Summary of probability theory

- conditional probability
  probability of event A assuming event B

$$Pr(A \mid B) = \frac{Pr(A,B)}{Pr(B)}$$

- complete set of conditions

  – in case of a countable disjunct division of the event space (S), i.e. $\cup_i B_i = S$, and $B_i \cap B_j = \emptyset, \forall i \neq j$:

$$Pr(A) = \sum_i Pr(A \mid B_i) Pr(B_i)$$

  – in case of an uncountable disjunct division of the event space (S)

$$Pr(A) = \int_t Pr(A \mid X = t) dF_X(t) = \int_t Pr(A \mid X = t) f_X(t) dt$$

  comment: interpret the expression $Pr(A) = E(Pr(A|X))$ where A denotes an event and X denotes a random variable.

  – with independent conditions the above expressions can be arbitrarily nested.

  – with dependent conditions the dependence of the conditions has to be considered:

$$Pr(A) = \sum_i \sum_j Pr(A \mid B_i, D_j) Pr(B_i \mid D_j) Pr(D_j)$$

  or

$$Pr(A) = \int_x \int_y Pr(A \mid X = x, Y = y) f_{X|Y=y}(x) f_Y(y) dy dx$$

- remaining lifetime
  X is a r.v. with CDF $F_X(t)$ assuming $X > \tau$ evaluate the distribution of the *remaining lifetime* of X from time τ, i.e. what is the distribution of $X' = X - \tau$ if $X > \tau$

$$F_{X'}(t) = Pr(X' < t \mid X > \tau) = Pr(X < t + \tau \mid X > \tau) = \frac{Pr(X < t + \tau, X > \tau)}{Pr(X > \tau)} =$$

$$\frac{F_X(t + \tau) - F_X(\tau)}{1 - F_X(\tau)}$$

- memoryless property
  X is memoryless if $\forall \tau$ the remaining lifetime X' has the same distribution as X.
  check the memoryless property of the exponential distribution.

- Exercise 1.: two competing activities with exponential distribution

At $t = 0$ two *independent* activities start. Each of them takes an exponentially distributed random time to complete. These random times are denoted by X and Y, where X is exponentially distributed with parameter λ and Y is exponentially distributed with parameter μ. Let $V = min(X, Y)$, $Z = max(X, Y)$ and $W = Z - V$.

Problems:

a) when does complete the shorter activity (i.e. what is the distribution and the mean of V),

b) what is the probability that X completes first ($Pr(X < Y)$),

c) what is the time between the completion of the first and the second activity (distribution and mean of W),

d) what is the probability that in a given time instant t

  * X is already completed but Y is not completed yet ($Pr(X < t < Y)$)
  * the first activity is completed but the second is not completed yet ($Pr(V < t < Z)$)
  * both activity are completed ($Pr(X < t, Y < t) = Pr(Z < t)$)

e) what is the distribution of $W + V$

f) what is the distribution of X assuming that $X < \tau$ ($Pr(X < t \mid X < \tau)$),

g) what is the distribution of X assuming that $X < Y$ ($Pr(X < t \mid X < Y)$),

h) what is the distribution of X assuming that $X > Y$ ($Pr(X < t \mid X > Y)$).

Solution:

a) distribution of the time of the shorter activity

$$Pr(V < t) = 1 - Pr(V > t) = 1 - Pr(X > t, Y > t) =$$

$$= 1 - Pr(X > t)Pr(Y > t) = 1 - e^{-\lambda t}e^{-\mu t} = 1 - e^{-(\lambda+\mu)t}$$

V is exponentially distributed with parameter $\lambda + \mu$, $E(V) = \dfrac{1}{\lambda + \mu}$

b) probability that X completes first

$$Pr(X < Y) = \int_{y=0}^{\infty} Pr(X < y) f_Y(y) dy = \int_{y=0}^{\infty} (1 - e^{-\lambda y}) \mu e^{-\mu y} dy =$$

$$= 1 - \frac{\mu}{\lambda + \mu} = \frac{\lambda}{\lambda + \mu}$$

c) the time between the completion of first and second activity
solution 1. (rude forth)

$$Pr(W < t) = \int_{y=0}^{\infty} \int_{x=0}^{\infty} Pr(W < t \mid X = x, Y = y) f_X(x) f_Y(y) dx dy =$$

$$= \int_{y=0}^{\infty} \int_{x=0}^{\infty} Pr(|x-y| < t) f_X(x) f_Y(y) dx dy =$$

$$= \int_{y=0}^{t} \int_{x=0}^{y+t} \lambda e^{-\lambda x} dx \mu e^{-\mu y} dy + \int_{y=t}^{\infty} \int_{x=y-t}^{y+t} \lambda e^{-\lambda x} dx \mu e^{-\mu y} dy = ...$$

solution 2.

