Numerical Analysis of Large Markov Reward Models

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Abstract

First analysis of Markov Reward Models (*MRM*) resulted in a double transform expression, whose numerical solution is based on the inverse transformations both in time and reward variable domain. Better numerical methods were proposed based on the time domain properties of these models, such as the set of partial differential equation describes the process evolution in time.

This paper introduces an effective numerical method for the analysis of MRMs based on the transform domain description of the system, which allows the evaluation of models with large state space ($\sim 10^6$ states). The proposed method provides the moments of reward measures on the same computational cost and memory requirement as the transient analysis of the underlying Continuous Time Markov Chain and benefits from the advantages of the Randomization method, which avoids numerical instabilities and provides global error bound in advance of the computation. Implementation notes and numerical examples demonstrate the numerical properties of the proposed method are also provided.

Key words: Markov Reward Models, Performability, Completion Time, Randomization.

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1 Introduction

The stochastic reward processes have been studied since a long time [13,9], because the possibility of associating a reward variable to each system state increases the descriptive power and the modeling flexibility. However, only recently, stochastic reward models (*SRM*) have received attention as a modeling tool in performance evaluation of computer and communication systems. Common assignments of the reward rates are: execution rates of tasks in computing systems (the computational capacity) [1,20], number of active processors (or processing power) [3,8], throughput [14], available bandwidth [2], or average response time [12].

Two main different points of view have been assumed in the literature when dealing with *SRM* [11]. In the *system oriented* point of view the most significant measure is the total amount of work done by the system in a finite interval. This measure is often referred to as *performability* [14]. In the *user oriented* (or *task oriented*) point of view the system is regarded as a server, and the emphasis of the analysis is on the ability of the system to accomplish an assigned task in due time. Consequently, the most characterizing measure becomes the probability of accomplishing an assigned service in a given time.

A unified formulation to the system oriented and the user oriented point of view was provided in [11,16] together with the double Laplace transform expression of the completion time for the case when the underlying stochastic process $\mathcal{Z}(t)$ is a Continuous Time Markov Chain (*CTMC*). This case is referred to as Markov Reward Model (*MRM*).

Various numerical techniques were proposed for the evaluation of the system and the user oriented measures of *MRM*s. Some of these methods calculate the distribution of reward measures. The distribution, in double transform domain, can be obtained by a symbolic matrix inversion. If the size of the state space allows to obtain the solution of the symbolic matrix inversion then multi-dimensional numerical inverse transform methods [22] can provide the time domain results, but, due to the computational complexity of the symbolic inversion of matrices, this approach is not applicable for models with more than 20 states.

In time domain, reward measures can be described either by a set of equations with convolution integrals, or by a set of partial differential equations, but the numerical methods compute the distribution in time domain are usually based on the evaluation of a double summation, where both of the summation parameters increase to infinity. The discrete summations are obtained by adopting the randomization technique [19]. The randomization technique usually provides nice numerical properties and an overall error bound. The numerical methods based on this approach [6,5,15] differ in the complexity and memory requirement of one iteration step. The methods in [5,15] are with polynomial complexity with respect to the size of the state space.

*MRM*s with special features allow special, effective numerical approaches. In the case when the underlying *CTMC* has an absorbing state, in which no useful work is performed, it is easy to evaluate the limiting distribution of performability [1]. The numerical method in [7] makes use of a special structure of the underlying *CTMC*.

The numerical analysis of the distribution of reward measures is, in general, more complex than the computation of the moments of those measures. The mean of performability can be obtained by the transient analysis of the underlying *CTMC*. A numerical convolution approach is proposed in [10] to evaluated the $(n+1)$-th moment of performability based on its n-th moment. A similar approach is followed in [21] to calculate the moments of the user oriented measures, but the high computational complexity of the numerical convolution does not allow to apply this approach for the analysis of *MRM* with large (> 100) state spaces. Other *direct* methods make use of a spectral - or partial fraction decomposition, which is relatively easy for acyclic *CTMC*s, since the eigenvalues of the generator matrix are available in its diagonal [18]. The subclass of *MRM*s where the user has an associated Phase-type distributed random work requirement was studied in [4]. In this case the completion time is Phase type distributed, i.e., an "extended" *CTMC* can be defined which characterize the distribution of the completion time.

There are very few general numerical methods applicable for the reward analysis of *MRM*s with more than 10^5 states, while there are effective numerical methods to compute the steady state, the transient and the cumulative transient measures of large *CTMC*s [19,17]. It seems, only those reward measures of large *MRM*s can be evaluated which are associated (with simple computation) with the steady state, the transient or the cumulative transient measures of a *CTMC* of the same size.

In this paper, we provide a method based on the transform domain description of MRMs which allows the reward analysis of large models. Indeed, the proposed method evaluates each required moments of reward measures on the same computational cost as the transient analysis of the underlying CMTC, hence, it outperforms all the above mentioned general methods, at least, regarding the size of the models for which the numerical analysis is feasible.

The paper is organized as follows. Section 2 provides a summary of results about *MRM*s. In Section 3 the analysis of the accumulated reward while in Section 4 the completion time analysis of *MRM*s is presented. Section 5 gives some implementation issues of the proposed computational approach. In Section 6 two numerical examples are investigated and the paper is concluded in Section 7.

2 Markov Reward Models

In this section we provide the definitions and the well known results about MRMs, but following a different (may be simpler) way of reasoning than the one in the original papers.

Let $\{\mathcal{Z}(t),\ t \geq 0\}$ be a *CTMC* over the finite state space $S = \{1, 2, \ldots, M\}$ with generator $\mathbf{Q} = [q_{ij}]$ and initial distribution $\underline{P} = [p_i]$. A non-negative real constant $(r_i, i \in S)$ is associated to each state of the process representing the reward rate (the performance index) in state i. Let $\mathbf{R}$ be the diagonal matrix of the reward rates (i.e., $\mathbf{R} = \text{diag}(r_1, r_2, \ldots, r_M)$).

Let $\underline{\ell}(t) = [\ell_i(t)]$ denote the transient state probability vector ($\ell_i(t) = Pr\{\mathcal{Z}(t) = i\}$) and $\underline{L}(t) = [L_i(t)]$ denote the cumulative state probability vector ($L_i(t) = \int_0^t \ell_i(\tau) d\tau$). It is known that $\underline{\ell}(t) = \underline{P} e^{\mathbf{Q}t}$ and $\underline{L}(t) = \underline{P} \int_0^t e^{\mathbf{Q}\tau} d\tau$.

Definition 1 The **accumulated reward** $\mathcal{B}(t)$ is the random variable which represents the accumulation of reward in time:

$$\mathcal{B}(t) = \int_0^t r_{\mathcal{Z}(\tau)} \, d\tau \tag{1}$$

and

$$\mathcal{B}_i(t) = \int_0^t r_{\mathcal{Z}(\tau)} \, d\tau \ , \qquad \text{if} \ \ \mathcal{Z}(0) = i \ . \tag{2}$$

By this definition, $\mathcal{B}(t)$ is a stochastic process that depends on $\mathcal{Z}(u)$ for $0 \leq u \leq t$ and $\mathcal{B}(0) = 0$. According to Definition 1 this paper restricts the attention to the class of models in which no state transition can entail to a loss of the accumulated reward. This kind of process is called preemptive resume model. The distribution of the accumulated reward is defined by

$$B(t, w) = Pr\{\mathcal{B}(t) \leq w\} \tag{3}$$

and

$$B_i(t, w) = Pr\{\mathcal{B}_i(t) \leq w\} \ . \tag{4}$$

Note that

$$B(t,w) = \sum_{i \in S} p_i \; B_i(t,w) \; , \tag{5}$$

hence, in the rest of this paper, we use the initial state dependent measures and the global measures can always be evaluated by the mean of this relation.

