Analysis of the Completion Time of Markov Reward Models and its Application*

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Abstract

*Analysis of Markov Reward Models (*MRM*) with preemptive resume (prs) policy usually results in a double transform expression, whose solution is based on the inverse transformations both in time and reward variable domain. This paper discusses the case when the reward rates can be either* 0 *or positive, and analyses the completion time of* MRM*s. We present a symbolic expression of moments of the completion time, from which a computationally effective recursive numerical method can be obtained. As a numerical example the mean and the standard deviation of the completion time of a Carnegie-Mellon multiprocessor system are evaluated by the proposed method.*

Key words: Markov Reward Models, Preemptive Resume Policy, Completion Time.

1 Introduction

The properties of stochastic reward processes have been studied since a long time [9]. However, only recently, stochastic reward models (*SRM*) have received attention as a modeling tool in performance and reliability evaluation. Indeed, the possibility of associating a reward variable to each structure state increases the descriptive power and the flexibility of the model.

Different interpretations of the structure-state process and of the associated reward structure give rise to different applications. Common assignments of the reward rates are: execution rates of tasks in computing systems (the computational capacity) [1], number of active processors (or processing power), throughput [12], available bandwidth average response time or response time distribution.

Two main different points of view have been assumed in the literature when dealing with *SRM* for degradable systems [11]. In the *system oriented* point of view the most

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significant measure is the total amount of work done by the system in a finite interval. The accumulated reward is a random variable whose distribution function is sometimes called *performability* [12]. Various numerical techniques for the evaluation of the performability have been investigated in recent papers: [10, 7, 8]. In the *user oriented* (or *task oriented*) point of view the system is regarded as a server, and the emphasis of the analysis is on the ability of the system to accomplish an assigned task in due time. Consequently, the most characterizing measure becomes the probability of accomplishing an assigned service in a given time. The task oriented point of view is a more direct representation of the quality of service. Asymptotic behaviour of some task oriented measures is studied in [16] under the assumption of fast service (or repair).

A unified formulation to the system oriented and the user oriented point of view was provided by Kulkarni et al. in [11]. An alternative interpretation of the completion time problem can be given in terms of the hitting time of an appropriate cumulative functional [6] against an absorbing barrier equal to the work requirement. The definition of a cumulative functional was first suggested by Kulkarni et al. [11] and then explicitly exploited in [4], where the completion time was modelled as a first hitting time against an absorbing barrier. The subclass of *MRM*s with Phase-type distributed random work requirement was studied by Bobbio and Trivedi [5]. In this case the completion time is Phase type distributed and they defined the "extended" Continuous Time Markov Chain (*CTMC*) which characterize the distribution of the completion time.

In this paper, we improve the results of [2, 13] and propose a computationally effective approach not only to calculate the mean completion time of on-off *MRM*s (i.e. *MRM*s with reward rates equal to 0 or 1), but to obtain all the moments of the completion time of *MRM*s with arbitrary non-negative reward rates.

The paper is organized as follows. Section 2 provides the formal definition of *SRM*s, and introduces the class of *MRM*s. In Section 3 the completion time analysis of *MRM*s is presented. Section 4 gives an application of the proposed computational approach to the task completion time analysis of a Carnegie-Mellon multiprocessor system. The paper is concluded in Section 5.

2 Stochastic Reward Models

The adopted modeling framework consists of describing the behaviour of the system configuration in time by means of a stochastic process, and by associating a non-negative real constant to each state of the structure-state process representing the effective working capacity or performance level or cost or stress of the system in that state. The variable associated to each structure-state is called the *reward rate* [9].

Let the *structure-state process* $Z(t)$ $(t \geq 0)$ be a (right continuous) stochastic process defined over a discrete and finite state space Ω of cardinality n. Let f be a non-negative real-valued function defined as:

$$f[Z(t)] = r_i \geq 0 \ , \qquad \text{if} \ \ Z(t) = i \ , \tag{1}$$

$f[Z(t)]$ represents the instantaneous reward rate associated to state i.

Definition 1 *The* **accumulated reward** $B(t)$ *is a random variable which represents the accumulation of reward in time:*

$$B(t) = \int_0^t f[Z(\tau)]d\tau = \int_0^t r_{Z(\tau)}d\tau.$$

$B(t)$ is a stochastic process that depends on $Z(u)$ for $0 \leq u \leq t$. According to Definition 1 this paper restricts the attention to the class of models in which no state transition can entail to a loss of the accumulated reward. A *SRM* of this kind is called *preemptive resume* (prs) model. The distribution of the accumulated reward is defined as

$$B(t,w) = Pr\{B(t) \leq w\}.$$

The complementary question concerning the reward accumulation of *SRM*s is the time needed to complete a given (possibly random) work requirement (i.e., the time to accumulate the required amount of reward).

