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ON THE CANONICAL REPRESENTATION OF HOMOGENEOUS MARKOV PROCESSES MODELLING FAILURE - TIME DISTRIBUTIONS

ALDO CUMANI

Gruppo Sistemi E Circuiti,
Istituto Elettrotecnico Nazionale "Galileo Ferraris",
Strada delle Cacce 91 - I - 10135,
TORINO,
Italy.

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ABSTRACT

This paper deals with the use of Multistate Homogeneous Markov Models (MHMM) to represent failure-time distributions in reliability analysis, with particular emphasis on MHMM's representable by acyclic transition graphs (Triangular MHMM's). It is shown that a generic TMHMM can be transformed into an equivalent minimum-parameter form (canonical form). This result is used to characterize the class of distributions representable by TMHMM's. It is shown that, although not all distributions which are linear combinations of exponential terms can be exactly represented by a TMHMM, such models can nevertheless be used to approximate as closely as desired any reasonable failure-time distribution.

1. INTRODUCTION

Although widely employed in the reliability analysis of complex systems, the theory of homogeneous Markov processes is somewhat limited in its application by the essential assumption that the life and repair times of each component be exponentially distributed (constant failure/repair rate). Many distributions often used in reliability analysis do not follow this simple exponential model: for instance, components either exhibit degradation (increasing failure rate), or burn-in (decreasing failure rate), or both. The most obvious way to take into account such behaviours is to pass to a nonhomogeneous model, by letting the transition rates depend upon time; in this way, however, both the theoretical elegance and the computatio

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nal convenience of the homogeneous model are lost.

A different approach consists in representing each component by a Multistate Homogeneous Markov Model (MHMM) [1] [2], whose stochastic behaviour approximates, according to some given criterion, that of the original component. This approach is a generalization of well-known techniques for approximating non-exponential distributions by combinations of series and/or parallel configurations with constant transition rates (also known as "stage device", see e.g. [3] [4]). One of the major advantages of this approach is that the overall system is thus still represented by a homogeneous Markov process, so allowing the use of standard techniques for the analysis of its behaviour.

In this context, the following questions arise naturally:

a) What kind of distributions can be represented by MHMM's?
b) Can we use MHMM's to approximate an arbitrary distribution as close as we want?
c) Can a generic MHMM be transformed into an equivalent canonical form, i.e. a form having the minimum number of free parameters?

The answer to question c) is of noticeable practical importance, for at least two reasons: first, a canonical form would simplify the computation of the best approximation for a given distribution, by not taking into account redundant parameters; second, the use of a minimal structure for each component would also help to control the complexity of the overall system model.

This paper presents some partial answers to the above questions, by considering mainly MHMM's representable by acyclic transition graphs; they will be referred to as Triangular MHMM's (TMHMM) since in that case the transition matrix can be put in triangular form by a suitable reordering of the states.

The paper is organized as follows. Section 2 reports some basic definitions and the major properties of MHMM's and TMHMM's. In section 3 we show that a generic TMHMM can always be transformed into any one of three canonical forms, the choice among them being a matter of convenience. Section 4 deals with the computation of canonical forms and the related subject of their uniqueness. In section 5 the above results are used to characterize the class of distributions representable by TMHMM's. In particular, we show that there are distributions which, although being linear combinations of exponentials, cannot be generated by a TMHMM (nor by a generic MHMM). This negative result is, however, of little practical significance since we also show that any reasonably well-behaved distribution can be approximated as well as desired by a TMHMM of sufficiently high order.

2. BASIC DEFINITIONS AND PROPERTIES OF MHMM'S

Definition 1. An n-state MHMM (shortly, n-MHMM) is a time-continuous homogeneous Markov process with n discrete states represented by the triple: (Λ, Ω, C), where

- Λ is the transition rate matrix, i.e. a square matrix of order n satisfying:

$$\Lambda_{ik} \geq 0 \qquad \forall i \neq k, \qquad \sum_{i=1}^{n} \Lambda_{ik} = 0 \quad \forall k$$

We adhere to the convention of representing probability vectors by column vectors; so, Λ_{ik} is the transition rate from state k to state i;

- Ω is the initial probability vector, i.e. a column vector of dimension n satisfying:

$$\Omega_i \geq 0 \quad \forall i, \qquad \sum_{i=1}^{n} \Omega_i = 1$$

- C is the structure vector, i.e. a colum vector of dimension n with 0/1 - valued entries which represents a partitioning of the set of n states into two mutually disjoint subsets U and D, such that $i \in U$ if $C_i = 0$ and $i \in D$ if $C_i = 1$. U is the set of "up" states and D the set of "down" states.

With this definition, the state probability vector P(t) is obtained by solving the standard Markov equation:

$$\frac{dP}{dt} = \Lambda P \tag{1}$$

under the initial condition $P(0) = \Omega$. The formal solution of (1) is :

$$P(t) = e^{\Lambda t}\, \Omega, \quad t \geq 0 \tag{2}$$

The time function F(t) defined by:

$$F(t) = C^T P(t) \tag{3}$$

is the probability of the system being in some state $i \in D$ at time t. Since we are dealing only with the use of MHMM's for approximating failure-time distributions, we shall assume that the D set is ergodic, so that the down states can be grouped together into a single absorbing state which shall be identified with state n; so, the structure vector C will always be equal to $\delta^{(n,n)}$ and will often be omitted for brevity*. We shall furthermore assume that $C^T\Omega = 0$, i.e. that the component is initially "good"; under these conditions, (3) represents the cumulative distribution function (cdf) of the transition time from the U set to the D set, i.e. of the failure time of the component modelled by (Λ, Ω, C); it will be referred to as the cdf of the MHMM. We shall sometimes use the notation $F(t;\ \Lambda, \Omega)$ in order to make the dependence on Λ and Ω explicit.