$$Pr(W < t) =$$

$$= Pr(W < t \mid X < Y)\ Pr(X < Y) +\ Pr(W < t \mid X > Y)\ Pr(X > Y) =$$

if $(X < Y)$ then W is the remaining lifetime of Y which is exponentially distributed with the same parameter (memoryless property of the exponential distribution)

$$= Pr(Y < t)\ Pr(X < Y) +\ Pr(X < t)\ Pr(X > Y) =$$

$$\frac{\lambda}{\lambda+\mu}(1 - e^{-\mu t}) + \frac{\mu}{\lambda+\mu}(1 - e^{-\lambda t})$$

mean of W

$$E(W) = E(Y)\ Pr(X < Y) +\ E(X)\ Pr(X > Y) = \frac{\lambda}{\mu(\lambda+\mu)} + \frac{\mu}{\lambda(\lambda+\mu)}$$

d) * at time t X is completed but Y is not completed yet

$$Pr(X < t < Y) = Pr(X < t) Pr(t < Y) = (1 - e^{-\lambda t}) e^{-\mu t}$$

* at time t the first activity is completed but the second is not completed yet

$$Pr(V < t < Z) =$$

$$= Pr(V < t < Z \mid X < Y) Pr(X < Y) +$$
$$+ Pr(V < t < Z \mid X > Y) Pr(X > Y) =$$
$$= Pr(X < t < Y \mid X < Y) Pr(X < Y) +$$
$$+ Pr(Y < t < X \mid X > Y) Pr(X > Y) =$$
$$= Pr(X < t < Y, X < Y) + Pr(Y < t < X, X > Y) =$$
$$= Pr(X < t < Y) + Pr(Y < t < X) =$$
$$= (1 - e^{-\lambda t}) e^{-\mu t} + (1 - e^{-\mu t}) e^{-\lambda t}$$

* at time t both activity is completed

$$Pr(X < t, Y < t) = Pr(Z < t) = Pr(X < t)\ Pr(Y < t) =$$

$$= (1 - e^{-\lambda t})(1 - e^{-\mu t})$$

e) the distribution of W plus V
Since W and V are dependent r.v. convolution can not be used !

$$Pr(W + V < t) = Pr(Z < t) = (1 - e^{-\lambda t})(1 - e^{-\mu t})$$

f) distribution of X if $X < \tau$

$$Pr(X < t \mid X < \tau) = \frac{Pr(X < t, X < \tau)}{Pr(X < \tau)} =$$

from which the modified CDF is

$$\begin{cases} \dfrac{Pr(X < t)}{Pr(X < \tau)} = \dfrac{1 - e^{-\lambda t}}{1 - e^{-\lambda \tau}} & 0 < t < \tau \\ 1 & t > \tau \end{cases}$$

g) distribution of X if $X < Y$

$$Pr(X < t \mid X < Y) = \frac{Pr(X < t, X < Y)}{Pr(X < Y)} = \frac{\int_{y=0}^{\infty} Pr(X < t, X < y) f_Y(y) dy}{Pr(X < Y)} =$$

$$= \frac{\int_{y=0}^{t} Pr(X < y) f_Y(y) dy}{Pr(X < Y)} + \frac{\int_{y=t}^{\infty} Pr(X < t) f_Y(y) dy}{Pr(X < Y)} = 1 - e^{-(\lambda+\mu)t}$$

which is an exponential distribution with parameter $\lambda + \mu$.

h) distribution of X if $X > Y$

$$Pr(X < t \mid X > Y) = \frac{Pr(X < t, X > Y)}{Pr(X > Y)} = \frac{\int_{y=0}^{\infty} Pr(X < t, X > y) f_Y(y) dy}{Pr(X > Y)} =$$

$$= \frac{\int_{y=0}^{t} Pr(y < X < t) f_Y(y) dy}{Pr(X > Y)} = \frac{\int_{y=0}^{t} (F_X(t) - F_X(y)) f_Y(y) dy}{Pr(X > Y)} =$$

$$1 - e^{-\lambda t} - \frac{\lambda}{\mu} e^{-\lambda t} (1 - e^{-\mu t})$$

- mean of the geometric distribution
  The probability mass is defined as $Pr(X = i) = (1-p)p^{i-1}, \quad i = 1, 2, ..$ from which

  $$E(X) = \sum_{i=1}^{\infty} i(1-p)p^{i-1}$$

  but we can evaluate the same mean utilizing the memoryless property of the geometric distribution:

  $$E(X) = E(X \mid X = 1)Pr(X = 1) + E(X \mid X > 1)Pr(X > 1) = 1(1-p) + (1 + E(X))p$$