Definition 2 The **completion time**, $\mathcal{C}_i$, is the random variable representing the time to accumulate the random amount of reward $\mathcal{W}$

$$\mathcal{C}_i = \min[t \geq 0 \; : \; \mathcal{B}_i(t) = \mathcal{W}] \; . \tag{6}$$

The distribution of $\mathcal{C}_i$ is

$$C_i(t) = Pr\{\mathcal{C}_i \leq t\} \; . \tag{7}$$

Let $\mathcal{C}_i(w)$ be the random variable representing the time to accumulate w (fix) amount of reward and $C_i(t,w)$ its distribution, i.e.,

$$\mathcal{C}_i(w) = \min[t \geq 0 \; : \; \mathcal{B}_i(t) = w] \; , \tag{8}$$

$$C_i(t,w) = Pr\{\mathcal{C}_i(w) \leq t\} \; . \tag{9}$$

Let $G(w)$ be the distribution of $\mathcal{W}$ with support on $[0,\infty)$. By Definition 2,

$$C_i(t) = \int_0^\infty C_i(t,w) \; dG(w) \; . \tag{10}$$

The distribution of the completion time is closely related to the distribution of the accumulated reward by the mean of the following relation (see Figure 1.)

$$B_i(t,w) = Pr\{\mathcal{B}_i(t) \leq w\} = Pr\{\mathcal{C}_i(w) \geq t\} = 1 - C_i(t,w) \; . \tag{11}$$

Theorem 1 *The column vector of the distribution of the accumulated reward ($\underline{B}(t,w) = [B_i(t,w)]$) is defined as follows:*

$$\underline{B}^{\sim}(t,v) = e^{(\mathbf{Q}-v\mathbf{R})t} \cdot \underline{h} \tag{12}$$

where $^\sim$ denotes the Laplace-Stieltjes transform with respect to $w(\rightarrow v)$, and $\underline{h}$ is the column vector with all the entries equal to 1.

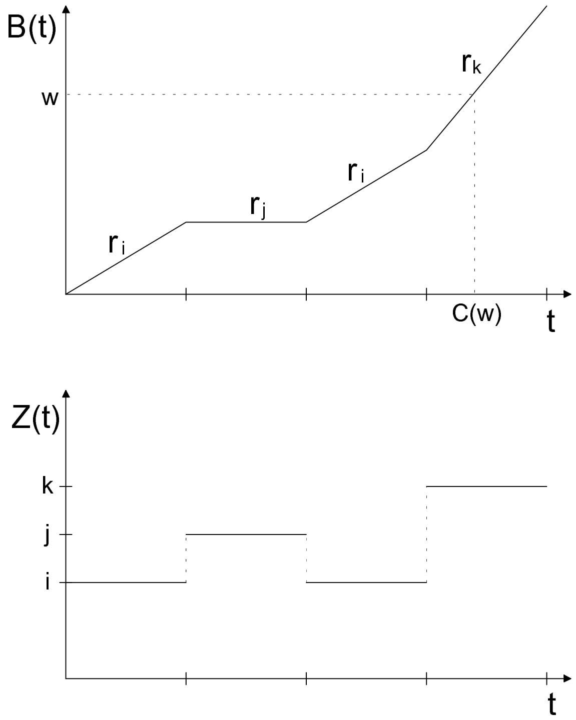

Fig. 1. A sample path of $\mathcal{Z}(t)$ and $\mathcal{B}(t)$.

Proof: Consider an exponentially distributed work requirement ($\mathcal{W}$) with parameter m. On the one hand, the completion time is characterized by the following distribution function

$$C_i(t) = \int_0^\infty C_i(t,w)\ dG(w) = \int_0^\infty \Big(1 - B_i(t,w)\Big)\ dG(w) \tag{13}$$

$$= m \int_0^\infty \Big(1 - B_i(t,x)\Big) e^{-mx}\ dx = 1 - B_i^\sim(t,v)\Big|_{v=m}$$

which, in vector form, is

$$\underline{C}(t) = \underline{h} - \underline{B}^\sim(t,v)\Big|_{v=m} . \tag{14}$$

On the other hand, $C_i(t)$ is phase type distributed and its distribution can be obtained by the representation of the phase type distribution (the original *CTMC* plus an absorbing state to which transition from state $i \in \mathcal{S}$ is at rate $m\ r_i$) [4]:

$$\underline{C}(t) = \underline{h} - e^{(\mathbf{Q} - m\mathbf{R})t} \cdot \underline{h} . \tag{15}$$

And since (12) is analytical for $\Re(v) \geq 0$ the theorem is given. □

A further Laplace-Stieltjes transform of (12) with respect to t results:

$$\underline{B}^{\sim\sim}(s,v) = s(s\mathbf{I} + v\mathbf{R} - \mathbf{Q})^{-1} \cdot \underline{h} \tag{16}$$

In order to simplify the transform domain expressions, in the rest of the paper, we apply the most convenient version of them using the $F^{\sim}(a) = aF^{*}(a)$ rule [2]. Detailed derivations in [10] resulted in the same expression for distribution of the accumulated reward based on different approaches. From (11), (16), using $\mathbf{Q} \cdot \underline{h} = \underline{0}$, we have:

$$\begin{aligned}
\underline{C}^{\sim\sim}(s,v) &= \underline{h} - \underline{B}^{\sim\sim}(s,v) \\
&= [\mathbf{I} - s(s\mathbf{I} + v\mathbf{R} - \mathbf{Q})^{-1}] \cdot \underline{h} \\
&= [(s\mathbf{I} + v\mathbf{R} - \mathbf{Q})^{-1} \cdot (s\mathbf{I} + v\mathbf{R} - \mathbf{Q}) - s(s\mathbf{I} + v\mathbf{R} - \mathbf{Q})^{-1}] \cdot \underline{h} \\
&= (s\mathbf{I} + v\mathbf{R} - \mathbf{Q})^{-1} \cdot (v\mathbf{R} - \mathbf{Q}) \cdot \underline{h} \\
&= v(s\mathbf{I} + v\mathbf{R} - \mathbf{Q})^{-1} \cdot \mathbf{R} \cdot \underline{h}
\end{aligned} \tag{17}$$

which was obtained with a different way of reasoning in [11]. Suppose $\mathbf{R}^{-1}$ exists, i.e., $r_i > 0, \forall i \in \mathcal{S}$, (17) can be inverse transformed with respect to the reward variable as follows:

$$\begin{aligned}
\underline{C}^{\sim *}(s,v) &= (s\mathbf{I} + v\mathbf{R} - \mathbf{Q})^{-1} \cdot (\mathbf{R}^{-1})^{-1} \cdot \underline{h} \\
&= (s\mathbf{R}^{-1} + v\mathbf{I} - \mathbf{R}^{-1}\mathbf{Q})^{-1} \cdot \underline{h} \ ,
\end{aligned} \tag{18}$$

from which

$$\underline{C}^{\sim}(s,w) = e^{(\mathbf{R}^{-1}\mathbf{Q} - s\mathbf{R}^{-1})w} \cdot \underline{h} \ . \tag{19}$$

