaluated as follows:

$$\mathbf{E}\left[C(w)^k\right] = LST^{-1}_{v \to w}\left\{ (-1)^{k+1} \frac{\partial^k}{\partial\, s^k} \underline{P}^T(0)\, \mathbf{R}^{\sim\sim}(s,v)\, \underline{h}\Big|_{s \to 0} \right\} \tag{9}$$

When the reward requirement is a phase type (*PH*) random variable the moments of the completion time can be evaluated by applying the results of Theorem 1 in [19]. As a simple example we consider the case when the reward requirement W is an exponentially distributed r.v. with parameter μ. In that case

$$\mathbf{E}\left[C^k\right] = \mu \quad (-1)^{k+1} \frac{\partial^k}{\partial\, s^k} \underline{P}^T(0)\, \mathbf{R}^{\sim\sim}(s,\mu)\, \underline{h}\Big|_{s \to 0} \tag{10}$$

5 Analysis of subordinated processes

*MRRM*s can be analyzed based on Theorem 1, when the reward kernels are known. This section provides results for $\mathbf{S}^{\sim\sim}(s,v)$ and $\mathbf{G}^{\sim\sim}(s,v)$ in case of some simple subordinated processes and rate reward accumulation.

Consider a subordinated *SMP* with state space Ω, kernel $\mathbf{Q}(t)$ and reward rates $r_i, i \in \Omega$. The regenerative period is concluded by the expiration of the random delay θ which is distributed according to $T(\tau)$ (independent of the subordinated process). At the end of the regeneration period a state transition from state i to state j can take place with probability Δ_{ij}. $\Delta = \{\Delta_{ij}\}$ is called the branching probability matrix.

To analyze an *MRRM* with this kind of subordinated processes one has to evaluate $\mathbf{S}^{\sim\sim}(s,v)$ and $\mathbf{G}^{\sim\sim}(s,v)$. Since they depend on τ we introduce

$$G_{ij}(t,w,\tau) = \Pr\{B(t) \leq w,\ Z(t) = j,\ T_1 > t \mid Z(0) = i, \theta = \tau\}, \quad (11)$$
$$S_{ij}(t,w,\tau) = \Pr\{B(T_1) \leq w,\ Z(T_1) = j,\ T_1 \leq t \mid Z(0) = i, \theta = \tau\}, \quad (12)$$

from which $S_{ij}^{\sim\sim}(s,v)$ and $G_{ij}^{\sim\sim}(s,v)$ can be obtained as:

$$G_{ij}(t,w) = \int_{\tau=0}^{\infty} G_{ij}(t,w,\tau)\ dT(\tau), \qquad S_{ij}(t,w) = \int_{\tau=0}^{\infty} S_{ij}(t,w,\tau)\ dT(\tau).$$

Theorem 2 *The distribution of the accumulated reward of a complete regenerative period, $S_{ij}(t,w,\tau)$ satisfies the following transform domain equation:*

$$\begin{aligned} &S_{ij}^{\sim **}(s,v,\chi) = \\ &\quad \Delta_{ij}\ \tfrac{1-Q_i^{\sim}(s+r_iv+\chi)}{v(s+r_iv+\chi)} + \textstyle\sum_{k\in\Omega} Q_{ik}^{\sim}(s+r_iv+\chi)\ S_{kj}^{\sim **}(s,v,\chi) \end{aligned} \quad (13)$$

where $*$ denotes the Laplace transform (LT).

Proof: Conditioning on the sojourn time h in state i we have:

$$S_{ij}(t,w,\tau \mid h) = \begin{cases} 0 & \text{if } r_i\tau > w \text{ and } h \geq \tau \\ \Delta_{ij}\ U(t-\tau) & \text{if } r_i\tau \leq w \text{ and } h \geq \tau \\ 0 & \text{if } r_ih > w \text{ and } h < \tau \\ \displaystyle\sum_{k\in\Omega} \frac{dQ_{ik}(h)}{dQ_i(h)}\ S_{kj}(t-h, w-r_ih, \tau-h) & \text{if } r_ih \leq w \text{ and } h < \tau \end{cases} \quad (14)$$

where $U(.)$ denotes the unit step function.

In Equation (14) the condition $h \geq \tau$ means that there is no state transition before τ (the actual value of the random delay). In this case the relation of the accumulated reward $r_i\tau$ and the reward bound w determine the probability defined in (12). When the accumulated reward exceeds the reward bound, i.e. $r_i\tau > w$, $S_{ij}(t, w, \tau)$ equals to 0. Otherwise it depends on state j and time t. $S_{ij}(t, w, \tau)$ equals to the probability that the next regeneration period start from state j, i.e. Δ_{ij}, if $t > \tau$ and it is 0 for $t < \tau$.