  where $E(X \mid X = 1)$ equals to 1 plus the mean of the remaining lifetime of the geometric distribution which equals to the original mean. By this

  $$E(X) = \frac{1}{1-p}$$

  which plugged into the original definition of the mean gives

  $$\sum_{i=1}^{\infty} i \; p^{i-1} = \frac{1}{(1-p)^2}$$

- Exercise 2.: two competing activities with geometric distribution
  Solve the problems of Exercise 1. with discrete 'time scale', when two *independent* activities start at $t = 0$. One of them completes in X time unit where $Pr(X = i) = (1-p)p^{i-1}, \ \ i = 1, 2, ..$ the other one of them completes in Y time unit with distribution $Pr(Y = i) = (1-q)q^{i-1}, \ \ i = 1, 2, ...$ Let $V = min(X, Y)$, $Z = max(X, Y)$, $W = Z - V$ as before.

  a) the time of the shorter activity

  $$Pr(V = t) = Pr(V = t \mid X < Y) \ Pr(X < Y) +$$

  $$+ Pr(V = t \mid X > Y) \ Pr(X > Y) + Pr(V = t \mid X = Y) \ Pr(X = Y) =$$

  $$Pr(X = t, t < Y) + Pr(Y = t, X > t) + Pr(X = t, Y = t) =$$

  $$Pr(X = t) \ Pr(t < Y) + Pr(Y = t) \ Pr(X > t) + Pr(X = t) \ Pr(Y = t) =$$

  $$= Pr(X = t) \ Pr(t \geq Y) + Pr(Y = t) \ Pr(X > t) =$$

  $$= (1-p)p^{t-1}q^{t-1} + (1-q)q^{t-1}p^t = (1-pq)(pq)^{t-1}$$

  which is a geometric distribution with parameter pq.

  b) X completes before

  $$Pr(X < Y) = \sum_{x=1}^{\infty} Pr(x < Y) \ Pr(X = x) = \sum_{x=1}^{\infty} q^x (1-p)p^{x-1} = \frac{q(1-p)}{1-pq}$$

  the two activity completes at the same time

  $$Pr(X = Y) = \sum_{x=1}^{\infty} Pr(X = x) \ Pr(Y = x) = \sum_{x=1}^{\infty} (1-q)q^{x-1}(1-p)p^{x-1} =$$

  $$= \frac{(1-q)(1-p)}{1-pq}$$

  X completes later

  $$Pr(X \leq Y) = \frac{1-p}{1-pq}$$

  c) time between the completion of the first and the second activity
  the possible values of W are $0, 1, 2, \ldots$

  $$Pr(W = 0) = Pr(X = Y) = \frac{(1-q)(1-p)}{1-pq}$$

  If $t > 0$ then solution 1. (rude forth)

  $$Pr(W = t) =$$

  $$\sum_{y=1}^{\infty} \sum_{x=1}^{\infty} Pr(W = t \mid X = x, Y = y) Pr(X = x) Pr(Y = y) =$$

  $$= \sum_{y=1}^{\infty} Pr(X = y + t) Pr(Y = y) + \sum_{x=1}^{\infty} Pr(X = x) Pr(Y = y + t) = ...$$

solution 2. $(t > 0)$

$$Pr(W = t) =$$
$$= Pr(W = t \mid X < Y)\ Pr(X < Y) + Pr(W = t \mid X > Y)\ Pr(X > Y)$$

if $(X < Y)$ then W is the remainig lifetime of Y and using the memoryless property of the geometric distribution

$$= Pr(Y = t)\ Pr(X < Y) + Pr(X = t)\ Pr(X > Y) =$$
$$= \frac{q(1-p)}{1-pq}(1-q)q^{t-1} + \frac{p(1-q)}{1-pq}(1-p)p^{t-1}$$

the mean of W is

$$E(W) = E(Y)\ Pr(X < Y) + E(X)\ Pr(X > Y) + 0\ Pr(X = Y) =$$
$$= \frac{q(1-p)}{1-pq}\frac{1}{1-q} + \frac{p(1-q)}{1-pq}\frac{1}{1-p}$$

d) * X is completed and Y is not completed at time t

$$Pr(X \le t < Y) = Pr(X \le t)\ Pr(t < Y) = (1 - p^t)q^t$$

* the first activity is completed and the second is not completed at time t

$$\begin{aligned}
&Pr(V \le t < Z) = \\
&= Pr(V \le t < Z \mid X < Y)Pr(X < Y) + \\
&\quad + Pr(V \le t < Z \mid X > Y)Pr(X > Y) = \\
&= Pr(X \le t < Y \mid X < Y)Pr(X < Y) + \\
&\quad + Pr(Y \le t < X \mid X > Y)Pr(X > Y) = \\
&= Pr(X \le t < Y, X < Y) + Pr(Y \le t < X, X > Y) = \\
&= Pr(X \le t < Y) + Pr(Y \le t < X) = \\
&= Pr(X \le t)Pr(t < Y) + Pr(Y \le t)Pr(t < X) = \\
&= (1 - p^t)q^t + p^t(1 - q^t)
\end{aligned}$$