It will be useful to consider the relations corresponding to (2) and (3) in the Laplace transform domain. Let $P_s(s)$ be the transform of P(t); then eq. (2) rewrites as:

$$P_s(s) = (sI - \Lambda)^{-1}\,\Omega \tag{4}$$

where I is the identity matrix of order n, and (3) rewrites as:

$$F_s(s) = C^T(sI - \Lambda)^{-1}\Omega = \frac{N(s)}{Q(s)} \tag{5}$$

where N and Q are polynomials in s and $Q(s) = \det\,(sI - \Lambda) = \prod_{i=1}^{n} (s - \mu_i)$, μ_i being the eigenvalues of Λ.

Remark 2.1: Notice that Q(s) is specified by n-1 parameters since one of the eigenvalues of Λ, by its definition, must be zero. Furthermore, the condition $C^T\Omega = 0$ implies $F(0) = 0$ and so $\deg\,(N) \le n-2$. Taking also into account the condition $F(+\infty) = \lim_{s\to 0} sF_s(s) = 1$ (nondegeneracy of the cdf), it can be easily checked that (5) is completely specified by no more than 2 n - 3 parameters, i.e. that the cdf of an n-MHMM has 2 n - 3 degrees of freedom.

In the particular case of an MHMM whose transition graph has no cycles, the Λ matrix can always be put in lower triangular form by a suitable reordering of the states. For this reason such a model is called Triangular MHMM (TMHMM). The major properties of TMHMM's have been stated in [2]; in particular, from the properties of triangular matrices we get the following

Property 1. The Λ matrix of an n - TMHMM has n - 1 negative real eigenvalues and a single zero eigenvalue; they coincide with

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* $\delta^{(i,k)}$ represents a column vector of dimension k with the i-th entry equal to 1 and all other entries equal to 0.

the diagonal entries of Λ. From this property it follows that for an n - TMHMM the denominator of (5) has the simple form:

$$Q(s) = s\prod_{i=1}^{n}(s + \lambda_i)\ , \qquad \lambda_i = -\Lambda_{ii}$$

Definition 2. In an MHMM: (Λ, Ω, C) a path is a sequence of m states $i_1, i_2, \ldots, i_m$ such that

$$\Lambda_{i_{k+1}, i_k} \ne 0 \qquad k = 1, 2, \ldots, m-1$$

In other words, a path is a sequence of connected states in the transition graph corresponding to Λ. Notice that for a generic MHMM some states may appear more than once in a path, while this obviously does not happen for a TMHMM.

Definition 3. In an MHMM a state i is called essential if either $\Omega_i \ne 0$ or it belongs to some path starting from another state k with $\Omega_k \ne 0$.

Definition 4. An MHMM is termed irreducible if all its states are essential; otherwise it is reducible.

The proof of the following property is almost trivial:

Property 2. A reducible MHMM is cdf-equivalent to (i.e. has the same cdf of) an irreducible MHMM of lower order, obtained by deleting all non-essential states in the former.

Notice that series and parallel configurations of n states are particular cases of n - TMHMM's. In particular, for a series the Λ matrix is bidiagonal, so that it is completely specified by its n - 1 nonzero diagonal entries. Such a matrix will be termed an s-matrix.

Definition 5. For a given n - TMHMM: (Λ, Ω) an elementary series (ES) of order $m \le n$ is an m - TMHMM: $(\bar{\Lambda}, \delta^{(1,m)})$ where $\bar{\Lambda}$ is an s-matrix of order m with

$$\bar{\Lambda}_{k,k} = -\lambda_{i_k} = \Lambda_{i_k, i_k}$$

where $i_1, i_2, \ldots, i_m$ form a path of Λ and $i_m \equiv n$. Fig. 1 shows a 4-TMHMM together with its ES's. A simple calculus shows that the number of ES's for given n is at most $2^{n-1} - 1$. An ES may be represented by the notation:

$$E = \langle \lambda_{i_1}\ \lambda_{i_2}\ \cdots\ \lambda_{i_{m-1}} \rangle$$

(recall that $\lambda_{i_m} \equiv 0$). It is immediate that the cdf of an ES has Laplace transform:

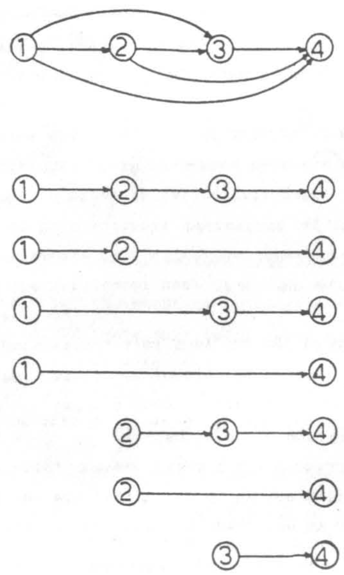

Fig. 1 - A 4-TMHMM and its 7 elementary series.