Definition 2. *The* **completion time** C *is the random variable representing the time to accumulate a reward requirement equal to a random variable* W*:*

$$C = min\,[t \geq 0 \ : \ B(t) = W] \ .$$

C is the time instant at which the work accumulated by the system reaches the value W for the first time. Assume, in general, that W is a random variable independent from $Z(t)$ with distribution $G(w)$ with support on $(0, \ \infty)$. The degenerate case, in which W is deterministic and the distribution $G(w)$ becomes the unit step function $U(w - w_d)$, can be considered as well. For a given sample of $W = w$, the completion time $C(w)$ and its cumulative distribution function $C(t,w)$ are defined as:

$$C(w) = \min\,[t \geq 0 \ : \ B(t) = w] \ \ ; \qquad C(t,w) = Pr\,\{C(w) \leq t\} \ . \tag{2}$$

The completion time C is characterized by the following distribution:

$$\hat{C}(t) = Pr\,\{\, C \leq t\} = \int_0^\infty C(t,w)\, dG(w) \ . \tag{3}$$

The distribution of the completion time of a *prs SRM* is closely related to the distribution of the accumulated reward by means of the following relation:

$$B(t,w) = Pr\{B(t) \le w\} = Pr\{C(w) \ge t\} = 1 - C(t,w) . \tag{4}$$

For the purposes of the subsequent analysis below we define the following matrix functions $\mathbf{P}(t,w) = \{P_{ij}(t,w)\}$ and $\mathbf{F}(t,w) = \{F_{ij}(t,w)\}$ as:

$$P_{ij}(t,w) = Pr\{Z(t) = j \, , \, B(t) \le w \,|\, Z(0) = i\} \, , \tag{5}$$

$$F_{ij}(t,w) = Pr\{Z(C(w)) = j \, , \, C(w) \le t \,|\, Z(0) = i\} \, , \tag{6}$$

- $P_{ij}(t,w)$ is the joint distribution of the accumulated reward and the structure state at time t supposed that the initial state of the structure state process is i,
- $F_{ij}(t,w)$ is the joint distribution of the completion time and the structure state at completion supposed that the initial state of the structure state process is i.

We assume (5) and (6), it follows for any t and i that $\sum_{j\in\Omega} [\, P_{ij}(t,w) + F_{ij}(t,w)] = 1$. By these definitions:

$$B(t,w) = \underline{P}(0)\, \mathbf{P}(t,w)\, \underline{h}^T \quad \text{and} \quad C(t,w) = \underline{P}(0)\, \mathbf{F}(t,w)\, \underline{h}^T \, ,$$

where $\underline{P}(0) = \{P_i(0)\}$ is the row vector of the initial state probabilities of the structure-state process ($Pr\{Z(0) = i\} = P_i(0)$), and $\underline{h}^T$ is the column vector with all the entries equal to 1.

Given that $G(w)$ is the cumulative distribution function of the random work requirement W, the distribution of the completion time is:

$$\hat{C}(t) = \int_{w=0}^{\infty} \left[\sum_{i\in\Omega} \sum_{j\in\Omega} P_i(0) F_{ij}(t,w) \right] dG(w) = \int_{w=0}^{\infty} \underline{P}(0)\, \mathbf{F}(t,w)\, \underline{h}^T \, dG(w) \, . \tag{7}$$

2.1 Markov Reward Models

Definition 3. *The subclass of SRMs in which the structure state process $(Z(t))$ is an ergodic* CTMC *with any initial probability distribution is called Markov Reward Models (*MRM*).*

The introduced matrix functions of a *MRM* can be described in double transform domain based on the infinitesimal generator ($\mathbf{A}$) of the structure state process and the reward rates. Detailed derivations presented in [11, 17] results in:

$$F_{ij}^{\sim *}(s,v) = \delta_{ij} \, \frac{r_i}{s + v r_i - a_{ii}} + \sum_{k\in R, k\neq i} \frac{a_{ik}}{s + v r_i - a_{ii}} F_{kj}^{\sim *}(s,v) \tag{8}$$

$$P_{ij}^{\sim *}(s,v) \;=\; \delta_{ij}\;\frac{s}{v(s+vr_i-a_{ii})}\;+\sum_{k\in R,k\neq i}\frac{a_{ik}}{s+vr_i-a_{ii}}\,P_{kj}^{\sim *}(s,v) \tag{9}$$

where δ_{ij} is the Kronecker delta.