When a state transition takes place before τ ($h < \tau$) the following cases have to be considered. If the accumulated reward up to the state transition exceeds the reward bound ($r_i h > w$) then $S_{ij}(t, w, \tau)$ equals to 0, otherwise a state transition from state i to state k takes place at time h with probability $dQ_{ik}(h)/dQ_i(h)$, and a similar analysis problem arises from that point on.

Unconditioning according to the distribution of the sojourn time ($Q_i(t) = \sum_j Q_{ij}(t)$) yields:

$$\begin{aligned} S_{ij}(t, w, \tau) &= \Delta_{ij}\ [1 - Q_i(\tau)]\ U(t-\tau)\ U(w - r_i\tau) \\ &+ \sum_{k\in\Omega} \int_{h=0}^{\tau} S_{kj}(t-h, w - r_i h, \tau - h)\ U(w - r_i\tau)\ dQ_{ik}(h) \end{aligned} \tag{15}$$

An LST with respect to t, denoting the transform variable by s, results in:

$$\begin{aligned} S_{ij}^{\sim}(s, w, \tau) &= \Delta_{ij}\ [1 - Q_i(\tau)]\ e^{-s\tau}\ U(w - r_i\tau) \\ &+ \sum_{k\in\Omega} \int_{h=0}^{\tau} e^{-sh}\ S_{kj}^{\sim}(s, w - r_i h, \tau - h)\ U(w - r_i\tau)\ dQ_{ik}(h) \end{aligned} \tag{16}$$

An LT with respect to w, denoting the transform variable by v, results in:

$$\begin{aligned} S_{ij}^{\sim *}(s, v, \tau) &= \Delta_{ij}\ [1 - Q_i(\tau)]\ \tfrac{1}{v}\ e^{-(s+r_i v)\tau} \\ &+ \sum_{k\in\Omega} \int_{h=0}^{\tau} e^{-(s+r_i v)h}\ S_{kj}^{\sim *}(s, v, \tau - h)\ dQ_{ik}(h) \end{aligned} \tag{17}$$

And finally another LT with respect to τ, denoting the transform variable by χ provides the theorem. □

Theorem 3 *The distribution of the accumulated reward inside a regenerative period, $G_{ij}(t, w, \tau)$ satisfies the following transform domain equation:*

$$\begin{aligned} G_{ij}^{\sim **}(s, v, \chi) &= \delta_{ij}\ \tfrac{s[1 - Q_i^{\sim}(s + r_i v + \chi)]}{v\chi(s + r_i v + \chi)} \\ &+ \textstyle\sum_{k\in\Omega} Q_{ik}^{\sim}(s + r_i v + \chi)\ G_{kj}^{\sim **}(s, v, \chi) \end{aligned} \tag{18}$$

where δ_{ij} is the Kronecker delta.

Proof: Conditioning on the sojourn time h in state i we have:

$$
G_{ij}(t,w,\tau \mid h) =
\begin{cases}
\delta_{ij}[U(t) - U(t - w/r_i)] & \text{if } r_i\tau > w \text{ and } h \geq \tau \\
\delta_{ij}[U(t) - U(t - \tau)] & \text{if } r_i\tau \leq w \text{ and } h \geq \tau \\
\delta_{ij}[U(t) - U(t - w/r_i)] & \text{if } r_i h > w \text{ and } h < \tau \\
\delta_{ij}[U(t) - U(t - h)] + & \\
\quad \sum\limits_{k \in \Omega} \dfrac{dQ_{ik}(h)}{dQ_i(h)} \; G_{kj}(t-h, w - r_i h, \tau - h) & \text{if } r_i h \leq w \text{ and } h < \tau
\end{cases}
\tag{19}
$$

Similar to Equation (14) in Equation (19) the condition $h \geq \tau$ means that there is no state transition before τ. In these cases the probability defined in (11) equals to 1 if $t < \tau$ and the accumulated reward is less than the reward bound, i.e. $r_i t < w$ and it equals to 0 otherwise.

When we have a state transition before τ and the accumulated reward exceeds the reward bound before $(r_i h > w)$ $G_{ij}(t,w,\tau)$ equals to 1 up to time $t = w/r_i$ and it equals to 0 after that. When we have a state transition before τ but the accumulated reward does not exceed the reward bound before $(r_i h < w)$ $G_{ij}(t,w,\tau)$ equals to 1 up to the state transition $t = h$ and a similar analysis problem arises from that point on.