$$F_s(s) = \frac{\lambda_{i_1} \dots \lambda_{i_{m-1}}}{s(s+\lambda_{i_1})\dots(s+\lambda_{i_{m-1}})} = \frac{1}{s}\prod_{k=1}^{m-1} \frac{\lambda_{i_k}}{s+\lambda_{i_k}} \qquad (6)$$

3. CANONICAL FORMS OF TMHMM'S

In the previous section we have seen that the cdf of a n - MHMM has at most 2 n - 3 free parameters; this figure should be compared with the number of parameters needed to specify an n - MHMM, which is easily computed as $N_G = (n-1)^2 + n - 2$ for a generic model and $N_T = n(n-1)/2 + n - 2$ for the triangular case. Since the representation of a cdf by an MHMM is so highly redundant, we suspect the existence of some cdf-preserving transformation able to reduce a given model to a form of minimal complexity (canonical form). The existence of such forms will be stated in the following for the triangular case.

Theorem 1. The cdf of an n - TMHMM: (Λ, Ω) is a mixture of the cdf's of its elementary series, where each ES has a weight proportional to the product of the transition rates along the

corresponding path and to the initial probability of the first state in the path.

The proof of Th. 1 is given in the Appendix. This result is useful also because it allows, at least for moderate values of n, to write out $F_s(s)$ of eq. (5) by simple inspection of the transition graph.

To proceed further, we need the following definition and lemma.

Definition 6. Given n positive real numbers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$, their basic series (BS) are the n series of 2, 3, ..., n+1 states

$$BS_1 = < \lambda_1 >$$
$$BS_2 = < \lambda_1 \; \lambda_2 >$$
$$\dots\dots\dots$$
$$BS_n = < \lambda_1 \; \lambda_2 \dots \lambda_n >$$

For a given n - TMHMM: (Λ, Ω) its n - 1 basic series are similarly defined using as λ_i's the ordered set of the eigenvalues of $-\Lambda$.

Lemma 1. Given an n - TMHMM: (Λ, Ω), the cdf of each of its elementary series is a mixture of the cdf's of its basic series.

The proof of this lemma is rather involved and is reported in the Appendix.

It basically relies on the following identity: given two positive real numbers a and b, with $a \leq b$,

$$\frac{a}{s+a} = w\frac{b}{s+b} + (1-w)\frac{ab}{(s+a)(s+b)} \qquad (7)$$

where $w = \frac{a}{b} \in (0,1]$. This identity shows that an elementary series containing a stage with transition rate a can be substituted (as long as the cdf is concerned) with a mixture of two series, one containing a stage with transition rate b and the other containing both a and b, provided that $b \geq a$. It is therefore intuitive that by repeated use of (7) one can transform an ES into a mixture of BS's.

We now state the following

Theorem 2. The cdf of an n - TMHMM: (Λ, Ω) is a mixture of the cdf's of its basic series.

Proof: from Theorem 1 and Lemma 1.

From Th. 2 stems the following important

Corollary 2.1 (series canonical form). Any n - TMHMM: (Λ, Ω) is cdf-equivalent to a series configuration $(\hat{\Lambda}, \hat{\Omega})$ with transition rates $\hat{\Lambda}_{k+1,k} = \lambda_{n-k}$ equal to the eigenvalues of $-\Lambda$, so ordered that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1}$ (see Fig. 2). In other words, the schema of Fig. 2 is a canonical form for TMHMM's.

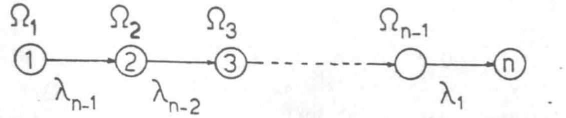

Fig. 2 - The series canonical form.

The proof is immediate since from Th. 2 it follows that for any (Λ, Ω) there must exist nonnegative real numbers β_i, $i = 1, 2, \ldots, n-1$ such that

$$F(t; \Lambda, \Omega) = \sum_{i=1}^{n-1} \beta_i F_i(t, \Lambda) \tag{8}$$

where $\Sigma\beta_i = 1$ and $F_i(t; \Lambda)$ is the cdf of the i-th basic series of Λ. But it is easy to see that the r.h.s. of (8) is the cdf of the series configuration in Fig. 2, provided that $\beta_i = \hat{\Omega}_{n-i}$

q.e.d.

Remark 3.1: It can be easily checked that (8) has the right number of degrees of freedom to be a minimal representation of (Λ, Ω); indeed, (8) is specified by $2n - 3$ parameters, namely $n - 1$ transition rates and $n - 2$ independent initial probabilities.

Although the above series form is probably the most compact representation of a TMHMM, there are at least two other forms which have the advantage that the initial probability is concentrated in the first state (i.e. $\Omega = \delta^{(1,n)}$). This property is particularly useful when using the TMHMM as a failure model for a component imbedded in a larger system since it allows, e.g., to represent a repair action (with the repaired component "as good as new") by a simple transition from state n to state 1.