The final expressions take the following matrix forms:

$$\mathbf{F}^{\sim *}(s,v) = (s\mathbf{I}+v\mathbf{R}-\mathbf{A})^{-1}\mathbf{R} \tag{10}$$

$$\mathbf{P}^{\sim *}(s,v) = \frac{s}{v}\,(s\mathbf{I}+v\mathbf{R}-\mathbf{A})^{-1} \tag{11}$$

where $^\sim$ denotes the Laplace-Stieltjes transform with respect to $t(\rightarrow s)$, * denotes the Laplace transform with respect to $w(\rightarrow v)$, $\mathbf{I}$ is the identity matrix and $\mathbf{R}$ is the diagonal matrix of the reward rates (r_i); the dimensions of $\mathbf{I}$, $\mathbf{R}$, $\mathbf{A}$, $\mathbf{F}$ and $\mathbf{P}$ are $(n\times n)$.

Starting from equations (10-11), the evaluation of the reward measures of a *MRM* requires the following steps:

1. symbolic evaluation of the entries of the $\mathbf{P}^{*\sim}(s,v)$ and $\mathbf{F}^{*\sim}(s,v)$ matrices in the double transform domain according to (10) and (11), which requires a symbolic inversion of an $n\times n$ size matrix;

2. symbolic inverse Laplace-Stieltjes transformation of $\mathbf{P}^{*\sim}(s,v)$ and/or $\mathbf{F}^{*\sim}(s,v)$ with respect to s;

3. numerical inverse Laplace transformation with respect to v;

4. unconditioning of the result by a numerical integration according to the distribution of the work requirement defined by (7).

However, this way of the analysis contains some computationally intensive steps, and the whole procedure can be applied to very small scale problems (less than 6-8 states) only.

3 Completion time analysis of *MRM*s

According to the associated reward rates the states of *MRM*s can be divided into two parts, namely S and $S^c=\Omega-S$, where S contains the states with positive reward rates and S^c with zero reward rates, i.e., $\forall i\in S,\ r_i>0$ and $\forall i\in S^c,\ r_i=0$. Suppose that S contains m states out of n. Thus we can renumber the states in Ω in a way that the states numbered $1,2,\ldots,m$ belong to S and the states numbered $m+1,m+2,\ldots,n$ belong to S^c. By this ordering of the states, $\mathbf{A}$ can be partitioned into the following form $\mathbf{A}=\begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_3 & \mathbf{A}_4\end{bmatrix}$, where $\mathbf{A}_1$ describes the transitions inside S, $\mathbf{A}_2$ contains the intensity of the transitions from S to S^c, $\mathbf{A}_3$ the transitions from S^c to S, and $\mathbf{A}_4$ the transitions inside S^c. Note that according to Definition 3 $Z(t)$ is an ergodic *CTMC*, hence the completion time of a finite work requirement w is finite with probability 1 and $\mathbf{A}_4^{-1}$ exists. By the renumbering of states the diagonal matrix of the reward rates has the form $\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where $\mathbf{R}_1 = \mathrm{Diag}_{i \in S} < r_i >$ is the diagonal matrix of the reward rates in S with cardinality $m \times m$.

3.1 Moments of the completion time of *MRM*s

In this section we calculate the moments of the completion time using the Laplace-Stieltjes transform, and we propose a recursive method to calculate the moments in a computationally effective way. We make use of the idea proposed by Iyer et al. for the analysis of the accumulated reward [10]. The nth moment of the completion time of w amount of work is defined by

$$M_{(n)}(w) = E\{C(w)^n\} = \int_{t=0}^{\infty} t^n \; d\,C(t,w) \; .$$

Theorem 1. *The nth moment of the completion time of an* MRM *with work requirement* w *is:*

$$M_{(n)}(w) = n! \; \underline{P}(0) \; \mathrm{LT}^{-1} \left[(\mathbf{R}v - \mathbf{A})^{-(n+1)} \mathbf{R} \right] \; \underline{h}^T \qquad (12)$$

where LT^{-1} *means the inverse Laplace transformation with respect to* v*.*

Proof: The moments can be calculated using the Laplace-Stieltjes transform of the completion time and substituting equation (10):