To simplify the notation we introduce $\rho = min(\tau, w/r_i)$. Unconditioning yields:

$$
\begin{aligned}
G_{ij}(t,w,\tau) = {} & \delta_{ij} \; [1 - Q_i(\rho)] \; [U(t) - U(t-\rho)] \\
& + \int\limits_{h=0}^{\rho} \delta_{ij} \; [U(t) - U(t-h)] \; dQ_i(h) \\
& + \sum_{k \in \Omega} \int\limits_{h=0}^{\rho} \; G_{kj}(t-h, w - r_i h, \tau - h) \; dQ_{ik}(h)
\end{aligned}
\tag{20}
$$

An LST with respect to t, denoting the transform variable by s, results in:

$$
\begin{aligned}
G^{\sim}_{ij}(s,w,\tau) = {} & \delta_{ij}[1 - e^{-s\rho} + e^{-s\rho} \; Q_i(\rho) - \int\limits_{h=0}^{\rho} \; e^{-sh} \; dQ_i(h)] \\
& + \sum_{k \in \Omega} \int\limits_{h=0}^{\rho} e^{-sh} \; G^{\sim}_{kj}(s, w - r_i h, \tau - h) \; dQ_{ik}(h)
\end{aligned}
\tag{21}
$$

An LT with respect to w, denoting the transform variable by v, and taking care of the dependence of ρ on w and τ, results in:

$$\begin{aligned} G_{ij}^{\sim *}(s,v,\tau) = \delta_{ij} \Bigg[& \frac{s}{v(s+r_i v)} \left(1 - e^{-(s+r_i v)\tau}\right) + \frac{1}{v} e^{-(s+r_i v)\tau} Q_i(\tau) \\ & + \int_{w=0}^{r_i \tau} e^{-(s/r_i+v)w} Q_i(w/r_i)\, dw - \frac{1}{v} \int_{h=0}^{\tau} e^{-(s+r_i v)h}\, dQ_i(h) \Bigg] \\ & + \sum_{k\in\Omega} \int_{h=0}^{\tau} e^{-(s+r_i v)h} G_{kj}^{\sim *}(s,v,\tau-h)\, dQ_{ik}(h) \end{aligned} \tag{22}$$

And finally an other LT with respect to τ, denoting the transform variable by χ provides the theorem. □

To evaluate the accumulated reward of an *MRRM* based on Equation (2) an inverse Laplace transformation of $S_{ij}^{\sim **}(s,v,\chi)$ and $G_{ij}^{\sim **}(s,v,\chi)$ is necessary with respect to χ.

The execution of the inverse transformation depends on the particular *SMP* described by $Q_{ij}(t)$. Below we consider the special case when the subordinated process is a *CTMC*.

5.2 CTMC *subordinated process with random delay*

Suppose the subordinated process is a *CTMC* with infinitesimal generator $\mathbf{A} = \{a_{ij}\}$ and the diagonal matrix of the reward rates is denoted by $\widehat{\mathbf{R}} = \langle r_i \rangle$ the reward measures are characterized by the following theorems.

Theorem 4 *The distribution of the accumulated reward of a complete regenerative period can be evaluated as follows:*

$$\mathbf{S}^{\sim\sim}(s,v) = \int_{\tau=0}^{\infty} e^{-\tau(s\mathbf{I}+v\widehat{\mathbf{R}}-\mathbf{A})}\, \Delta\, dT(\tau) \tag{23}$$

Proof: Substituting $Q_i^{\sim}(s+r_i v+\chi)$ by $\frac{-a_{ii}}{-a_{ii}+s+r_i v+\chi}$, $Q_{ik}^{\sim}(s+r_i v+\chi)$ by $\frac{a_{ik}}{-a_{ii}+s+r_i v+\chi}$ if $k \neq i$ and $Q_{ii}^{\sim}(s+r_i v+\chi)$ by 0 in Equation (13) results in:

$$\begin{aligned} S_{ij}^{\sim **}(s,v,\chi) = \Delta_{ij} & \frac{1}{v(-a_{ii}+s+r_i v+\chi)} \\ & + \textstyle\sum_{k\in R, k\neq i} \frac{a_{ik}}{-a_{ii}+s+r_i v+\chi} S_{kj}^{\sim **}(s,v,\chi) \end{aligned} \tag{24}$$

Which can be organized into matrix form as:

$$\mathbf{S}^{\sim**}(s,v,\chi) = \frac{1}{v}\ \left((s+\chi)\mathbf{I} + v\widehat{\mathbf{R}} - \mathbf{A})\right)^{-1}\ \Delta \tag{25}$$

From which an inverse Laplace transformation with respect to χ, a multiplication with v (to reach the LST with respect to v) and the integral according to the distribution of θ provides the theorem. □

Theorem 5 *The accumulated reward inside of a regenerative period can be evaluated as follows:*

$$\mathbf{G}^{\sim\sim}(s,v) = \int_{\tau=0}^{\infty} s\ \left(s\mathbf{I} + v\widehat{\mathbf{R}} - \mathbf{A})\right)^{-1}\ \left[\mathbf{I} - e^{-\tau(s\mathbf{I}+v\widehat{\mathbf{R}}-\mathbf{A})}\right]\ dT(\tau) \tag{26}$$

Proof: The same substitution of the kernel elements in Equation (18), and the same series of steps provide the theorem. □

5.3 Semi-Markov subordinated process with random delay and concluding state transitions

Consider a subordinated *SMP* over Ω with kernel $\mathbf{Q}(t)$. The regenerative period starts in $\Psi \subset \Omega$ and is concluded by the expiration of the random delay θ which is distributed according to $T(\tau)$ (independent of the subordinated process) or a preceding state transition to $\Psi^c = \Omega - \Psi$.