Canonical form A. Given the ordered set of $n-1$ positive real numbers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1}$, the form $(\Lambda^*, \delta^{(1,n)})$ is canonical for n-TMHMM's with eigenvalues $-\lambda_i$, where

$$\Lambda^* = \begin{bmatrix} -\lambda_1 & & & & & \\ x_{n-1} & -\lambda_{n-1} & & & & \\ x_{n-2} & \lambda_{n-1} & -\lambda_{n-2} & & & \\ \cdots & \cdots & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & -\lambda_2 & \\ x_1 & 0 & 0 & 0 & \lambda_2 & 0 \end{bmatrix}$$

$$x_i \in [0, \lambda_1] \quad i = 1, 2, \ldots, n-1$$

$$\sum_{i=1}^{n-1} x_i = \lambda_1$$

corresponding to the schema of Fig. 3. The proof is almost trivial since it is easy to see that the cdf of this form coincides with (8) when $x_i = \beta_i\lambda_1$.

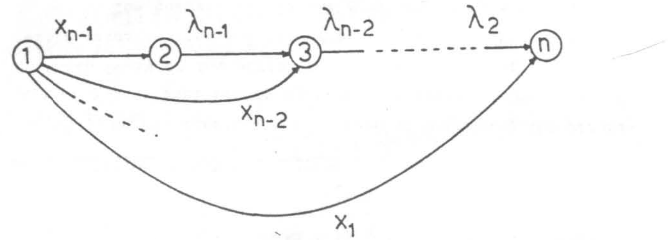

Fig. 3 - Canonical form A.

Canonical form B. Given the ordered set $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1}$, the form $(\tilde{\Lambda}, \delta^{(1,n)})$ is canonical for n-TMHMM's with the prescribed eigenvalues, where

$$\tilde{\Lambda} = \begin{bmatrix} -\lambda_1 & & & & & \\ x_1 & -\lambda_2 & & & & \\ 0 & x_2 & -\lambda_3 & & & \\ \cdots & \cdots & \cdots & & & \\ \cdots & \cdots & \cdots & \cdots & -\lambda_{n-1} & \\ \lambda_1 - x_1 & \lambda_2 - x_2 & \cdots & \cdots & \lambda_{n-1} & 0 \end{bmatrix}$$

$$x_i \in [0, \lambda_i], \quad i=1,2,\ldots,n-2$$

corresponding to the schema of Fig. 4. The proof of canonicity can be obtained by comparing the cdf of this form with that of the series form. Let $\gamma_i = x_i/\lambda_i$; then the cdf of $(\tilde{\Lambda}, \delta^{(1,n)})$ is given by (8) provided that

$$\begin{aligned} \beta_1 &= 1 - \gamma_1 \\ \beta_2 &= \gamma_1(1 - \gamma_2) \\ &\cdots\cdots\cdots \\ \beta_k &= \gamma_1 \gamma_2 \cdots \gamma_{k-1}(1-\gamma_k) \end{aligned} \tag{9}$$

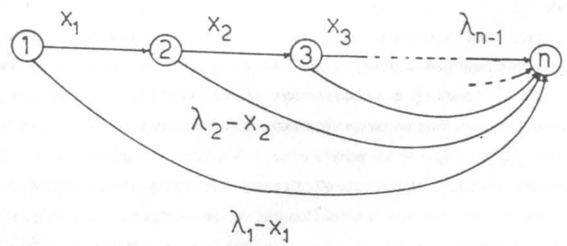

Fig. 4 - Canonical form B.

After some algebraic manipulation, one gets the solution of (9) with respect to the γ_i's as

$$\gamma_i = \frac{1 - \sum_{k=1}^{i} \beta_k}{1 - \sum_{k=1}^{i-1} \beta_k} \tag{10}$$

provided that $\sum_{k=1}^{i-1} \beta_k \neq 1$. Let us suppose that this last condition is satisfied for all $i = 1, 2, \ldots, n-2$; then it is easy to see that for any choice of β_i such that $\beta_i \geq 0$, $\Sigma\beta_i = 1$, the condition $0 \leq \gamma_i \leq 1$ is satisfied for all i, so that $\tilde{\Lambda}$ is a legitimate transition rate matrix. Since there is a one-to-one correspondence between the series canonical form and $(\tilde{\Lambda}, \delta^{(1,n)})$, then the latter is canonical too, q.e.d.

Remark 3.2: It is interesting to notice what happens if, for some $i < n-1$, $\sum_{k=1}^{i} \beta_k = 1$ while $\sum_{k=1}^{i-1} \beta_k \neq 1$. In this case $\gamma_i = 0$ and γ_k for $k > i$ as given by (10) is undefined; however, this is only possible if $\beta_k = 0$ $\forall k > i$, so that (9) is satisfied by any choice of γ_k's for $k > i$. Indeed, in this case the canonical form becomes a reducible TMHMM since the states $i+1$, $i+2, \ldots, n-1$ are no more reachable from state 1. We should notice, however, that under such conditions also the other two forms are reducible: in the series form, if $\beta_k = 0$ for $k = i+1, \ldots, n-1$, the states $1, 2, \ldots, n-i-1$ are not essential, and in form A the same happens for states $2, 3, \ldots, n-i$.

Remark 3.3: Notice that if an MHMM is reducible, then there must be some pole-zero cancellation in the corresponding $F_s(s)$ (i.e., $P(s)$ and $Q(s)$ must have some common factors) while the converse is not true, in general. For example, the 3- TMHMM of Fig. 5 a (which is in canonical form) has

$$F_s(s) = \frac{2}{(s+3)(s+1)} + \frac{1}{s+3} = \frac{s+3}{(s+3)(s+1)} = \frac{1}{s+1}$$

and is therefore cdf-equivalent to the 2- TMHMM of Fig. 5 b. However, the schema of Fig. 5 a is not reducible in the sense of Definition 4.