$$\begin{aligned} M_{(n)}(w) &= (-1)^n \left. \frac{\partial^n \, \mathrm{LT}^{-1} \left[C^{\sim *}(s,v) \right]}{\partial s^n} \right|_{s=0} \\ &= (-1)^n \left. \frac{\partial^n \, \mathrm{LT}^{-1} \left[\underline{P}(0) \; \mathbf{F}^{\sim *}(s,v) \; \underline{h}^T \right]}{\partial s^n} \right|_{s=0} \\ &= (-1)^n \underline{P}(0) \left. \frac{\partial^n \, \mathrm{LT}^{-1} \left[\mathbf{F}^{\sim *}(s,v) \right]}{\partial s^n} \, \underline{h}^T \right|_{s=0} \\ &= (-1)^n \underline{P}(0) \left. \frac{\partial^n \, \mathrm{LT}^{-1} \left[(s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1} \mathbf{R} \right]}{\partial s^n} \right|_{s=0} \underline{h}^T \, . \end{aligned} \qquad (13)$$

We assume in the above formula that the order of the inversion and the derivation can be changed:

$$M_{(n)}(w) = (-1)^n \underline{P}(0) \, \mathrm{LT}^{-1} \left[\left. \frac{\partial^n \, (s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1} \mathbf{R}}{\partial s^n} \right|_{s=0} \right] \, \underline{h}^T .$$

The derivation can be accomplished using Leibniz's rule, and setting the value of s to 0:

$$M_{(n)}(w) = n! \; \underline{P}(0) \; \mathrm{LT}^{-1} \left[(v\mathbf{R} - \mathbf{A})^{-(n+1)} \mathbf{R} \right] \; \underline{h}^T . \quad \blacksquare$$

3.2 Analysis of the mean completion time of *MRM*s

Because of the inverse Laplace transformation and matrix inversion contained in equation (12) the calculation of the moments is a computationally intensive task. Begain et al. [2] proposed an effective method to calculate the first moment, i.e., the mean value of the completion time of on-off reward models. Here we generalize that result for the mean completion time of *MRM*s with general reward structure.

Theorem 2. *The expected time while a* MRM *with general reward rates completes w amount of work is:*

$$E\{C(w)\} = \underline{P}(0) \begin{bmatrix} \mathbf{L}(w) & -\mathbf{L}(w)\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3\mathbf{L}(w) & -\mathbf{A}_4^{-1} + \mathbf{A}_4^{-1}\mathbf{A}_3\mathbf{L}(w)\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix} \underline{h}^T \ , \tag{14}$$

where

$$\mathbf{L}(w) = \int_0^w e^{u\mathbf{R}_1^{-1}\boldsymbol{\beta}} du \ \mathbf{R}_1^{-1} \quad \textit{and} \qquad \boldsymbol{\beta} = \mathbf{A}_1 - \mathbf{A}_2\mathbf{A}_4^{-1}\mathbf{A}_3.$$

Proof:

$$\begin{aligned} E\{C(w)\} &= \int_{t=0}^{\infty}(1 - C(t,w))\, dt = \int_{t=0}^{\infty} B(t,w)\, dt \\ &= \lim_{s\to 0} \frac{1}{s} B^{\sim}(s,w) = \lim_{s\to 0} \underline{P}(0)^T \mathbf{P}^{\sim}(s,w)\underline{h}^T \\ &= \underline{P}(0)\ \mathrm{LT}^{-1}\left[\frac{1}{v}\,(v\mathbf{R} - \mathbf{A})^{-1}\right]\ \underline{h}^T \ . \end{aligned} \tag{15}$$

Let us consider the term $\mathrm{LT}^{-1}\left[\frac{1}{v}\,(v\mathbf{R} - \mathbf{A})^{-1}\right]$ separately using the partitioned form $\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

$$\mathrm{LT}^{-1}\left[\frac{1}{v}\,(v\mathbf{R} - \mathbf{A})^{-1}\right] = \mathrm{LT}^{-1}\left\{\frac{1}{v}\begin{bmatrix} v\mathbf{R}_1 - \mathbf{A}_1 & -\mathbf{A}_2 \\ -\mathbf{A}_3 & -\mathbf{A}_4 \end{bmatrix}^{-1}\right\}$$

$$= \mathrm{LT}^{-1}\left\{\frac{1}{v}\begin{bmatrix} (v\mathbf{R}_1 - \boldsymbol{\beta})^{-1} & -(v\mathbf{R}_1 - \boldsymbol{\beta})^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3(v\mathbf{R}_1 - \boldsymbol{\beta})^{-1} & \mathbf{A}_4^{-1} + \mathrm{A}_4{}^{-1}\mathrm{A}_3(v\mathbf{R}_1 - \boldsymbol{\beta})^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix}\right\}$$