Theorem 6 *The distribution of the accumulated reward of a complete regenerative period, $S_{ij}(t,w,\tau)$ satisfies the following transform domain equation:*

$$\begin{aligned} S_{ij}^{\sim**}(s,v,\chi) &= \Delta_{ij}\ \frac{1 - Q_i^{\sim}(s + r_i v + \chi)}{v(s + r_i v + \chi)} \\ &+ \sum_{k\in\Psi} Q_{ik}^{\sim}(s + r_i v + \chi)\ S_{kj}^{\sim**}(s,v,\chi) + I_{j\in\Psi^c}\frac{1}{v\chi}\ Q_{ij}^{\sim}(s+\chi) \end{aligned} \tag{27}$$

where $I_{j\in\Psi^c}$ is the indicator that state j is in Ψ^c.

Proof: Conditioning on the sojourn time h in state i we have:

$$S_{ij}(t,w,\tau \mid h) = \begin{cases} 0 & \text{if } r_i\tau > w \text{ and } h \geq \tau \\ \Delta_{ij}\, U(t-\tau) & \text{if } r_i\tau \leq w \text{ and } h \geq \tau \\ 0 & \text{if } r_i h > w \text{ and } h < \tau \\ I_{j\in\Psi^c}\, \dfrac{dQ_{ij}(h)}{dQ_i(h)}\, U(t-h) + & \\ \displaystyle\sum_{k\in\Psi} \frac{dQ_{ik}(h)}{dQ_i(h)}\, S_{kj}(t-h, w-r_i h, \tau-h) & \text{if } r_i h \leq w \text{ and } h < \tau \end{cases} \tag{28}$$

The effect of a concluding state transition is captured by the last condition, where different cases arise for state transitions out of Ψ and inside Ψ.

The same series of steps as in Theorem 2 results in the theorem. □

Theorem 7 *The distribution of the accumulated reward inside a regenerative period, $G_{ij}(t,w,\tau)$ satisfies the following transform domain equation:*

$$\begin{aligned} G_{ij}^{\sim**}(s,v,\chi) = \delta_{ij}\, & \tfrac{s[1-Q_i^{\sim}(s+r_i v+\chi)]}{v\chi(s+r_i v+\chi)} \\ & + \textstyle\sum_{k\in\Psi} Q_{ik}^{\sim}(s+r_i v+\chi)\, G_{kj}^{\sim**}(s,v,\chi) \end{aligned} \tag{29}$$

Proof: Conditioning on the sojourn time h in state i we have:

$$G_{ij}(t,w,\tau \mid h) = \begin{cases} \delta_{ij}[U(t) - U(t-w/r_i)] & \text{if } r_i\tau > w \text{ and } h \geq \tau \\ \delta_{ij}[U(t) - U(t-\tau)] & \text{if } r_i\tau \leq w \text{ and } h \geq \tau \\ \delta_{ij}[U(t) - U(t-w/r_i)] & \text{if } r_i h > w \text{ and } h < \tau \\ \delta_{ij}[U(t) - U(t-h)] + & \\ \quad \displaystyle\sum_{k\in\Psi} \frac{dQ_{ik}(h)}{dQ_i(h)}\, G_{kj}(t-h, w-r_i h, \tau-h) & \text{if } r_i h \leq w \text{ and } h < \tau \end{cases} \tag{30}$$

Note that state transitions only inside Ψ are summed up in the latest case. The same series of steps as in Theorem 3 result in the theorem. □

5.4 CTMC *subordinated process with random delay and concluding state transitions*

Consider a subordinated *CTMC* with infinitesimal generator $\mathbf{A}$. The regenerative period starts in $\Psi \subset \Omega$ and is concluded by the expiration of the random delay θ which is distributed according to $T(\tau)$ or a preceding state transition to $\Psi^c = \Omega - \Psi$, so $\mathbf{A}$ can be partitioned as $\mathbf{A} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right]$, where $\mathbf{A}_{11}$ describes the transitions inside Ψ, $\mathbf{A}_{12}$ contains the intensity of the transitions from Ψ to Ψ^c, $\mathbf{A}_{21}$ the transitions from Ψ^c to Ψ, and $\mathbf{A}_{22}$ the transitions inside Ψ^c, however $\mathbf{A}_{21}$ and $\mathbf{A}_{22}$ are irrelevant since the subordinated process is concluded by the state transition out of Ψ.