* both activities are completed at time t

$$Pr(X \le t, Y \le t) = Pr(Z \le t) = Pr(X \le t)Pr(Y \le t) = (1 - p^t)(1 - q^t)$$

e) the distribution of W plus V

$$\begin{aligned}
&Pr(W + V = t) = Pr(Z = t) = \\
&= Pr(Z = t \mid X \le Y)Pr(X \le Y) + Pr(Z = t \mid X > Y)Pr(X > Y) = \\
&= Pr(Y = t \mid X \le Y)Pr(X \le Y) + Pr(X = t \mid X > Y)Pr(X > Y) = \\
&= Pr(Y = t, X \le Y) + Pr(X = t, X > Y) = \\
&= Pr(Y = t, X \le t) + Pr(X = t, t > Y) = \\
&= Pr(Y = t)Pr(X \le t) + Pr(X = t)Pr(Y < t) = \\
&= (1 - q)q^{t-1}(1 - p^t) + (1 - p)p^{t-1}(1 - q^{t-1})
\end{aligned}$$

f) distribution of X if $X < \tau$
the possible values of X are $1, 2, 3, \ldots, \tau-2, \tau-1$

$$Pr(X = t \mid X < \tau) = \frac{Pr(X = t, X < \tau)}{Pr(X < \tau)} =$$

if $t < \tau$ it is

$$\frac{Pr(X = t)}{Pr(X < \tau)} = \frac{(1-p)p^{t-1}}{1-p^{\tau-1}}$$

g) distribution of X if $X < Y$
the possible values of X are $1, 2, 3, \ldots$

$$Pr(X = t \mid X < Y) = \frac{Pr(X = t, X < Y)}{Pr(X < Y)} =$$

$$= \frac{\sum_{y=1}^{\infty} Pr(X = t, X < y)\ Pr(Y = y)}{Pr(X < Y)} =$$

$$= \frac{\sum_{y=t+1}^{\infty} Pr(X = t)\ Pr(Y = y)}{Pr(X < Y)} = \frac{Pr(X = t)\ Pr(Y > t)}{Pr(X < Y)} =$$

$$\frac{(1-p)p^{t-1}q^t}{\frac{q(1-p)}{1-pq}} = (1-pq)(pq)^{t-1}$$

which is a geometric distribution with parameter pq.

h) distribution of X if $X > Y$
the possible values of X are $2, 3, \ldots$

$$Pr(X = t \mid X > Y) = \frac{Pr(X = t, X > Y)}{Pr(X > Y)} =$$

$$= \frac{\sum_{y=1}^{\infty} Pr(X = t, X > y)\ Pr(Y = y)}{Pr(X > Y)} =$$

$$= \frac{\sum_{y=1}^{t-1} Pr(X = t)\ Pr(Y = y)}{Pr(X > Y)} = \frac{Pr(X = t)\ Pr(Y < t)}{Pr(X > Y)} =$$

$$\frac{(1-p)p^{t-1}(1-q^{t-1})}{\frac{p(1-q)}{1-pq}} = \frac{(1-pq)(1-p)}{p(1-q)}(p^{t-1} - (pq)^{t-1})$$

additional problems: check if it is a real distribution; evaluate the generator function (z-transform) of this distribution; evaluate the mean of this distribution.

- Excercise 3.: rude forth
  Solve all the above problems with the use of the rude forth approach. Practice the interchange of the order of summation (integration).
- Exercise 4.: mixed distributions
  Solve the probles of Exercise 1. when
  - X is exponentially and Y is continuous uniformly distributed,
  - X is exponentially and Y is discrete uniformly distributed,
  - X is geometrically and Y is discrete uniformly distributed,
  - X is geometrically and Y is continuous uniformly distributed,
- Summary of conclusions
  independent exponentially (geometrically) distributed r.v.:
  - the minimum of two or more independent exponentially (geometrically) distributed r.v. is exponentially (geometrically) distributed whose parameter is the sum (product) of the parameters of the independent r.v.
  - the probability that a particular r. v. is the less is proportional to the parameter of the r.v.
  - the probability that a particular r. v. is the less is independent of the value of the minimum.
  - the remaining lifetime of the r.v. after the minimum is exponentially (geometrically) distributed with the same parameter.