$$= \mathrm{LT}^{-1}\left\{\frac{1}{v}\begin{bmatrix} (v\mathbf{I}_1 - \mathbf{R}_1^{-1}\boldsymbol{\beta})^{-1}\mathbf{R}_1^{-1} & -(v\mathbf{I}_1 - \mathbf{R}_1^{-1}\boldsymbol{\beta})^{-1}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3(v\mathbf{I}_1 - \mathbf{R}_1^{-1}\boldsymbol{\beta})^{-1}\mathbf{R}_1^{-1} & \mathbf{A}_4^{-1} + \mathrm{A}_4{}^{-1}\mathrm{A}_3(v\mathbf{I}_1 - \mathbf{R}_1^{-1}\boldsymbol{\beta})^{-1}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \mathbf{L}(w) & -\mathbf{L}(w)\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3\mathbf{L}(w) & \mathbf{A}_4^{-1} + \mathbf{A}_4^{-1}\mathbf{A}_3\mathbf{L}(w)\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix} . \tag{16}$$

In (16), the first step is the partitioned description based on the block structure of matrices $\mathbf{A}$ and $\mathbf{R}$; the second step is the application of the inverse of a partitioned matrix ([3, 14]); the third step comes as the product of inverse matrices ([3, 14]); while the fourth step is because the Laplace transform of $\mathbf{L}(w)$ has the form

$$\mathbf{L}^*(v) = \frac{1}{v} \, (v\mathbf{I}_1 - \mathbf{R}_1^{-1}\boldsymbol{\beta})^{-1} \, \mathbf{R}_1^{-1}.$$

From (16) the theorem follows. ■

An intuitive proof of Theorem 2 is possible based on the interpretation of matrix $\boldsymbol{\beta}$. Define $Z'(t')$ a *CTMC* over S based on the original structure state process $Z(t) \in \Omega$ as follows:

$$Z'(t') = Z(t); \qquad \frac{dt'}{dt} = 1; \qquad \text{if } Z(t) \in S \ ,$$
$$\frac{dt'}{dt} = 0; \qquad \text{if } Z(t) \in \Omega - S \ ,$$

i.e., $Z'(t')$ takes the same state as $Z(t)$ when $Z(t) \in S$ and the clock t' is switched on (off) when $Z(t) \in S$ ($Z(t) \in \Omega - S$). $\boldsymbol{\beta}$ is the infinitesimal generator of *CTMC* $Z'(t')$ over S (with the usual properties: $\forall i,j \in S$, $\beta_{ij}|_{i\neq j} \geq 0$ and $\sum_{j\in S} \beta_{ij} = 0$). The multiplication with $\mathbf{R}_1^{-1}$ stands for scaling and rescaling the time providing a constant reward increment rate as proposed by Beaudry [1]. $Z'(t')$ is the stochastic process which characterize the reward accumulation as captured by $\mathbf{L}(w)$. The submatrices in (14) account for the time $Z(t)$ spends out of S.

3.3 A recursive analysis of higher moments

Here we propose a recursive method to calculate the higher moments. First we introduce some notation. Let $M_{ij\,(n)}(w)$ be the nth moment of the completion time assuming that the process was started in state i, the work requirement was completed in state j and the work requirement was w. Let $\mathbf{M}_{(n)}(w)$ be a matrix with entries $M_{ij\,(n)}(w)$, and $\mathbf{M}^*_{(n)}(v)$ be the Laplace transform of $\mathbf{M}_{(n)}(w)$. Let

$$\mathbf{F}^{\sim *\,(n)}(0,v) = \left. \frac{\partial^n \mathbf{F}^{\sim *}(s,v)}{\partial s^n} \right|_{s=0} .$$

Theorem 3. *The nth moment ($n \geq 2$) of the completion time of an* MRM *with work requirement w can be obtained as*

$$\begin{aligned} M_{(n)}(w) &= \underline{P}(0) \; \mathbf{M}_{(n)}(w) \; \underline{h}^T \\ &= n \; \underline{P}(0) \int_{y=0}^{w} \boldsymbol{\Theta}(w-y) \, \mathbf{M}_{(n-1)}(y) \; \underline{h}^T \; dy + n \; \underline{P}(0) \; \hat{\mathbf{A}} \, \mathbf{M}_{(n-1)}(w) \; \underline{h}^T \ , \end{aligned} \tag{17}$$

where

$$\boldsymbol{\Theta}(w) = \begin{bmatrix} e^{w\mathbf{R}_1^{-1}\boldsymbol{\beta}}\mathbf{R}_1^{-1} & -e^{w\mathbf{R}_1^{-1}\boldsymbol{\beta}}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3 e^{w\mathbf{R}_1^{-1}\boldsymbol{\beta}}\mathbf{R}_1^{-1} & \mathbf{A}_4^{-1}\mathbf{A}_3 e^{w\mathbf{R}_1^{-1}\boldsymbol{\beta}}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix} \quad \textit{and} \quad \hat{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}_4^{-1} \end{bmatrix}.$$