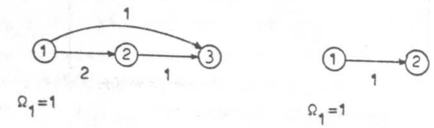

Fig. 5 - a) an irreducible 3-TMHMM;
b) a 2-TMHMM cdf-equivalent to the former.

4. COMPUTATION OF CANONICAL FORMS

We now consider the problem of computing the parameters of a canonical representation of a given n- TMHMM: (Λ, Ω). We shall focus on the series form, since the other two are easily obtained from the former by simple relations.

The computation of the β_i's appearing in (8) is best done in the Laplace transform domain. To this purpose, rewrite (8) as:

$$F_s(s; \Lambda, \Omega) = \sum_{i=1}^{n-1} \beta_i F_{si}(s; \Lambda) \tag{11}$$

where

$$F_{si}(s; \Lambda) = \frac{\lambda_1 \lambda_2 \cdots \lambda_i}{s(s+\lambda_1)\cdots(s+\lambda_i)} = \frac{\lambda_1 \lambda_2 \cdots \lambda_i (s+\lambda_{i+1})\cdots(s+\lambda_{n-2})}{s(s+\lambda_1)\cdots(s+\lambda_{n-1})} = \frac{N_i(s)}{Q(s)} \tag{12}$$

where $N_i(s)$ is a polynomial of degree $n-i-1$ in s. By equating (11) and (5) one gets:

$$N(s) = \sum_{i=1}^{n-1} \beta_i N_i(s)$$

which, after equating separately the coefficients of s^k, $k = n-2, n-3, \ldots, 1, 0$, becomes a system of n-1 linear equations in the β_i's which turns out to be in triangular form with nonzero diagonal coefficients, hence nonsingular and easily solvable by Gaussian elimination.

It should be remarked that nonsingularity of the above defined system implies the uniqueness of the solution vector $\hat{\Omega} = [\beta_{n-1}\ \beta_{n-2}\ \ldots \beta_1\ 0]^T$ for given N(s) and Q(s); but, since Q(s) is fixed by the eigenvalues of Λ, N(s) is itself unique so that we conclude that the series canonical form of (Λ, Ω) is unique. This uniqueness property may be transferred immediately to canonical form A, and also to form B provided that there is no pole-zero cancellation as mentioned in Remarks 3.1 and 3.2 (in that case we need some convention for uniquely defining the undefined γ_i's, e.g. $\gamma_i = 0$).

5. EXACT AND APPROXIMATE REPRESENTATION OF CDF'S BY TMHMM'S

We now consider the problem of characterizing the class of cdf's generated by TMHMM's.

Definition 7. A real-valued function F(t) over $[0, +\infty)$ is of class $R_c(n)$ (Rational Laplace Transform cdf of order n) iff it is a cdf and its Laplace transform is a rational function, i.e. a ratio of two polynomials in s:

$$F_s(s) = L\{F(t)\} = \frac{N(s)}{Q(s)}$$

where deg (Q) = n.

The n zeros of the denominator Q(s) are the poles of $F_s(s)$. Notice that for F(t) to be an honest cdf, $F_s(s)$ must have a single pole at s = 0 with unit residue, and the other poles must have negative real part. We shall also assume that deg (N) $\leq$ deg (Q)-2 in order to have F(0) = 0, i.e. no mass at the origin.

Definition 8. A function F(t) is of class $N_c(n)$ iff it is of class $R_c(n)$ and its n-1 non-zero poles are real (negative, by the above remark).

By eq. (5) and Prop. 1 one immediately gets

Property 3. The cdf of an n-MHMM is of class $R_c(n)$

Property 4. The cdf of an n-TMHMM is of class $N_c(n)$

We notice that the so defined classes contain many distributions often used in reliability analysis, e.g. the simple exponential which is $N_c(1)$ and the gamma distributions with integral parameter θ which are $N_c(\theta)$.

It should be noticed that any $F(t) \in R_c(n)$ is also $R_c(k)$ $\forall k > n$, since one can always multiply both numerator and denominator of $F_s(s)$ by a common factor of degree $k - n > 0$ without affecting the cdf. The same holds for $N_c(n)$.

Definition 9. $T_c(n)$ is the class of cdf's realizable by n-TMHMM's. Obviously, $T_c(n) \subset N_c(n)$.

We are now able to answer the question: what cdf's are representable by an n-TMHMM (i.e., what cdf's belong to $T_c(n)$)? By the results of sections 3 and 4, we have the

Property 5. $T_c(n)$ is the subset of $N_c(n)$ whose elements admit a series canonical representation with $\beta_i \geq 0$.

This proposition is less trivial than it appears, since we can give examples of cdf's which are $N_c(n)$ but not $T_c(n)$.