A kind of duality can be observed comparing (12) and (19). Assume that $\{\mathcal{Z}'(w),\ w \geq 0\}$ is a *CTMC* over S with generator $\mathbf{Q}' = \mathbf{R}^{-1} \cdot \mathbf{Q}$ (which is a proper generator matrix). The mean reward accumulated up to time t (w) by $\mathcal{Z}(t)$ ($\mathcal{Z}'(w)$) with reward rate matrix $\mathbf{R}$ ($\mathbf{R}' = \mathbf{R}^{-1}$) can be evaluated

[2] E.g., $\underline{B}^{*\sim}(s,v) = (s\mathbf{I} + v\mathbf{R} - \mathbf{Q})^{-1} \cdot \underline{h}$ and $\underline{B}^{\sim *}(s,v) = \frac{s}{v}\,(s\mathbf{I} + v\mathbf{R} - \mathbf{Q})^{-1} \cdot \underline{h}$

by multiplying the cumulative state probabilities with the associated reward rates:

$$E\{\underline{\mathcal{B}}(t)\} = \int_0^t e^{\mathbf{Q}\,\tau}\,d\tau \cdot \mathbf{R} \cdot \underline{h} \quad \text{and} \quad E\{\underline{\mathcal{B}}'(w)\} = \int_0^w e^{\mathbf{Q}'\,\tau}\,d\tau \cdot \mathbf{R}' \cdot \underline{h}\ . \tag{20}$$

Now, by (12) and (19), one can see that the mean time to accumulate w unit of reward by $\mathcal{Z}(t)$ equals to $\underline{\mathcal{B}}'(w)$ and vice-versa, i.e.,

$$E\{\underline{\mathcal{C}}(w)\} = E\{\underline{\mathcal{B}}'(w)\} \quad \text{and} \quad E\{\underline{\mathcal{C}}'(t)\} = E\{\underline{\mathcal{B}}(t)\}\ . \tag{21}$$

Note that, we did not restrict the class of *MRM*s till (18), hence the results are valid for any reducible and irreducible underlying *CTMC* and any non-negative reward rates. In (18) – (21), the only restriction is that $\mathbf{R}$ must be invertable, i.e., strictly positive reward rates are only allowed.

3 Moments of the accumulated reward

Let $m_i^{(n)}(t) = E\{\mathcal{B}_i(t)^n\}$ be the n-th moment of the reward accumulated in $[0,t)$. The column vector $\underline{m}^{(n)}(t) = [m_i^{(n)}(t)]$ can be evaluated based on $\underline{B}^\sim(t,v)$ as

$$\underline{m}^{(n)}(t) = (-1)^n \frac{\partial^n \underline{B}^\sim(t,v)}{\partial v^n}\bigg|_{v=0}\ . \tag{22}$$

The following theorem provides a computationally effective, recursive method for the numerical analysis of the moments of accumulated reward.

Theorem 2 *The n-th moment ($n \geq 1$) of the accumulated reward is*

$$\underline{m}^{(n)}(t) = (-1)^n \sum_{i=0}^{\infty} \frac{t^i}{i!} \mathbf{N}^{(n)}(i) \cdot \underline{h} \tag{23}$$

where $\mathbf{N}^{(n)}(i)$ is defined as

$$\mathbf{N}^{(n)}(i) = \begin{cases} \mathbf{I}\ , & \text{if} \quad i = n = 0\ , \\ \mathbf{0}\ , & \text{if} \quad i = 0, n \geq 1\ , \\ \mathbf{Q}^i\ , & \text{if} \quad i \geq 1, n = 0\ , \\ \mathbf{Q} \cdot \mathbf{N}^{(n)}(i-1) - n\,\mathbf{R} \cdot \mathbf{N}^{(n-1)}(i-1)\ , & \text{if} \quad i \geq 1, n \geq 1\ . \end{cases} \tag{24}$$

To prove the theorem we need the following results.

Lemma 1 If $\mathbf{F}(t)$ and $\mathbf{G}(t)$ are real-valued, n times derivable matrix functions and $\mathbf{F}''(t) = \mathbf{0}$, then

$$(\mathbf{F}(t) \cdot \mathbf{G}(t))^{(n)} = \mathbf{F}(t) \cdot \mathbf{G}^{(n)}(t) + n\ \mathbf{F}'(t) \cdot \mathbf{G}^{(n-1)}(t), \quad n \geq 1\ . \tag{25}$$

Proof of Lemma 1

1. For $n = 1$

$$(\mathbf{F}(t) \cdot \mathbf{G}(t))' = \mathbf{F}(t) \cdot \mathbf{G}'(t) + \mathbf{F}'(t) \cdot \mathbf{G}(t) \tag{26}$$

holds.

2. Assuming (25) holds for $n = k$, it follows

$$\begin{aligned}(\mathbf{F}(t) \cdot \mathbf{G}(t))^{(k+1)} &= \sum_{l=0}^{k+1} \binom{k+1}{l} \mathbf{F}^{(l)}(t) \cdot \mathbf{G}^{(k+1-l)}(t) \\ &= \mathbf{F}(t) \cdot \mathbf{G}^{(k+1)}(t) + (k+1)\ \mathbf{F}'(t) \cdot \mathbf{G}^{(k)}(t)\end{aligned} \tag{27}$$

where the assumption for $n = k$ and $\mathbf{F}''(t) = \mathbf{0}$ is used. □

Lemma 2 If $i, n \geq 1$ then

$$\begin{aligned}&\frac{\partial^n}{\partial v^n}(\mathbf{Q} - v\mathbf{R})^i \Big|_{v=0} = \\ &\quad \mathbf{Q} \cdot \frac{\partial^n}{\partial v^n}(\mathbf{Q} - v\mathbf{R})^{i-1} \Big|_{v=0} - n\ \mathbf{R} \cdot \frac{\partial^{n-1}}{\partial v^{n-1}}(\mathbf{Q} - v\mathbf{R})^{i-1} \Big|_{v=0}\end{aligned} \tag{28}$$

Proof of Lemma 2 Let $\mathbf{F}(v) = (\mathbf{Q} - v\mathbf{R})$ and $\mathbf{G}(v) = (\mathbf{Q} - v\mathbf{R})^{i-1}$. From Lemma 1

$$\begin{aligned}&\frac{\partial^n}{\partial v^n}(\mathbf{Q} - v\mathbf{R})^i = \\ &\quad (\mathbf{Q} - v\mathbf{R}) \cdot \frac{\partial^n}{\partial v^n}(\mathbf{Q} - v\mathbf{R})^{i-1} - n\ \mathbf{R} \cdot \frac{\partial^{n-1}}{\partial v^{n-1}}(\mathbf{Q} - v\mathbf{R})^{i-1}\end{aligned} \tag{29}$$

which implies the Lemma. □

Proof of Theorem 2 From (22) and (12)

$$\begin{aligned} \underline{m}^{(n)}(t) &= (-1)^n \frac{\partial^n e^{(\mathbf{Q}-v\mathbf{R})t}}{\partial v^n}\bigg|_{v=0} \cdot \underline{h} \\ &= (-1)^n \frac{\partial^n}{\partial v^n} \sum_{i=0}^{\infty} \frac{t^i}{i!} (\mathbf{Q}-v\mathbf{R})^i \bigg|_{v=0} \cdot \underline{h} \\ &= (-1)^n \sum_{i=0}^{\infty} \frac{t^i}{i!} \;\; \frac{\partial^n}{\partial v^n} (\mathbf{Q}-v\mathbf{R})^i \bigg|_{v=0} \cdot \underline{h} \ . \end{aligned} \tag{30}$$