Proof: From equation (10)

$$(s\mathbf{I} + v\mathbf{R} - \mathbf{A})\mathbf{F}^{\sim *}(s, v) = \mathbf{R} \ . \tag{18}$$

Using Leibniz's rule, the differentiation of equation (18) $n+1$ times with respect to s and setting $s = 0$ yields

$$\mathbf{F}^{\sim * (n+1)}(0, v) = -(n+1)(\mathbf{R}v - \mathbf{A})^{-1}\mathbf{F}^{\sim * (n)}(0, v) \ . \tag{19}$$

Because $\mathbf{M}^*_{(n)}(v) = (-1)^n \mathbf{F}^{\sim * (n)}(0, v)$ according to equation (13), equation (19) can be rewritten as

$$\mathbf{M}^*_{(n+1)}(v) = (n+1)\,(\mathbf{R}v - \mathbf{A})^{-1}\mathbf{M}^*_{(n)}(v). \tag{20}$$

Let us consider the term $\mathrm{LT}^{-1}\left[(v\mathbf{R} - \mathbf{A})^{-1}\right]$ separately.

$$\mathrm{LT}^{-1}\left[(v\mathbf{R} - \mathbf{A})^{-1}\right] = \mathrm{LT}^{-1}\left\{ \begin{bmatrix} v\mathbf{R}_1 - \mathbf{A}_1 & -\mathbf{A}_2 \\ -\mathbf{A}_3 & -\mathbf{A}_4 \end{bmatrix}^{-1} \right\}$$

$$= \mathrm{LT}^{-1}\begin{bmatrix} (v\mathbf{R}_1 - \boldsymbol{\beta})^{-1} & (v\mathbf{R}_1 - \boldsymbol{\beta})^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3(v\mathbf{R}_1 - \boldsymbol{\beta})^{-1} & \mathbf{A}_4^{-1} + \mathbf{A}_4^{-1}\mathbf{A}_3(v\mathbf{R}_1 - \boldsymbol{\beta})^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix}$$

$$= \mathrm{LT}^{-1}\begin{bmatrix} (v\mathbf{I}_1 - \mathbf{R}_1^{-1}\boldsymbol{\beta})^{-1}\mathbf{R}_1^{-1} & (v\mathbf{I}_1 - \mathbf{R}_1^{-1}\boldsymbol{\beta})^{-1}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3(v\mathbf{I}_1 - \mathbf{R}_1^{-1}\boldsymbol{\beta})^{-1}\mathbf{R}_1^{-1} & \mathbf{A}_4^{-1} + \mathbf{A}_4^{-1}\mathbf{A}_3(v\mathbf{I}_1 - \mathbf{R}_1^{-1}\boldsymbol{\beta})^{-1}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} e^{w\mathbf{R}_1^{-1}\boldsymbol{\beta}}\mathbf{R}_1^{-1} & e^{w\mathbf{R}_1^{-1}\boldsymbol{\beta}}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3 e^{w\mathbf{R}_1^{-1}\boldsymbol{\beta}}\mathbf{R}_1^{-1} & \mathbf{A}_4^{-1}\,\delta(w) + \mathbf{A}_4^{-1}\mathbf{A}_3 e^{w\mathbf{R}_1^{-1}\boldsymbol{\beta}}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix} . \tag{21}$$

The steps in (21) are similar to the steps in (16); the only difference is that here we have the inverse Laplace transform of $(v\mathbf{I}_1 - \mathbf{R}_1^{-1}\boldsymbol{\beta})^{-1}\ \mathbf{R}_1^{-1}$ which is $e^{w\mathbf{R}_1^{-1}\boldsymbol{\beta}}\mathbf{R}_1^{-1}$.