Theorem 8 *The distribution of the accumulated reward of a complete regenerative period, $S_{ij}(t, w, \tau)$ satisfies the following transform domain equation:*

$$\mathbf{S}^{\sim\sim}(s,v) = \int_{\tau=0}^{\infty} e^{-\tau(s\mathbf{I}_\Psi + v\widehat{\mathbf{R}}_\Psi - \mathbf{A}_{11})} \left[\mathbf{I}_\Psi \;\; \mathbf{0}\right] \; \Delta \\ +(s\mathbf{I}_\Psi + v\widehat{\mathbf{R}}_\Psi - \mathbf{A}_{11})^{-1}(\mathbf{I}_\Psi - e^{-\tau(s\mathbf{I}_\Psi + v\widehat{\mathbf{R}}_\Psi - \mathbf{A}_{11})})\mathbf{A}_{12}\left[\mathbf{0} \;\; \mathbf{I}_{\Psi^c}\right] \; dT(\tau) \tag{31}$$

where $\mathbf{I}_\Psi$ and $\mathbf{I}_{\Psi^c}$ are identity matrices of dimension $\#\Psi$ and $\#\Psi^c$, respectively, $\mathbf{0}$ is the matrix of zeroes of the necessary size and $\widehat{\mathbf{R}}_\Psi = \langle r_i \rangle$, $i \in \Psi$ is the diagonal matrix of the reward rates associated to the states of Ψ.

Proof: Substituting the entries of the kernel in Equation (27) following the way described in Theorem 4 yields an equation, from which an inverse Laplace transformation with respect to χ, a multiplication with v (to reach the LST with respect to v) and the integral according to the distribution of θ provides the theorem. □

Theorem 9 *The distribution of the accumulated reward inside a regenerative period, $G_{ij}(t, w, \tau)$ satisfies the following transform domain equation:*

$$\mathbf{G}^{\sim\sim}(s,v) = \left[(s\mathbf{I}_\Psi + v\widehat{\mathbf{R}}_\Psi - \mathbf{A}_{11})^{-1} \left(\mathbf{I}_\Psi - e^{-\tau(s\mathbf{I}_\Psi + v\widehat{\mathbf{R}}_\Psi - \mathbf{A}_{11})}\right) \;\; \mathbf{0}\right]. \tag{32}$$

Proof: Repeating the algebraic transformations used to prove Theorem 8 on Equation (29) will yield the statement of the theorem. □

The most complex regeneration periods of Deterministic and Stochastic Petri Nets [1,3] and MRSPN* (as it is defined in [4]) fall into this class, where the random delay is the firing time of a general transition and the concluding state transitions are caused by disabling this transition before firing.

5.5 *Subordinated process without internal state transition*

An *MRGP* often has a simple subordinated process without internal state transition. This special case is considered below.

Theorem 10 *When there is no state transition during the subordinated process and the distribution of the time to the next regeneration epoch is $T(\tau)$, $S_{ij}^{\sim *}(s,v)$ and $G_{ij}^{\sim *}(s,v)$ satisfy the following equations:*

$$S_{ij}^{\sim *}(s,v) = \int\limits_{\tau=0}^{\infty} \Delta_{ij}\ \frac{1}{v} e^{-\tau(s+r_i v)}\ dT(\tau); \tag{33}$$

$$G_{ij}^{\sim *}(s,v) = \int\limits_{\tau=0}^{\infty} \delta_{ij}\ \frac{s}{v(s+r_i v)} \left[1 - e^{-\tau(s+r_i v)}\right]\ dT(\tau). \tag{34}$$

Proof: Substituting $Q_i^{\sim}(s + r_i v + \chi)$ and $Q_{ik}^{\sim}(s + r_i v + \chi)$ by 0, in Equation (13) and (18), inverse Laplace transforming the results with respect to χ, and integrating according to the distribution of θ gives the theorem. □

Two often applied special cases are the exponentially distributed and the deterministic delay of the subordinated process. In the first case when θ has an exponential distribution with parameter λ:

$$S_{ij}^{\sim\sim}(s,v) = \ \Delta_{ij}\ \frac{\lambda}{\lambda + s + r_i v} \quad ; \quad G_{ij}^{\sim\sim}(s,v) = \ \delta_{ij}\ \frac{s}{\lambda + s + r_i v}. \tag{35}$$

In the second case when θ is deterministic, i.e. $\theta = \tau$:

$$\begin{aligned} S_{ij}^{\sim\sim}(s,v) &= \ \Delta_{ij} e^{-\tau(s+r_i v)}; \\ G_{ij}^{\sim\sim}(s,v) &= \ \delta_{ij}\ \frac{s}{s + r_i v} \left[1 - e^{-\tau(s+r_i v)}\right]. \end{aligned} \tag{36}$$

The multiplication of Equation (33) and Equation (34) by v is necessary to get Laplace-Stieltjes transform from the Laplace transform.