Example 1. Let

$$F(t) = 1 - e^{-t}(1 + \frac{1}{2}t^2)$$

then

$$F_s(s) = \frac{s^2 + s + 1}{s(s+1)^3} \tag{13}$$

hence $F(t) \in N_c(4)$. Now, if there is a canonical 4-TMHMM yielding F(t), it must have $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and

$$\beta_1 = 1 \qquad \beta_2 = -1 \qquad \beta_3 = 1$$

We see that $[\beta_3, \beta_2, \beta_1, 0]^T$ is not a probability vector. But we have proved that the series canonical form of an n-TMHMM is unique, so we conclude that there is no 4-TMHMM yielding F(t), since any such TMHMM should yield nonnegative β_i's when reduced to canonical form. Hence $F(t) \in N_c(4)$ but $F(t) \notin T_c(4)$.

For this particular case, the problem may be circumvented by raising the order of the model, i.e. by introducing "dummy" poles in $F_s(s)$. For example, if both numerator and denominator of (13) are multiplied by s + 2, we may show that $F(t) \in T_c(5)$ by constructing the 5-TMHMM of Fig. 6 with

$$\lambda_1 = 2 \qquad \lambda_2 = \lambda_3 = \lambda_4 = 1$$
$$\beta_1 = 1/2 \quad \beta_2 = \beta_3 = 0 \qquad \beta_4 = 1/2$$

which yields F(t) as its cdf.

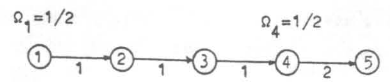

Fig. 6 - The 5-TMHMM which realizes the cdf of Example 1.

Example 2. Let

$$F(t) = 1 - e^{-t}(1 + t^2) \tag{14}$$

then

$$F_s(s) = \frac{s^2 + 1}{s(s + 1)^3}$$

hence again $F(t) \in N_c(4)$. But we can show that $F(t) \notin T_c(n)$ for any finite order n. Indeed, we have the following

Property 6. Given the cdf of an n-MHMM, the corresponding density is nonzero for any finite $t > 0$.

The proof is given in the Appendix. Now, the density of (14) is

$$f(t) = e^{-t}(1 - t)^2$$

and so $f(t) = 0$ for $t = 1$. Hence, there can be no n-TMHMM with finite n yielding F(t) as its cdf, q.e.d.

The above examples should not induce, however, pessimistic conclusions about the usefulness of the Markov approach. Indeed, we can show that

Property 7. Any reasonably well-behaved cdf can be approximated as close as desired by an n-TMHMM for sufficiently large n.

bármennyire se jól't- ható a közelí- tés.

Let F(t) be a cdf such that $F(0) = 0$ and $R(t) = 1 - F(t)$ the corresponding survival function; define

$$R_\lambda(t) = \sum_{k=0}^{\infty} R(k/\lambda)\, e^{-\lambda t} \frac{(\lambda t)^k}{k!} \tag{15}$$

then

$$\lim_{\lambda \to \infty} R_\lambda(t) = R(t)$$

uniformly in every finite t-interval [5]. Now define $R_{n\lambda}(t)$ as

$$R_{n\lambda}(t) = \sum_{k=0}^{n-1} a_k\, e^{-\lambda t} \frac{(\lambda t)^k}{k!} \tag{16}$$

where

$$a_k = R(k/\lambda).$$

then it is clear that $F_{n\lambda}(t) = 1 - R_{n\lambda}(t)$ is the cdf of the series (n+1)-TMHMM of Fig. 7, provided that $\beta_i = a_{i-1} - a_i$,

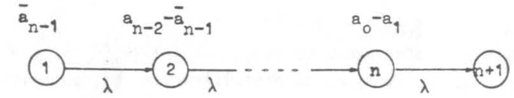

Fig. 7 - Approximation of a cdf by an n-TMHMM.

$i=1, \dots n-1$, $\beta_n = a_{n-1}$; furthermore,

$$\lim_{\substack{n \to \infty \\ \lambda \to \infty}} F_{n\lambda}(t) = F(t)$$

which implies that F(t) can be approximated as closely as desired by $F_{n\lambda}(t)$ in any finite t-interval by choosing sufficiently large values of n and λ.

It should be noticed that for a cdf which is the output of some MHMM: (Λ, Ω, C) the above result can be somewhat strengthened by using, instead of (15),

$$R^*(t) = \sum_{k=0}^{\infty} q_k\, e^{-\lambda t} \frac{(\lambda t)^k}{k!} \tag{17}$$

with

$$q_k = \bar{C}^T\left(I + \frac{\Lambda}{\lambda}\right)^k \Omega, \quad \bar{C}_i = 1 - C_i, \; i = 1, 2, \dots, n$$

It can be shown that if $\lambda \geq \max_{i,k} |\Lambda_{ik}|$, then $q_k \geq 0$ and (17) is an exact representation of R(t) (not only in the limit $\lambda \to \infty$) [6].

It should be remarked that the above is not meant to be an efficient way to approximate a given cdf by an n-TMHMM. In a practical case, we would use a canonical model of some given order n in an optimization procedure, such as the one reported in [2], in order to get "optimal" values of the $2n-3$ parameters $\lambda_1 \dots \lambda_{n-1}$ and $\beta_1 \dots \beta_{n-2}$. In most cases of interest such a procedure is likely to produce a good approximation of F(t) even for small values of n, and we refer the reader to the cited work for examples justifying this assertion.