Hence $\mathrm{LT}^{-1}\left[\,(v\mathbf{R} - \mathbf{A})^{-1}\right] = \boldsymbol{\Theta}(w) + \hat{\mathbf{A}}\delta(w)$, where $\delta(w)$ denotes the Dirac delta function (a Dirac impulse at $w = 0$), the inversion and the integration yields the theorem. ■

To apply the result of Theorem 3 for the evaluation of the first moment we shall define in accordance with equation (13)

$$M_{(0)}(w) = \mathrm{LT}^{-1}\left[C^{\sim *}(0, v)\right] = \mathrm{LT}^{-1}\left[\underline{P}(0)\ \mathbf{F}^{\sim *}(0, v)\ \underline{h}^T\right]$$

and

$$\mathbf{M}^*_{(0)}(v) = \mathbf{F}^{\sim *}(0, v).$$

To express the first moment we use equation (17) and then equation (20) to obtain

$$\begin{aligned} M_{(1)}(w) &= \mathrm{LT}^{-1}\left[\underline{P}(0)\ \mathbf{M}^*_{(1)}(v)\ \underline{h}^T\right] \\ &= \mathrm{LT}^{-1}\left[\underline{P}(0)\ (\mathbf{R}v - \mathbf{A})^{-1}\,\mathbf{M}^*_{(0)}(v)\ \underline{h}^T\right], \end{aligned}$$

which is by definition

$$\begin{aligned} M^{(1)}(w) &= \mathrm{LT}^{-1}\left[\underline{P}(0)\ (\mathbf{R}v - \mathbf{A})^{-1}\,\mathbf{F}^{\sim *}(0, v)\ \underline{h}^T\right] \\ &= \mathrm{LT}^{-1}\left[\underline{P}(0)\ (\mathbf{R}v - \mathbf{A})^{-2}\,\mathbf{R}\ \underline{h}^T\right] \\ &= \mathrm{LT}^{-1}\left[\underline{P}(0)\ \frac{1}{v}\,(\mathbf{R}v - \mathbf{A})^{-1}\ \underline{h}^T\right], \end{aligned}$$

since $(\mathbf{R}v - \mathbf{A})^{-2}\,\mathbf{R}\ \underline{h}^T = 1/v\ (\mathbf{R}v - \mathbf{A})^{-1}\ \underline{h}^T$, because $\mathbf{A}\,\underline{h}^T = \underline{0}^T$. The inverse transform gives the result of Theorem 2.

If the system is started from operational states, which is a rather realistic assumption, (i.e., $\forall i \in S^c, P_i(0) = 0$), then one can neglect the second term of equation (17). This term stands for the time needed to start the reward accumulation (i.e., to enter S) when the system starts from S^c.

Another important analysis problem of *MRM*s is the probability distribution of the structure state process at completion, i.e., $P^c_{ij} = Pr\{Z(C) = j | Z(0) = i\}$. For example, the required maintenance after a mission of a system which started from a particular state can be estimated based on this performance measure. A closed form expression of the probability distribution at completion, by which its effective computation is possible, comes by the following theorem.

Theorem 4. *The probability of being in state j at completion given that the process started from state i can be computed as follows:*

$$P^c_{ij} = \int_{w=0}^{\infty} \begin{bmatrix} e^{w\mathbf{R}_1^{-1}\boldsymbol{\beta}} & \mathbf{0} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3 e^{w\mathbf{R}_1^{-1}\boldsymbol{\beta}} & \mathbf{0} \end{bmatrix}_{ij} dG(w) \tag{22}$$

Proof: By the known transform domain measures we have:

$$\begin{aligned} P^c_{ij} &= \lim_{t\to\infty} \int_{w=0}^{\infty} F_{ij}(t, w)\, dG(w) = \lim_{s\to 0} \int_{w=0}^{\infty} F^{\sim}_{ij}(s, w)\, dG(w) \\ &= \int_{w=0}^{\infty} \left\{ \mathrm{LT}^{-1}\left[(v\mathbf{R} - \mathbf{A})^{-1}\right] \begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\}_{ij} dG(w) \end{aligned} \tag{23}$$

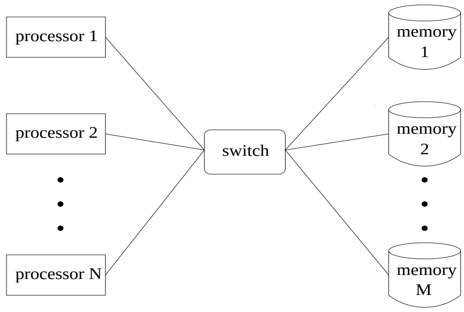

Figure 1: Example system structure

From (23) and (21) the theorem comes. ∎

From Theorem 4, $P^c_{ij} = 0$ if $j \in S^c$. It is the consequence of that the accumulated reward does not increase in S^c and the completion can not occur while $Z(t) \in S^c$.

4 Numerical Example

The results of this paper are demonstrated by the analysis of a simple multiprocessor system. The system is similar to the Carnegie-Mellon multiprocessor system, presented in [15]. The system consists of N processors, M memories, and an interconnection network (i.e., a crossbar switch) that allows any processor to access any memory (Figure 1). The failure rates per hour for the system are set to be 0.2, 0.1 and 0.05 for the processors, memories and the switch, respectively.