6 Numerical Example

As a simple example to illustrate the analysis steps of the proposed method we consider an M/D/1/2/2 queueing system. This is a closed queueing system with at most two customers in it and non-preemptive service mechanism. The steady state behaviour of this system was studied in [1], while the transient analysis was accomplished in [3].

The Petri net description for the system, proposed in [1], is reported in Figure 1. Place p_1 contains the "thinking" customers, i.e. the customers waiting to submit a job, and transition t_1 represents the submission of a job. Tokens in place p_2 represent the jobs queueing for service. A token in p_3 means that the server is busy while a token in p_4 means that the server is idle. Transition t_2 represents the service of a job; when the job is completed the customer returns to his thinking state. Transition t_3 is an immediate transition modeling the start of service, i.e. the transfer of the job from the queue to the server, this transfer becomes possible when the service unit becomes free.

We make the same assumptions as in [1,3]. The firing time of t_1 is exponentially distributed with rate $m_1 \cdot \lambda$ being m_1 is the number of tokens in p_1 and $\lambda = 0.5$ *job/hour*. t_2 is a *DET* transition modeling a constant service time of duration $d = 1.0$ *hour*. We augment this description by the reward rates, $r_1 = 0$, $r_2 = 1$, $r_3 = 0.8$, representing the idle server, the busy server and and the busy server with some penalty charged because a job is waiting for service, respectively.

The reduced state space of the system eliminating the vanishing markings arising from the immediate transition t_3 is composed of three states, called s_1, s_2 and s_3 in Figure 1b.

There are three regenerative transitions (i.e. state transitions result in a regeneration epoch): $s_1 \to s_2$, $s_2 \to s_1$, $s_3 \to s_2$. Hence s_3 can never be a regeneration state when the process is started from state s_1 or s_2. We denote by $Z^i(t)$ the subordinated process started in state i. In the sequel we determine the corresponding $\mathbf{S}^{\sim\sim}(s,v)$ and $\mathbf{G}^{\sim\sim}(s,v)$ matrices, or the relevant rows of these matrices to build the $\mathbf{S}^{\sim\sim}(s,v)$ and $\mathbf{G}^{\sim\sim}(s,v)$ matrices of $Z(t)$.

(i) $Z^1(t)$: When the process starts from state s_1, the subordinated process will contain no internal state transitions, since the state transition to state s_2 terminates the process, while the distribution of the delay is exponential. Thus we can directly apply Equation (35) to determine the only relevant first row of the matrices:

$$S^{\sim\sim}_{11}(s,v) = 0,\ S^{\sim\sim}_{12}(s,v) = \tfrac{\lambda}{\lambda+s+vr_1},\ S^{\sim\sim}_{13}(s,v) = 0,$$
$$G^{\sim\sim}_{11}(s,v) = \tfrac{s}{\lambda+s+vr_1},\ G^{\sim\sim}_{12}(s,v) = 0,\ G^{\sim\sim}_{13}(s,v) = 0.$$

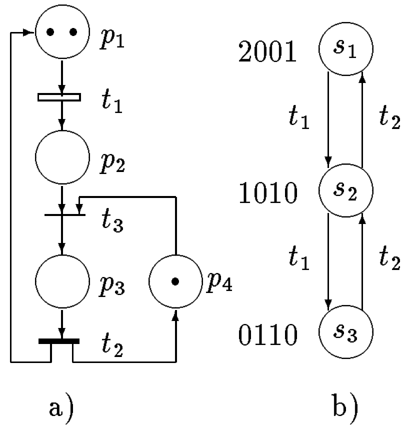

Fig. 1. a) PN modeling of a M/D/1/2/2; b) corresponding reduced reachability graph.

(ii) $Z^2(t)$: When the process starts from state s_2, the subordinated process will be a one-step *CTMC*, i.e. the only possible state transition in the subordinated process is a transition from state s_2 to state s_3, while the delay is deterministic (τ). Thus we can apply Theorem (4) and Theorem (5) to determine the relevant second row of the matrices. First we determine the corresponding $\widehat{\mathbf{R}}$, $\mathbf{A}$ and Δ matrices:

$$\widehat{\mathbf{R}} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & 0 \end{bmatrix} \Delta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{37}$$