Let

$$\mathbf{N}^{(n)}(i) = \frac{\partial^n}{\partial v^n} (\mathbf{Q}-v\mathbf{R})^i \bigg|_{v=0} , \qquad \text{for } \forall n, i \geq 1. \tag{31}$$

From Lemma 2 it follows

$$\mathbf{N}^{(n)}(i) = \mathbf{Q} \cdot \mathbf{N}^{(n)}(i-1) - n \; \mathbf{R} \cdot \mathbf{N}^{(n-1)}(i-1), \tag{32}$$

with the initial conditions $\mathbf{N}^{(0)}(0) = \mathbf{I}$, $\mathbf{N}^{(0)}(i) = \mathbf{Q}^i$ and $\mathbf{N}^{(n)}(0) = \mathbf{0}$. By this recursion $\mathbf{N}^{(n)}(i) = \mathbf{0}$, if $i < n$. This completes the proof of Theorem 2. □

The iterative procedure to evaluate $\mathbf{N}^{(n)}(i)$ has the following properties:

- it is not possible to evaluate the nth moment itself, but to obtain the nth moment all the previous moments (or at least the associated $\mathbf{N}^{(n)}(i)$ terms) must be computed;
- matrix-matrix multiplications are computed in each iteration steps;
- numerical problems are possible due to the repeated multiplication with $\mathbf{Q}$, which contains both positive and negative elements, hence Theorem 2 is not directly applicable for numerical analysis.

4 Moments of the completion time

Let $s_i^{(n)}(w) = E\{\mathcal{C}_i(w)^n\}$ be the n-th moment of the time to accumulate w amount of reward. The column vector $\underline{s}^{(n)}(w) = [s_i^{(n)}(w)]$ can be evaluated based on $\underline{C}^{\sim}(s,w)$ as

$$\underline{s}^{(n)}(w) = (-1)^n \frac{\partial^n \underline{C}^{\sim}(s,w)}{\partial s^n}\bigg|_{s=0} . \tag{33}$$

Theorem 3 *The n-th moment of completion time, $\underline{s}^{(n)}(w)$, satisfies the following equation*

$$\underline{s}^{(n)}(w) = (-1)^n \sum_{i=n}^{\infty} \frac{w^i}{i!} \mathbf{M}^{(n)}(i) \cdot \underline{h} \tag{34}$$

where $\mathbf{M}^{(n)}(i)$ is defined as

$$\mathbf{M}^{(n)}(i) = \begin{cases} \mathbf{I}, & i = n = 0 \ , \\ \mathbf{0}, & i = 0, n \geq 1 \ , \\ (\mathbf{R}^{-1} \cdot \mathbf{Q})^i, & i \geq 1, n = 0 \ , \\ \mathbf{R}^{-1} \left(\mathbf{Q} \cdot \mathbf{M}^{(n)}(i-1) - n \ \mathbf{M}^{(n-1)}(i-1) \right), & i, n \geq 1 \ . \end{cases} \tag{35}$$

Proof of Theorem 3 Using

$$\underline{s}^{(n)}(w) = (-1)^n \frac{\partial^n}{\partial s^n} e^{(\mathbf{R}^{-1} \cdot \mathbf{Q} - s\mathbf{R}^{-1})w} \bigg|_{s=0} \cdot \underline{h} \tag{36}$$

the proof follows the same pattern as the proof of Theorem 2. □

The numerical method based on Theorem 3 has the same properties as the one based on Theorem 2. In contrast with Theorem 2, the application of Theorem 3 is restricted to *MRM*s with strictly positive reward rates, while, as in Theorem 2, we do not have restriction on the underlying *CTMC*.

4.1 System with zero reward rates

In case of some of the reward rates are zero Theorem 3 can not be applied for computing the moments of completion time. In this section we give a method which can handle this case.

Let us partition the state space $\mathcal{S}$ into two disjoint sets $\mathcal{S}_+$ and $\mathcal{S}_0$. $\mathcal{S}_+$ ($\mathcal{S}_0$) contains the states with associated positive (0) reward rate, i.e., $r_i > 0; \forall i \in \mathcal{S}_+$ and $r_i = 0; \forall i \in \mathcal{S}_0$. The accumulated reward does not increase during the sojourn in $\mathcal{S}_0$. If $\mathcal{S}_0$ has got an absorbing subset then the distribution of the completion time is defective, i.e., there is a positive probability that $\mathcal{C}_i(w) = \infty$. In the subsequent analysis we do not allow this case.

Without loss of generality, we number the states in $\mathcal{S}$ such that $i < j, \forall i \in \mathcal{S}_+, \forall j \in \mathcal{S}_0$ By this partitioning of the state space the reward rate and and the generator matrix have the following sub-block structure:

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} , \qquad \mathbf{Q} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{Q}_3 & \mathbf{Q}_4 \end{pmatrix} . \tag{37}$$

Note that $\mathbf{Q}_4$ is invertable as a consequence of the requirement that $\mathcal{S}_0$ has no absorbing subset. The partitioned form of the performance vectors are:

$$\underline{C}^{\sim\sim}(s,v) = \begin{pmatrix} \underline{C}_1^{\sim\sim}(s,v) \\ \underline{C}_2^{\sim\sim}(s,v) \end{pmatrix} , \qquad \underline{s}^{(n)}(w) = \begin{pmatrix} \underline{s}_1^{(n)}(w) \\ \underline{s}_2^{(n)}(w) \end{pmatrix} . \tag{38}$$

Theorem 4 *The n-th moment of completion time, $\underline{s}^{(n)}(w)$, can be computed as follows:*

$$\underline{s}_1^{(n)}(w) = (-1)^n \sum_{i=0}^{\infty} \frac{w^i}{i!} \mathbf{L}^{(n)}(i) \cdot \underline{h} \tag{39}$$

$$\underline{s}_2^{(n)}(w) = (-1)^n \sum_{i=0}^{\infty} \frac{w^i}{i!} \mathbf{H}^{(n)}(i) \cdot \underline{h} \tag{40}$$

where

$$\mathbf{L}^{(n)}(i) = \begin{cases} \mathbf{0} \, , & i = 0, n > 0 \, , \\ (\mathbf{R}_1^{-1} \cdot \mathbf{Q}_1 - \mathbf{R}_1^{-1} \cdot \mathbf{Q}_2 \cdot \mathbf{Q}_4^{-1} \cdot \mathbf{Q}_3)^i \, , & i \geq 0, n = 0 \, , \\ -\mathbf{R}_1^{-1} \cdot \mathbf{Q}_2 \cdot \mathbf{Q}_4^{-2} \cdot \mathbf{Q}_3 - \mathbf{R}_1^{-1} \, , & i = 1, n = 1 \, , \\ (-1)^{n+1} \, n! \, \mathbf{R}_1^{-1} \cdot \mathbf{Q}_2 \cdot \mathbf{Q}_4^{-n-1} \cdot \mathbf{Q}_3 \, , & i = 1, n \geq 2 \, , \\ \sum_{\ell=0}^{n} \binom{n}{\ell} \mathbf{L}^{(\ell)}(1) \cdot \mathbf{L}^{(n-\ell)}(i-1) \, , & i \geq 2, n \geq 1 \, , \end{cases} \tag{41}$$

$$\mathbf{H}^{(n)}(i) = (-1) \sum_{\ell=0}^{n} \binom{n}{\ell} \ell! \, \mathbf{Q}_4^{-(\ell+1)} \cdot \mathbf{Q}_3 \cdot \mathbf{L}^{(n-\ell)}(i) \, , \quad i \geq 0, n \geq 0 \tag{42}$$