6. CONCLUDING REMARKS

From a practical point of view, the major result of this work is the existence, uniqueness and simple form of the canonical representation of TMHMM's. The use of triangular models in a special-purpose optimization program aimed at producing the best markovian approximation of a given cdf has been proposed in [2]; incorporation of a canonical structure in this program is expected to yield a significant improvement of its efficiency. Indeed, as long as the number of parameters of the model exceeds the number of degrees of freedom of its output, we expect the existence of many different sets of parameter values yielding the same output, and this is likely to cause problems such as slow convergence or even oscillations. For example, one may easily give examples of n-TMHMM's with the same cdf and the same eigenvalues, but differently ordered on the diagonal of the Λ matrix; so, it is well possible that the optimization procedure be trapped in an oscillation between two such configurations without ever reaching the optimal solution. This problem, however, is easily avoided if the canonical ordering of the λ_i's is incorporated as a constraint in the program.

Our results do not answer, however, the more general question of the existence of canonical forms for non-triangular MHMM's. The problem with these latter is that they may have complex poles, so that we lose the possibility of imposing upon them a strict ordering as in the real case. We should mention that the use of complex probabilities and/or transition rates has already been suggested in the literature [3], but in this way the model does no more represent a real Markov process. It should be remarked that, although the use of non-real models is perfectly legitimate until their input-output behaviour does represent a real process, in some cases it may be difficult to verify this last condition. For example, if an n-TMHMM in canonical form A is used to compute a best Markovian approximation of a given distribution, the condition of nonnegativity of the transition rates ensures, at least, that the resulting approximation is itself a distribution, while this is not guaranteed if the nonnegativity condition is dropped.

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REFERENCES

[1] A. Bobbio, A. Premoli, O. Saracco, "Multistate homogeneous Markov models in reliability analysis", Microelectron. Reliab. 20, 875-880 (1980).

[2] A. Bobbio, A. Cumani, A. Premoli, O. Saracco, "Modelling and identification of non-exponential distributions by homogeneous Markov processes", Proc. 6th Advances in Reliab. Technol. Symp., 373-392, Bradford (1980).

[3] C. Singh and R. Billinton, "System Reliability Modelling and Evaluation," Hutchinson: London (1977).

[4] D.R. Cox and H.D. Miller, "The Theory of Stochastic Processes", Chapman and Hall: London (1965).

[5] W. Feller, "An Introduction to Probability Theory and its Applications", 2, 220, Wiley: New York (1971).

[6] J. Keilson, "Markov Chain Models - Rarity and Exponentiality", p. 23, Springer-Verlag: Berlin (1979).

APPENDIX

A. Proof of Theorem 1.

We notice first that the thesis is trivially true for n = 2, since a 2-TMHMM coincides with its unique elementary series, so we proceed by induction. Let (Λ, Ω) be an n-TMHMM and let the thesis be true for arbitrary (n-1)-TMHMM: (Λ', Ω'). Partition Λ and Ω as

$$\Lambda = \left[\begin{array}{c|c} -\lambda_1 & \emptyset \\ \hline \lambda_1 X & \Lambda' \end{array}\right] \qquad \Omega = \left[\begin{array}{c} w_1 \\ \hline (1-w_1)\Omega' \end{array}\right]$$

where one easily checks that X and Ω' are (n-1)-dimensional probability vectors, so that (Λ', X) and (Λ', Ω') are (n-1)-TMHMM's. Now,

$$(sI - \Lambda)^{-1} = \left[\begin{array}{c|c} \dfrac{1}{s+\lambda_1} & 0 \\ \hline \dfrac{1}{s+\lambda_1}(sI'-\Lambda')^{-1}X & (sI'-\Lambda')^{-1} \end{array}\right]$$

where I' is the identity of order n-1. Hence

$$F_s(s;\Lambda,\Omega) = \delta^{(n,n)T}(sI-\Lambda)^{-1}\Omega = \frac{w_1\lambda_1}{s+\lambda_1}\delta^{(n-1,n-1)T}(sI'-\Lambda')^{-1}X +$$

$$+(1-w_1)\delta^{(n-1,n-1)T}(sI'-\Lambda')^{-1}\Omega' = w_1\frac{\lambda_1}{s+\lambda_1}F_s(s;\Lambda',X)+(1-w_1)F_s(s;\Lambda',\Omega') =$$

$$= w_1A(s)+(1-w_1)B(s)$$

Now, by the induction hypothesis $B(s)$ is a mixture of ES's of Λ' (which by definition are also ES's of Λ); but one easily sees that $A(s)$ is itself a mixture of ES's of Λ (with state 1 as the first state), since $F_s(s;\ \Lambda',\ X)$ is a mixture of those ES's of Λ' which are connected to state 1 (i.e., those which start from a state i such that $\Lambda_{i1} = \lambda_1 x_{i-1} \neq 0$). Hence if the thesis is true for $n-1$, it is true for n too, q.e.d.

The second part of the thesis (weight of each ES) can be easily proved by recursively applying the above formulae.