Viewing the interconnecting network as one switch and modeling the system at the processor-memory-switch level, the switch becomes essential for the system operation. It is also clear that a minimum number of processors and memories are necessary for the system to be operational. Each state is thus specified by a triple (i, j, k) indicating the number of operating processors, memories, and networks, respectively. We augment the states with the nonoperational state F. Events that decrease the number of operational units are the failures and events that increase the number of operational elements are the repairs. We assume that failures do not occur when the system is not operational. When a component fails, a recovery action must be taken (e.g., shutting down the a failed processor, etc.), or the whole system will fail and enter state F. The probability that the recovery action is successfully completed is known as *coverage*.

Two kinds of repair actions are considered, global repair which restores the system to state $(N, M, 1)$ with rate $\mu = 0.2$ per hour from state F, and local repair, which can be thought of as a repair person beginning to fix a component of the system as soon as a component failure occurs. We assume that there is only one repair person for each component type. Let the local repair rates be 2.0, 1.0 and 0.5 for the processors, memories and the switch, respectively.

The studied system has two processors, two memories, and one connections network, thus the state space consists of 13 states. For this case, the minimal configuration is supposed to have one processor, one memory and one interconnection switch. The value of the coverage was set to 0.90. This is a simple system, however a system of this size would be untractable using the double transformation method. We emphasize that it is just a demonstrative example, the performance of larger systems can also be calculated using the proposed method. More work has to be done to learn the limitations of the proposed method.

The mean value and the standard deviation of the completion time were calculated, the former using Theorem 2, the latter using Theorem 3 and the well known formula $\sigma(w) = (M_{(2)}(w) - (M_{(1)}(w))^2)^{1/2}$. The work requirement was chosen to take values from the interval $[1, 16]$ (in work hours). In Figures 2, 3 the mean value and the standard deviation of the completion time are shown, assuming that the system was started from the perfect state $(N, M, 1)$, from state F and from the steady state distribution. The integral values were calculated numerically in an iterative way. In each step twice as many sample points were evaluated, and the process was stopped when the maximal relative change of the values was less than 2%.

The mean completion time is higher if the system is started in the steady state instead of the perfect $(N, M, 1)$ state, or if the system is started in the F state instead of the steady state. The difference between the perfect and the F initial state curves refers to the mean time to get from state F to the perfect state. The curves of the standard deviation of the completion time show a similar picture. We have to note that the 2% accuracy limit brings more inaccuracy for higher values (8,16). The curve referring to the F state at time 0 takes the value of the standard deviation of the time to get from state F to the perfect state.

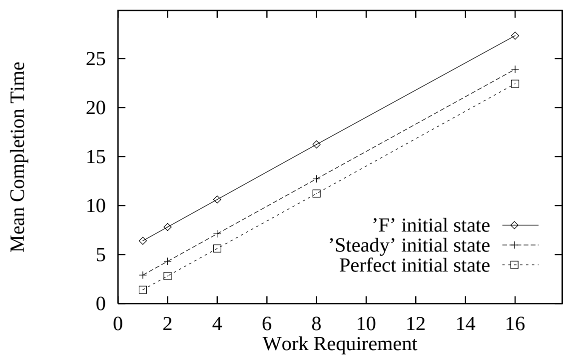

Figure 2: The mean value of the completion time

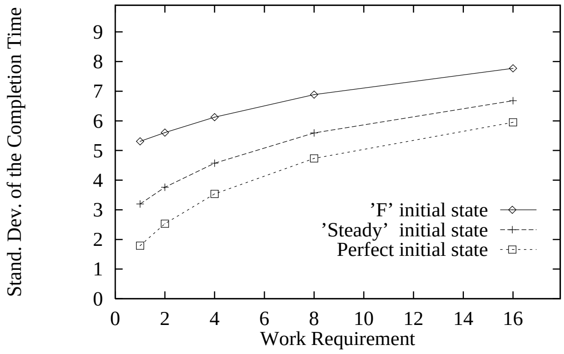

Figure 3: The standard deviation of the completion time

5 Conclusion

*MRM*s have been widely used to model performance and reliability of computer and communication systems. We discussed the analytical description of *MRM*s, allowing the assignment of 0 reward rates. A numerically effective computation method is described for the evaluation of the moments of the completion time of a *MRM*. Performance parameters of a Carnegie-Mellon multiprocessor system are evaluated by the proposed method as an application example.

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