Proof of Theorem 4 Substituting the vectors and matrices in (17) with their partitioned form and using the following form of matrix inverse

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \end{pmatrix}$$

where

$$\mathbf{A} = s\mathbf{I}_1 + v\mathbf{R}_1 - \mathbf{Q}_1 \ , \qquad \mathbf{B} = -\mathbf{Q}_2 \ ,$$
$$\mathbf{C} = -\mathbf{Q}_3 \ , \qquad \mathbf{D} = s\mathbf{I}_4 - \mathbf{Q}_4$$

for $\underline{C}_1^{\sim\sim}(s,v)$ we have:

$$\underline{C}_1^{\sim\sim}(s,v) = v[s\mathbf{I}_1 + v\mathbf{R}_1 - \mathbf{Q}_1 - \mathbf{Q}_2 \cdot (s\mathbf{I}_4 - \mathbf{Q}_4)^{-1} \cdot \mathbf{Q}_3]^{-1} \cdot \mathbf{R}_1 \cdot \underline{h} \ . \tag{43}$$

Since ${\mathbf{R}_1}^{-1}$ exists by its definition the inverse Laplace transform of (43) with respect to $v \rightarrow w$ gives

$$\underline{C}_1^{\sim}(s,w) = e^{\boldsymbol{\alpha}(s)w} \quad \cdot \underline{h} = \sum_{i=0}^{\infty} \frac{\boldsymbol{\alpha}(s)^i}{i!} w^i \cdot \underline{h} \tag{44}$$

where

$$\boldsymbol{\alpha}(s) = \mathbf{R}_1^{-1} \cdot \mathbf{Q}_1 + \mathbf{R}_1^{-1} \cdot \mathbf{Q}_2 \cdot (s\mathbf{I}_4 - \mathbf{Q}_4)^{-1} \cdot \mathbf{Q}_3 - s\mathbf{R}_1^{-1} \ . \tag{45}$$

The n-th moment of completion time is

$$\underline{s}_1^{(n)}(w) = (-1)^n \frac{\partial^n}{\partial s^n} \underline{C}_1^{\sim}(s,w)\bigg|_{s=0} = (-1)^n \sum_{i=0}^{\infty} \frac{w^i}{i!} \frac{\partial^n}{\partial s^n} \boldsymbol{\alpha}(s)^i \bigg|_{s=0} \cdot \underline{h} \tag{46}$$

where the n-th deviate of $\boldsymbol{\alpha}(s)^i$ can be evaluated using the Leibniz rule

$$(\boldsymbol{\alpha}(s) \cdot \boldsymbol{\alpha}(s)^{i-1})^{(n)} = \sum_{l=0}^{n} \binom{n}{l} \boldsymbol{\alpha}(s)^{(l)} \cdot \left(\boldsymbol{\alpha}\,(s)^{i-1}\right)^{(n-l)} \ . \tag{47}$$

Now $\mathbf{L}^{(n)}(i) = \dfrac{\partial^n}{\partial s^n} \boldsymbol{\alpha}(s)^i \Big|_{s=0}$, completes the proof for $\underline{s}_1^{(n)}(w)$.

The same partitioning of (17) gives

$$\begin{aligned} \underline{C}_2^{\sim}(s,w) &= (s\mathbf{I}_4 + \mathbf{Q}_4)^{-1} \cdot \mathbf{Q}_3 \cdot \underline{C}_1(s,w) \\ &= \sum_{i=0}^{\infty} \frac{w^i}{i!} (s\mathbf{I}_4 + \mathbf{Q}_4)^{-1} \cdot \mathbf{Q}_3 \cdot \boldsymbol{\alpha}(s)^i \cdot \underline{h} \end{aligned} \tag{48}$$

and applying the Leibniz-rule as before:

$$\underline{s}_2^{(n)}(x) = (-1)^n \cdot \frac{\partial^n}{\partial s^n} \underline{C}_2^{\sim}(s,x)\bigg|_{s=0} = (-1)^n \sum_{i=0}^{\infty} \frac{w^i}{i!} \mathbf{H}^{(n)}(i) \cdot \underline{h} \tag{49}$$

gives the theorem. $\square$

5 Numerical methods based on randomization

In the previous sections iterative procedures were provided to compute the moments of reward measures, but due to the properties of digital computers using floating point numbers a direct application of those methods would result in numerical problems such as instabilities, "ringing" (negative probabilities), etc. The main reason of these problems is that matrices with positive and negative elements (like $\mathbf{Q}$) are multiplied several times. To avoid these problems a modified procedure is proposed. Let

$$\mathbf{A} = \frac{\mathbf{Q}}{q} + \mathbf{I} \,, \qquad \mathbf{S} = \frac{\mathbf{R}}{qd} \tag{50}$$

where $q = \max_{i,j \in \mathcal{S}} (|q_{ij}|)$ and $d = \max_{i \in \mathcal{S}} (r_i)/q$. By this definition $\mathbf{A}$ is a stochastic matrix ($0 \leq a_{i,j} \leq 1, \forall i, j \in \mathcal{S}$ and $\sum_{j \in \mathcal{S}} a_{i,j} = 1, \forall i \in \mathcal{S}$) and $\mathbf{S}$ is a diagonal matrix such that $0 \leq s_{i,i} \leq 1, \forall i \in \mathcal{S}$. The dimension of d is unit of reward. d can be considered as a scaling factor of the accumulated reward. Using these matrices

$$\underline{B}^{\sim}(t, v) = e^{(\mathbf{Q} - v\mathbf{R})t} \cdot \underline{h} = e^{(\mathbf{A} - vd\mathbf{S})qt} \cdot \underline{h} e^{-qt} \ . \tag{51}$$

Theorem 5 *The moments of accumulated reward can be computed using only matrix-vector multiplications and saving only vectors of size $\#\mathcal{S}$ in each step of the iteration as*

$$\underline{m}^{(n)}(t) = n! \; d^n \; \sum_{i=0}^{\infty} \underline{U}^{(n)}(i) \frac{(qt)^i}{i!} e^{-qt} \tag{52}$$

where

$$\underline{U}^{(n)}(i) = \begin{cases} \underline{0} \ , & \text{if} \quad i = 0, n \geq 1 \ , \\ \underline{h} \ , & \text{if} \quad i \geq 0, n = 0 \ , \\ \mathbf{A} \cdot \underline{U}^{(n)}(i-1) + \mathbf{S} \cdot \underline{U}^{(n-1)}(i-1) \ , & \text{if} \quad i \geq 1, n \geq 1 \ . \end{cases} \tag{53}$$

Proof of Theorem 5 Starting from (51) the proof of Theorem 5 follows the same pattern as the proof of Theorem 2. $\square$

To demonstrate the iterative procedure of computing $\underline{U}^{(n)}(i)$ the first elements of $\underline{U}^{(n)}(i)$ evaluated based on (53) are provided in Table 1.