A substitution to Equation (23) and Equation (26) results in:

$$\begin{aligned}
&S^{\sim\sim}_{21}(s,v) = e^{-\tau(\lambda+s+r_2 v)}, \\
&S^{\sim\sim}_{22}(s,v) = \tfrac{\lambda}{(r_2-r_3)v+\lambda}(e^{-\tau(s+r_3 v)} - e^{-\tau(\lambda+s+r_2 v)}), \\
&S^{\sim\sim}_{23}(s,v) = 0,\ G^{\sim\sim}_{21}(s,v) = 0, \\
&G^{\sim\sim}_{22}(s,v) = \tfrac{s}{\lambda+s+vr_2}(1 - e^{-\tau(\lambda+s+r_2 v)}), \\
&G^{\sim\sim}_{23}(s,v) = \tfrac{\lambda s}{(s+r_3 v)} + \tfrac{\lambda s e^{-\tau(\lambda+s+r_2 v)}}{(\lambda+s+r_3 v)(\lambda+(r_2-r_3)v)} - \tfrac{\lambda s e^{-\tau(s+r_3 v)}}{(s+r_3 v)(\lambda+(r_2-r_3)v)}.
\end{aligned}$$

(iii) $Z^3(t)$: Finally, when the process starts from state s_3, the subordinated process will contain no internal state transitions, since the only possible state transition to state s_2 terminates the subordinated process. The distribution of the delay is deterministic. Thus we can apply Equation (36) to determine the relevant third row of the matrices:

$$\begin{aligned}
&S^{\sim\sim}_{31}(s,v) = 0,\ S^{\sim\sim}_{32}(s,v) = e^{-\tau(s+vr_2)},\ S^{\sim\sim}_{33}(s,v) = 0, \\
&G^{\sim\sim}_{31}(s,v) = 0,\ G^{\sim\sim}_{32}(s,v) = 0,\ G^{\sim\sim}_{33}(s,v) = \tfrac{s}{s+r_3 v}(1 - e^{-\tau(s+vr_3)}),
\end{aligned}$$

since in this case the only relevant non-zero entry of Δ is $\Delta_{23} = 1$.

Since the ith row of matrices $\mathbf{S}^{\sim\sim}(s,v)$ and $\mathbf{G}^{\sim\sim}(s,v)$ of the subordinated process $Z^i(t)$ gives the ith row of matrices $\mathbf{S}^{\sim\sim}(s,v)$ and $\mathbf{G}^{\sim\sim}(s,v)$, respectively, of the process $Z(t)$, we obtain the double Laplace-Stieltjes transform domain expression of $\mathbf{R}^{\sim\sim}(s,v)$ by substituting to Equation (2):

$$\mathbf{R}^{\sim\sim}(s,v) = \frac{1}{c}\begin{bmatrix} R_{11}^{\sim\sim}(s,v) & R_{12}^{\sim\sim}(s,v) & R_{13}^{\sim\sim}(s,v) \\ R_{21}^{\sim\sim}(s,v) & R_{22}^{\sim\sim}(s,v) & R_{23}^{\sim\sim}(s,v) \\ R_{31}^{\sim\sim}(s,v) & R_{32}^{\sim\sim}(s,v) & R_{33}^{\sim\sim}(s,v) \end{bmatrix}.$$

where

$$\begin{aligned}
c &= 1 - \frac{\lambda}{\lambda+s+r_1 v} e^{-\tau(\lambda+s+r_2 v)} \\
&\quad - \frac{\lambda}{\lambda+(r_2-r_3)v}\left(e^{-\tau(s+r_3 v)} - e^{-\tau(\lambda+s+r_2 v)}\right) \\
R_{11}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_1}\left(1 - \frac{\lambda}{(r_2-r_3)v+\lambda}\left(e^{-\tau(s+r_3 v)} - e^{-\tau(\lambda+s+r_2 v)}\right)\right) \\
R_{12}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_2}\,\frac{\lambda}{\lambda+s+vr_1}\left(1 - e^{-\tau(\lambda+s+r_2 v)}\right) \\
R_{13}^{\sim\sim}(s,v) &= \frac{\lambda}{\lambda+s+vr_1}\left(\frac{\lambda s}{(s+r_3 v)} + \frac{\lambda s e^{-\tau(\lambda+s+r_2 v)}}{(\lambda+s+r_3 v)(\lambda+(r_2-r_3)v)}\right. \\
&\qquad \left. - \frac{\lambda s e^{-\tau(s+r_3 v)}}{(s+r_3 v)(\lambda+(r_2-r_3)v)}\right) \\
R_{21}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_1} e^{-\tau(\lambda+s+r_2 v)} \\
R_{22}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_2}\left(1 - e^{-\tau(\lambda+s+r_2 v)}\right) \\
R_{23}^{\sim\sim}(s,v) &= \frac{\lambda s}{(s+r_3 v)} + \frac{\lambda s e^{-\tau(\lambda+s+r_2 v)}}{(\lambda+s+r_3 v)(\lambda+(r_2-r_3)v)} \\
&\quad - \frac{\lambda s e^{-\tau(s+r_3 v)}}{(s+r_3 v)(\lambda+(r_2-r_3)v)} \\
R_{31}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_1} e^{-\tau(\lambda+s+r_2 v)} e^{-\tau(s+vr_2)} \\
R_{32}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_2}\left(1 - e^{-\tau(\lambda+s+r_2 v)}\right) e^{-\tau(s+vr_2)} \\
R_{33}^{\sim\sim}(s,v) &= c\,\frac{s}{s+r_3 v}\left(1 - e^{-\tau(s+vr_3)}\right) + \frac{\lambda s e^{-\tau(\lambda+s+r_2 v)}}{(\lambda+s+r_3 v)(\lambda+(r_2-r_3)v)} \\
&\quad + \frac{\lambda s}{(s+r_3 v)} - \frac{\lambda s e^{-\tau(s+r_3 v)}}{(s+r_3 v)(\lambda+(r_2-r_3)v)} e^{-\tau(s+vr_2)}
\end{aligned}$$