B. Proof of Lemma 1.

Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1}$ be the ordered set of the eigenvalues of $-\Lambda$ and let E be an elementary series of Λ. We introduce a representation of E as the row vector $\hat{E} = [e_1\ e_2\ \ldots\ e_{n-1}]$ where $e_i = 1$ if $\lambda_i \in E$ and $e_i = 0$ otherwise. For example, let $n = 9$ and $E = < \lambda_4,\ \lambda_1,\ \lambda_5 >$; then

$$\hat{E} = [1\ 0\ 0\ 1\ 1\ 0\ 0\ 0]$$

Notice that the ordering of the λ_i's in the path corresponding to E is immaterial, since the cdf of an elementary series is invariant under permutation of the transition rates. Given this representation, define the following quantities:

5? $R(E)$ = index of the rightmost non-zero entry of $\hat{E}$

2? $Z(E)$ = number of zero entries between e_1 (inclusive) and e_R (obviously $Z(E) = 0$ iff E is a basic series)

3? $I(E)$ = index of the rightmost zero entry between e_1 and e_R (if $Z(E) = 0$, we define $I(E) = 0$)

$\Delta(E) = R(E) - I(E) \geq 1$

Now, let E be a non-basic series, hence $Z(E) \neq 0$. If $I(E) = k$, we apply identity (7) with $a = \lambda_{k+1}$ and $b = \lambda_k$ to represent E as the mixture of two series, say $S_1(E)$ and $S_2(E)$ where S_1 contains both λ_k and λ_{k+1} while S_2 contains only λ_k.

It is easy to see that for any E the following two cases apply:

I) if $\Delta(E) = 1$, then $Z[S_1(E)] = Z[S_2(E)] = Z(E) - 1$

II) if $\Delta(E) > 1$, then $Z[S_1(E)] = Z(E) - 1$ and $Z[S_2(E)] = Z(E)$ but $\Delta[S_2(E)] = \Delta(E) - 1$

We can now prove that the following procedure yields a representation of E as a finite mixture of basic series.

Start: $\nu_1 \leftarrow 1;\ k \leftarrow 1;\ \bar{Z}_k \leftarrow Z(E);\ E_1^1 \leftarrow E;$

Loop: $j \leftarrow 1;$
For $i = 1$ to ν_k do
Split: if $\Delta(E_i^k) = 1$
then begin $E_j^{k+1} \leftarrow S_1(E_i^k);\ E_{j+1}^{k+1} \leftarrow S_2(E_i^k);$ end
else begin $E_j^{k+1} \leftarrow S_1(E_i^k);\ j \leftarrow j+1;$
$E_i^k \leftarrow S_2(E_i^k);$ go to split; end
end;

Comment: at this point, the ν_k series E_i^k have been transformed into mixtures of $\nu_{k+1} = \sum_{i=1}^{\nu_k} [\Delta(E_i^k) + 1]$ series E_j^{k+1}, which satisfy $Z(E_j^{k+1}) = \bar{Z}_k - 1\ \ \forall j$; so

let $\bar{Z}_{k+1} \leftarrow \bar{Z}_k - 1$; if $\bar{Z}_{k+1} \neq 0$ then let $k \leftarrow k+1$; go to loop;
else stop

end.

It should be clear that each loop: step of this procedure involves a finite number of applications of (7) and therefore produces a finite number of terms in the expansion of E as a mixture of series; also, at each step $\bar{Z}_k$ is reduced by one, so that the process will ultimately stop after $k = Z(E)$ steps with $\bar{Z}_{k+1} = Z(E_j^{k+1}) = 0\ \ \forall j$, i.e. with a representation of E as a mixture of basic series, q.e.d.

C. Proof of Property 6.

The proof is given by the following two lemmas.

Lemma C1. For an n-MHMM, $P_j(t) \geq 0\ \ \forall i = 1, 2, \ldots, n,\ \forall t \geq \emptyset$.

The proof can be found in any textbook on Markov Processes, e.g. [4].

Lemma C2. For an irreducible n-MHMM, $P_i(t) \neq 0\ \ \forall i = 1, 2, \ldots, n,\ \forall t > \emptyset$.

Let by contradiction be some i and some $t_o > 0$ such that

$$P_i(t_o) = 0$$

Then

$$\dot{P}_i(t_o) = \sum_{k V_i} \Lambda_{ik} P_k(t_o)$$

where V_i is the set of states: $\{k:\ \Lambda_{ik} > 0\}$. Now either:

a) V_i is empty. Then either

a.1) $\Omega_i = 0$ but this contradicts irreducibility, or

a.2) $\Omega_i > 0$ but then

$$P_j(t) = \Omega_i \exp(\Lambda_{ii} t) > 0 \quad \forall t$$

which contradicts $P_i(t_o) = 0$.

b) V_i is nonempty. Then either

b.1) $P_k(t_o) \geq 0 \quad \forall k \in V_i$ and there is some $k \in V_i$ such that $P_k(t_o) > 0$;

hence

$$\dot{P}_i(t_o) > 0$$

but in that case there must be a left-neighborhood of t_o in which

$$P_i(t) < 0$$

which contradicts Lemma C1.

b.2) $P_k(t_o) = 0 \quad \forall k \in V_i$. In this case we repeat the above arguments for each $k \in V_i$; since the number of states is finite, we must ultimately reach a contradiction. Hence $P_i(t) \neq 0 \quad \forall t > 0$, q.e.d.

Now, for an n-MHMM the density of the cdf is given by

$$f(t) = \frac{d}{dt} F(t) = \frac{d}{dt} P_n(t) = M^T P(t)$$

where M^T is the last row of Λ. But M^T cannot be identically zero, since otherwise the final state would not be connected to the rest of the system. Since all components of $P(t)$ by lemma C2 are non-zero for $t > 0$, we conclude that $f(t) \neq 0$ $\forall t > 0$, q.e.d.