$\underline{U}^{(n)}(i)$	i=0	i=1	i=2	i=3
n=0	$\underline{h}$	$\underline{h}$	$\underline{h}$	$\underline{h}$
n=1	$\underline{0}$	$\mathbf{S}\underline{h}$	$\mathbf{AS}\underline{h}+\mathbf{S}\underline{h}$	$\mathbf{AAS}\underline{h}+\mathbf{AS}\underline{h}+\mathbf{S}\underline{h}$
n=2	$\underline{0}$	$\underline{0}$	$\mathbf{SS}\underline{h}$	$\mathbf{ASS}\underline{h}+\mathbf{SAS}\underline{h}+\mathbf{SS}\underline{h}$
n=3	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\mathbf{SSS}\underline{h}$

Table 1.

Suppose one is interested in the first 3 moments of the accumulated reward. To perform the computation 3 vectors of size $\#\mathcal{S}$ needs to store $\underline{U}^{(n)}(i), n = 1, 2, 3$. In each iteration step $i = 1, 2, 3, \ldots$ matrix-vector multiplications and vector summations has to be performed according to (53) using the vectors of the previous iteration step and the constant matrices $\mathbf{A}$ and $\mathbf{S}$. Figure 2. shows the dependency structure of the computation. One can recognize that only the $(i-1)$-th column (iteration) of $\underline{U}$ is used for calculating the i-th column of $\underline{U}$. Note that $\mathbf{S}$ is a diagonal matrix and $\mathbf{A}$ is as sparse as $\mathbf{Q}$ is. Further 3

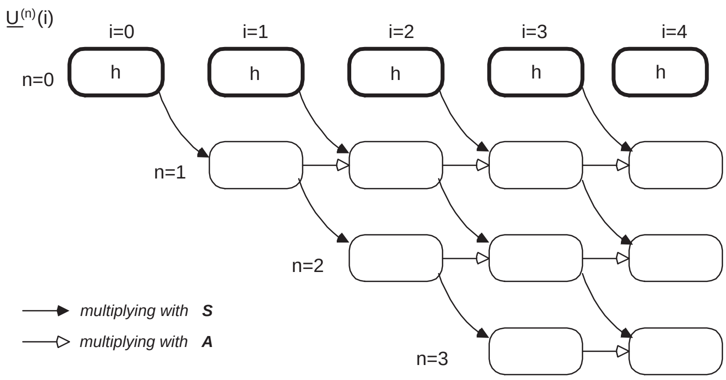

Fig. 2. The dependency structure of the iteration steps

vectors of the same size need to store the "actual value" of $\underline{m}^{(n)}(t), n = 1, 2, 3$ according to (52).

The following theorem provides a global error bound of the procedure.

Theorem 6 *The n-th moment of accumulated reward can be calculated as a finite sum and an error part, where the maximum allowed error is* ε

$$\underline{m}^{(n)}(t) = n!\ d^n \sum_{i=0}^{G-1} \underline{U}^{(n)}(i) \frac{(qt)^i}{i!} e^{-qt} + \underline{\xi}(G) \tag{54}$$

where

$$G = \min_{g \in \mathbb{N}} \left((qt)\ n!\ d^n \sum_{i=g-1}^{\infty} \frac{(qt)^i}{i!} e^{-qt} \leq \varepsilon \right) \tag{55}$$

and the $\underline{0} \le \underline{\xi}(G) \le \underline{h}\ \varepsilon$ *inequality holds for all the elements of the vectors.*

Proof of Theorem 6 By the definition of $\mathbf{S}$ and $\mathbf{A}$

$$\underline{0} \le \mathbf{S} \cdot \underline{h} \ \le \ \underline{h} \quad \text{and} \quad \underline{0} \le \mathbf{A} \cdot \mathbf{S} \cdot \underline{h} \ \le \ \underline{h} \tag{56}$$

hold piece-wise (as all the subsequent vector inequalities), hence $\underline{U}^{(n)}(i)$ is bounded by

$$\underline{0} \le \underline{U}^{(n)}(i) \le i\ \underline{h}. \tag{57}$$

The error $\underline{\xi}(g)$ incurred when eliminating the tale of the infinite sum is also bounded by

$$\begin{aligned}\underline{\xi}(g) &= n!\ d^n \sum_{i=g}^{\infty} \underline{U}^{(n)}(i)\ \frac{(qt)^i}{i!} e^{-qt} \le n!\ d^n \sum_{i=g}^{\infty} \underline{h}\ i\ \frac{(qt)^i}{i!} e^{-qt} \\ &\le (qt)\ n!\ d^n \sum_{i=g-1}^{\infty} \underline{h}\ \frac{(qt)^i}{i!} e^{-qt}\end{aligned} \tag{58}$$

which gives the theorem. □

The error bound provided by the theorem is the tail of a Poisson distribution with mean qt multiplied by a constant $(qt)\ n!\ d^n$. A Poisson distribution has a low squared coefficient of variation $(qt)^{-1}$, which decreases as qt increases, and its tail has an exponential decay. Hence, when qt is large (> 100) G is mainly determined by qt and it has only a logarithmic dependence on the constant $(qt)\ n!\ d^n$ and the precision requirement ε. In general, if $qt > 100$ then G and qt are of the same order of magnitude ($G > qt$). A high level description of the proposed method can be found in the appendix.

The same approach can be applied for the analysis of completion time, when all the reward rates are positive, i.e., $\mathbf{R}^{-1}$ exists. Let

$$\mathbf{B} = \frac{\mathbf{R}^{-1} \cdot \mathbf{Q}}{z} + \mathbf{I}\ , \qquad \mathbf{T} = \frac{\mathbf{R}^{-1}}{zf} \tag{59}$$

where $z = \max_{i,j \in \mathcal{S}} (|q_{ij}/r_i|)$ and $f = \max_{i \in \mathcal{S}}(1/r_i)/z$. By this definition $\mathbf{B}$ is a stochastic matrix ($0 \le b_{i,j} \le 1, \forall i,j \in \mathcal{S}$ and $\sum_{j \in \mathcal{S}} b_{i,j} = 1, \forall i \in \mathcal{S}$) and $\mathbf{T}$ is a diagonal matrix such that $0 \le t_{i,i} \le 1, \forall i \in \mathcal{S}$. f is a number with no dimension.

$$\underline{C}^{\sim}(s,w) = e^{(\mathbf{R}^{-1}\mathbf{Q} - s\mathbf{R}^{-1})w} \cdot \underline{h} \ = e^{(\mathbf{B} - sf\mathbf{T})zw} \cdot \underline{h}\ e^{-zw}\ . \tag{60}$$

Theorem 7 *The moments of the completion time can be computed using only matrix-vector multiplications and saving only vectors of size $\#\mathcal{S}$ as follows:*

$$\underline{s}^{(n)}(w) = n!\ f^n\ \sum_{i=0}^{\infty} \underline{V}^{(n)}(i) \frac{(zw)^i}{i!} e^{-zw} \tag{61}$$

where

$$\underline{V}^{(n)}(i) = \begin{cases} \underline{0} & \textit{if} \quad i = 0, n \geq 1\ , \\ \underline{h} & \textit{if} \quad i \geq 0, n = 0\ , \\ \mathbf{B} \cdot \underline{V}^{(n)}(i-1) + \mathbf{T} \cdot \underline{V}^{(n-1)}(i-1) & \textit{if} \quad i \geq 1, n \geq 1\ . \end{cases} \tag{62}$$

Proof of Theorem 7 From (60), Theorem 7 comes. □

Theorem 8 *The n-th moment of completion time can be calculated as a finite sum and an error part, where the maximum allowed error is ε*

$$\underline{s}^{(n)}(w) = n!\ f^n\ \sum_{i=0}^{G-1} \underline{V}^{(n)}(i) \frac{(zw)^i}{i!} e^{-zw} + \underline{\xi}(G) \tag{63}$$

$$\textit{where} \qquad G = \min_{g \in \mathbb{N}} \left((zw)\ n!\ f^n\ \sum_{i=g-1}^{\infty} \frac{(zw)^i}{i!} e^{-zw} \leq \varepsilon \right) \tag{64}$$

$$\textit{and} \qquad \underline{0} \leq \underline{\xi}(G) \leq \underline{h}\ \varepsilon\ . \tag{65}$$

Proof of Theorem 8 The proof of Theorem 8 follows the same pattern as the proof of Theorem 6. □.