To obtain the inverse transforms, at least one symbolic transform would be necessary, but it is not feasible even for this simple system. However, the

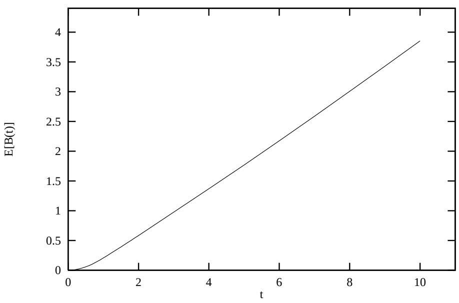

Fig. 2. Mean of the accumulated reward

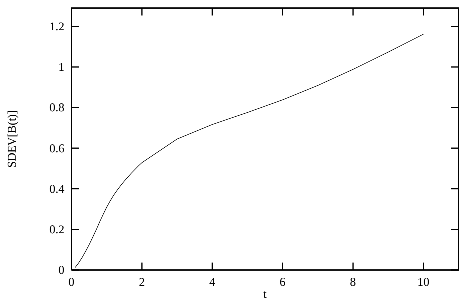

Fig. 3. Standard deviation of the accumulated reward

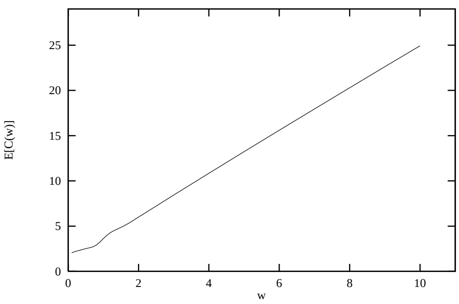

Fig. 4. Mean of the completion time

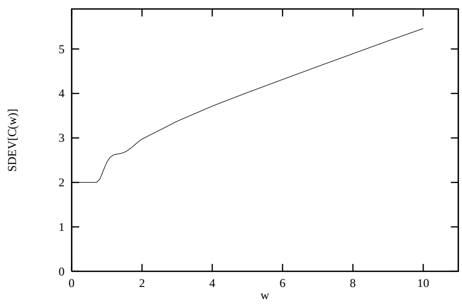

Fig. 5. Standard deviation of the completion time

moments of the corresponding distribution can be calculated based on Equation (5) or Equation (9) using the transform domain expression and a numerical inverse transformation method.

In Figures 2, 3, 4, 5 the numerical results are depicted for the mean and the standard deviation of the accumulated reward and the completion time, respectively, when the system was started in state s_1. We emphasize that any moments can be calculated using the proposed analytical method, however we chose to depict the most frequently used quantities, the mean and the standard deviation. The results were obtained by a numerical inverse transformation method written in Mathematica by resorting to the Jagerman method [10]. Some numerical uncertainties were experienced in the values close to zero especially when calculating the standard deviation of the completion time (Figure 5). The mean completion time tends to 2 as the work requirement goes to 0 (Figure 4), since the mean holding time in state s_1 is $1/\lambda = 2$, and the reward accumulation starts in state s_2.

7 Conclusion

MRGP with associated reward variable is a powerful modeling tool to evaluate the performance of stochastic systems with generally distributed (including deterministic) event time. To analyze such models a new approach is proposed based on the *reward kernels* description of *MRRMs*. Reward measures such as the distribution of the accumulated reward and the completion time and their moments are provided. A detailed analysis of a simple finite queue is given to demonstrate the steps of the proposed method.

In the paper we considered *MRRMs* with *preemptive resume* policy, but the approach can be applied for *preemptive repeat* policy (prd or pri) with some extension as well, if the reward loss can occur only at the regenerative epochs. Further research is required as well to provide automated tools and effective numerical methods to handle models much larger than the studied one, since the discussed solution contains computationally hard steps, such as the symbolic evaluation of the matrices $\mathbf{S}^{\sim\sim}(s,v)$ and $\mathbf{G}^{\sim\sim}(s,v)$, application of (2) and the transformation to the time domain from the double transform domain description ($\mathbf{R}^{\sim\sim}(s,v)$).

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