The numerical analysis of the completion time of large models when states with zero reward rate are present in the system is more complicated. A numerical procedure similar to the one in Theorem 8 can be obtained as well, but on the one hand it is very complicated, and on the other hand its applicability is strongly limited by the cardinality of $\mathcal{S}_0$. The $\mathbf{Q}_4$ matrix of cardinality $\#\mathcal{S}_0$ has to be inverted in this case. In general, the complexity of inverting a matrix of cardinality 10^4 has higher computational complexity and memory requirement than the proposed numerical method with 10^6 states.

6 Numerical examples

Example 1
Consider a *CTMC* with $n = 1,000,000$ states. Let the non-zero state transition

Mean value	$t = 0.02s$	$t = 0.1s$	$t = 0.2s$	$t = 1s$	$t = 2s$
$\mathcal{Z}(0) = 750,000$	$8.06 \cdot 10^{-12}$	$9.81 \cdot 10^{-8}$	$5.11 \cdot 10^{-6}$	0.022	0.33
$\mathcal{Z}(0) = 790,000$	0.00047	0.010	0.037	0.58	1.54
$\mathcal{Z}(0) = 800,000$	0.019	0.093	0.18	0.94	1.94

Table 2.

Variance	$t = 0.02s$	$t = 0.1s$	$t = 0.2s$	$t = 1s$	$t = 2s$
$\mathcal{Z}(0) = 750,000$	$4.61 \cdot 10^{-14}$	$2.73 \cdot 10^{-9}$	$5.03 \cdot 10^{-7}$	$7.73 \cdot 10{-}3$	0.17
$\mathcal{Z}(0) = 790,000$	$6.07 \cdot 10^{-6}$	$5.85 \cdot 10^{-4}$	$3.62 \cdot 10^{-3}$	0.096	0.16
$\mathcal{Z}(0) = 800,000$	$5.79 \cdot 10^{-6}$	$4.13 \cdot 10^{-4}$	$1.92 \cdot 10^{-3}$	0.018	0.022

Table 3.

rates the following:

$$q_{ij} = \begin{cases} 5 \ , & \text{if} \quad j = i + 1 \ , \\ 2.5 \ , & \text{if} \quad j = i + 10,000 \ , \\ 2.5 \ , & \text{if} \quad j = i - 1 \ . \end{cases} \tag{66}$$

The reward rate matrix $\mathbf{R}$ has the following structure:

$$r_{i,i} = \begin{cases} 0 & \text{if} \quad i < 800,000 \ , \\ 1 & \text{if} \quad i \geq 800,000 \ . \end{cases} \tag{67}$$

Figure 3. shows the structure of the underlying *CTMC*, where $u = 10,000$.

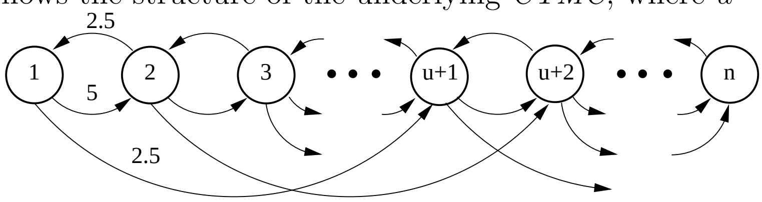

Fig. 3. The underlying *CTMC* of Example 1.

Table 2 and 3 contain the mean and the variance of the accumulated reward with different initial state. The accumulated reward represents the time the system spent in states $800,000, \ldots, 1,000,000$.

Example 2
In the second example, the performance parameters of a Carnegie-Mellon multiprocessor system are evaluated by the proposed method. The system is similar to the one presented in [18]. The system consists of N processors, M

memories, and an interconnection network (composed by switches) that allows any processor to access any memory (Figure 4). The failure rates per hour for the system are set to be 0.1, 0.05, 0.01 and 0.003 for the processors, memories, switches, and general failure, respectively.

Viewing the interconnecting network as S switches and modeling the system at the processor-memory-switch level, the system performance depends on the minimum of the number of operating processors, memories, and switches. Each state is thus specified by a triple (i, j, k) indicating the number of operating processors, memories, and switches, respectively. We augment the states with the nonoperational state F. Events that decrease the number of operational units are the failures and events that increase the number of operational elements are the repairs. We assume that failures do not occur when the system is not operational. When a component fails, a recovery action must be taken (e.g., shutting down the a failed processor, etc.), or the whole system will fail and enter state F.

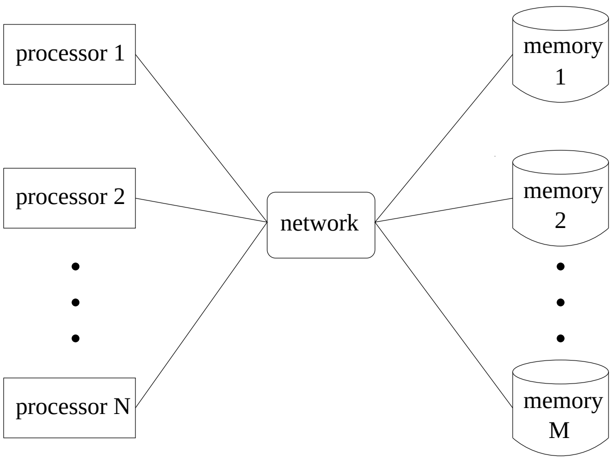

Fig. 4. Example system structure

Two kinds of repair actions are possible, global repair which restores the system to state (N, M, S) with rate $\mu = 0.01$ per hour from state F, and local repair, which can be thought of as a repair person beginning to fix a component of the system as soon as a component failure occurs. We assume that there is only one repair person for each component type. Let the local repair rates be 2.0, 2.0 and 0.1 for the processors, memories and the switch, respectively.

The system starts from the perfect state (N, M, S). The studied system has 32 processors, 64 memories, and 16 switches, thus the state space consists of 36,466 states (247,634 transitions). The performance of the system is proportional to the number of cooperating processors and memories, whose cooperation is provided by one switch. The reward rate is defined as the minimum of the operational processors, memories, and switches. The minimal operational configuration is supposed to have one processor, one memory and one interconnection switch.

t	$E(B(t))$	$E(B(t)^2)$	$E(B(t)^3)$	$E(B(t)^4)$	$E(B(t)^5)$	$E(B(t)^6)$
1	15.89	253.0	4030	$6.41 \cdot 10^4$	$1.02 \cdot 10^6$	$1.63 \cdot 10^7$
2	31.60	1001	$3.14 \cdot 10^4$	$1.00 \cdot 10^6$	$3.19 \cdot 10^7$	$1.01 \cdot 10^9$
5	77.70	6072	$4.75 \cdot 10^5$	$3.72 \cdot 10^7$	$2.92 \cdot 10^9$	$2.30 \cdot 10^{11}$
10	151.5	$2.32 \cdot 10^4$	$3.57 \cdot 10^6$	$5.51 \cdot 10^8$	$8.52 \cdot 10^{10}$	$1.31 \cdot 10^{13}$
20	289.5	$8.57 \cdot 10^4$	$2.55 \cdot 10^7$	$7.67 \cdot 10^9$	$2.30 \cdot 10^{12}$	$6.96 \cdot 10^{14}$
50	648.0	$4.42 \cdot 10^5$	$3.08 \cdot 10^8$	$2.16 \cdot 10^{11}$	$1.53 \cdot 10^{14}$	$1.09 \cdot 10^{17}$

Table 4.

t	$E(B(t))$	$E(B(t)^2)$	$E(B(t)^3)$	$E(B(t)^4)$	$E(B(t)^5)$	$E(B(t)^6)$
1	15.89	253.0	4030	$6.42 \cdot 10^4$	$1.02 \cdot 10^6$	$1.63 \cdot 10^7$
2	31.60	1001	$3.14 \cdot 10^4$	$1.00 \cdot 10^6$	$3.19 \cdot 10^7$	$1.01 \cdot 10^9$
5	77.70	6073	$4.75 \cdot 10^5$	$3.72 \cdot 10^7$	$2.92 \cdot 10^9$	$2.30 \cdot 10^{11}$
10	151.6	$2.32 \cdot 10^4$	$3.57 \cdot 10^6$	$5.51 \cdot 10^8$	$8.52 \cdot 10^{10}$	$1.31 \cdot 10^{13}$
20	290.1	$8.59 \cdot 10^4$	$2.56 \cdot 10^7$	$7.68 \cdot 10^9$	$2.31 \cdot 10^{12}$	$6.97 \cdot 10^{14}$
50	655.6	$4.48 \cdot 10^5$	$3.11 \cdot 10^8$	$2.19 \cdot 10^{11}$	$1.55 \cdot 10^{14}$	$1.10 \cdot 10^{17}$

Table 5.

The first 6 moments of the accumulated reward were calculated using Theorem 5 in two different cases. In the first case global repair was not possible, hence F was an absorbing state of the system. In the second case global repair was allowed at rate 0.01. Table 4 and 5 contain the results obtained at time $t = 1, 2, 5, 10, 20, 50$ for the case without and with global repair, respectively.

The mean and the variance of the accumulated reward of the two cases are compared in Figures 5, and 6, respectively. The dashed lines refer to the case when global repair is not possible. As it was expected, the mean accumulated reward of the case without global repair is less. The variance curves are misleading for the first sight. The second moment of the case without global repair is still less, but the relation of the variance parameters depend on the difference of the first two moments, and that is why the variance of the case without global repair is higher.

7 Conclusion

An iterative numerical method is introduced which can evaluate the moments of the accumulated reward and the completion time of *MRM*s with large 10^6

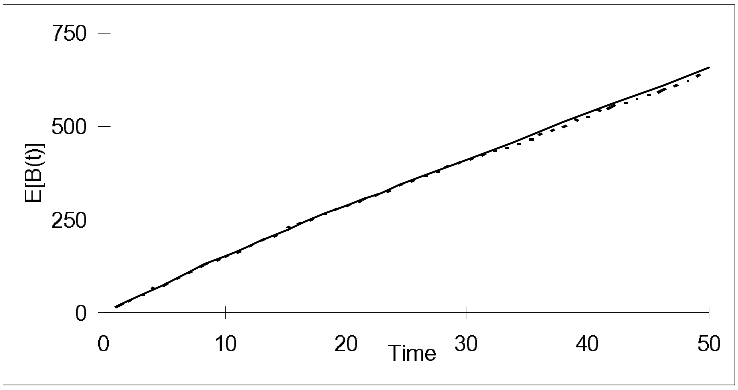

Fig. 5. Mean accumulated reward

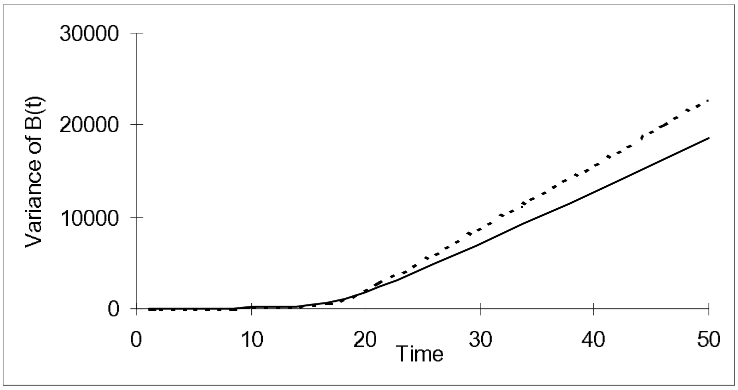

Fig. 6. Variance of the accumulated reward

state spaces. The proposed methods make use of the randomization technique, hence they are numerically stabile and allow the implementation of a global error bound.

A possible future extension of the proposed method is the automatic steady state detection. The computational complexity increases linearly with the time (in case of accumulated reward analysis) or with the work requirement w (in case of completion time analysis), but after the underlying $CTMC$ reached its steady state the reward measures can be computed in a simpler way.

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A Implementation of the numerical method

A formal description of the program that calculates the moments of accumulated reward according to Theorem 4 is provided. The memory requirement and number of required operations are calculated in advance.

Input	M	CARDINALITY OF THE STATE SPACE
	$\mathbf{Q}$	GENERATOR MATRIX OF UNDERLYING CTMC
	$\mathbf{R}$	DIAGONAL MATRIX OF THE REWARD RATES
	$\underline{P}$	INITIAL PROBABILITY VECTOR
	t	TIME OF ACCUMULATION
	n	ORDER OF MOMENT
	G	NUMBER OF ITERATIONS
	z	NUMBER OF NON-ZERO ELEMENTS IN $\mathbf{Q}$
Output	m	THE n-TH MOMENT OF ACCUMULATED REWARD
	mem	MEMORY REQUIREMENT
	mul	REQUIRED FLOATING POINT MULTIPLICATION
	add	REQUIRED FLOATING POINT ADDITION

1 $mem_A = z \cdot Size(double)$ — *storing elements of* $\mathbf{A}$

$mem_A = mem_A + (z + M) \cdot Size(int)$

$mem_S = M \cdot Size(double)$ — *storing* $\mathbf{S}$

$mem_P = M \cdot Size(double)$ — *storing* $\underline{P}$

$mem_N = M \cdot (n + 1) \cdot Size(double)$ — *temporary vectors*

$mem = mem_A + mem_S + mem_P + mem_N$

2 $add = o\,(G \cdot (2 \cdot n \cdot z + (n + 1) \cdot M))$ — *compute numerical complexity*

$mul = o\,(G \cdot (2 \cdot n \cdot z + M))$

3 $\underline{U}^{(0)} = \underline{h}; \quad \underline{U}^{(i)} = \underline{0}\ , \quad i : 1 \dots n;$ — *compute the n-th moment*

For $i := 1$ **To** G **Do**

  Begin

    For $j := n$ **DownTo** 0 **Do**

      $\underline{U}^{(j)} := \mathbf{S} \cdot \underline{U}^{(j-1)} + \mathbf{A} \cdot \underline{U}^{(j)};$

    $\underline{m} := \underline{m} + \underline{U}^{(j)} \cdot Poisson(i; qt);$

  End;

$m := m \cdot n! \cdot d^n$