Markovian techniques for performance analysis of computer and communication systems

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Chapter 1

Introduction

In the field of performance analysis of computer and communication systems, which is somewhere in between application driven methodology and applied research, the most important external motivations are industrial needs, feasibility, and computability. During the last 10 years all of these external motivations changed a lot.

All important performance indexes of electronic computers (CPU speed, memory and disk access speed, memory and disk size) increased by several orders of magnitude, and even more, personal computers, dedicated to single users, provides this high performance. This changes made possible to solve previously inconceivable problems. But, unfortunately, in performance analysis the complexity of several computational problems increases exponentially (or faster) with several model features like number of components, complexity of components, complexity of system behaviour, etc. Due to the intrinsic complexity of real systems behaviour the available computational power is still a bottleneck of performance analysis. The applied numerical methods have to be designed considering the available computational power. The main challenge of performance modeling and analysis of real computer and communication systems remained finding an optimal trade off between the accuracy and the computability of the model. The increased computing power enlarged the set of applicable numerical methods, but did not eliminate the computing capacity constraints.

Stochastic models has always been applied for system performance analysis. For the 80's queueing network models and discrete and continuous time Markov chains were widely applied together with some decision and reward processes models. For special problems more complex processes, e.g., semi-Markov or Markov regenerative processes, were applied. Naturally, the applicability of these models were always determined by the available computing capacity which significantly improved by time. There were theoretical results available for several other classes of stochastic models as well, but they did not got widely applied due to complexity constraints. A series of new trends become visible during the 90's.

In the late 80's the signs of revolution in telecommunication were not evident yet. At that time the introduction of computers to network planning and dimensioning was in progress. At the same time according to the common understanding of electronic equipment's behaviour the analysis of hardware reliability and performance was a major research challenge. Significant changes were experienced in both

fields. Since the early 90's the sharp development of the telecommunication network in Hungary and the introduction of new services initiated a very fertile research in this field. Instead, the improvement of semiconductor technologies, which resulted in significantly more reliable hardware, and the increasing software complexity of electronic equipments turned away the attention from hardware reliability analysis.

One source of the new priorities roots in the evolution of packet switched telecommunication services. Measurement of real communication systems indicated strange stochastic phenomenon. Distributions with heavy tail and point processes with slowly decaying correlation structures were identified in packet switched communication networks. Stochastic models which previously were not applied in queueing and performance analysis gain significant attention in these fields. Examples of such models are fractal models and models based on large deviation theory. The set of applied modeling approaches also changed due to the availability of new analysis methods. The most significant in performance analysis is the development of matrix geometric methods. These methods allows to describe complex stochastic models in a compact way and provides effective numerical methods for their analysis. The availability of matrix geometric methods made the state space based methods very popular in practical applications.

In spite of the mentioned significant changes the main approach of applied performance analysis remained the same:

- The modeler should understand the behaviour of the considered system from performance point of view. Commonly, the performance issues of system behaviour are quite far from other engineering issues of the same system, hence engineers familiar with the functional behaviour of a system are often unable to draw right consequences on its performance. The technical details which are crucial for functional correctness of a system might be irrelevant for its performance analysis and vice-versa. It is also quite often the case that a modeler should distinguish between important and the less important issues of system behaviour, because it is impossible to capture the entire system behaviour.

- It is also an essential question in applied analysis which are the available data. Completely different methods are applicable when the stochastic rules of system behaviour are complete or only partially known. The later one is quite often the case in practice.

- The modeler should also know the possible modeling approaches which can be used to describe the considered system behaviour. It is not enough to know these methods, but one should also know their limits of applicability and their possible extensibility utilizing special model features.

- The computation of performance parameters is the execution of a computer program. Most commonly this program is not a special one developed only for the particular problem attacked, but it is a general purpose "tool". The key features of these tools are the applied model description language (e.g., queueing network model, stochastic Petri net, Markov chain, etc.) and the built-in analysis procedure. The modeler should know the available tools, their

functionality and limits. If not a general purpose analysis tool is applied the development of a specific analysis program is based on available library functions. A modeler should also know the best way to utilize the set of available elements for the solution of the considered problem.
It is often hidden from the end-users of an analysis program, but the internal algorithms used in a program has significant consequences, which makes the application of a general purpose tool difficult in several cases. For example, if the steady state analysis of a stochastic model is based on direct methods like LU decomposition or Gauss elimination then the cardinality of the state space can not be larger then 10^4, while iterative methods like successive over-relaxation or Gauss-Seidel allows a cardinality of 10^7. The cardinality of a stochastic model is not visible from high level model descriptions (e.g., stochastic Petri net), hence the modeler should have a very clear idea what the applied analysis tool does for him/her and how to interpret the 'answers' (e.g., what to do in case of 'infinite' response time).

In applied performance analysis the technical details of practical systems often make very hard to apply the available theoretical results, but we should mention the advantage of this fact as well. There are several very successful examples of the symbiosis of technical development and theoretical research also in the field of performance analysis of computer and communication systems. Modeling needs of motivated by technical details of real systems often advanced the set of analysis techniques. The set of results introduces in this dissertation contain examples of both directions of the interaction. E.g., on the one side, the effort for the analysis of Markov regenerative reward models is motivated by the need of performance analysis of systems with non-exponential activities (Chapter 4), and on the other side, the availability of analysis results for partial loss reward models allowed us to introduce a new performance model of transaction processing systems (Chapter 7).

The practice of computer and communication engineering evolved very fast in the last decade. This fast evolution resulted in a wide range of performance analysis research problems. The considered research problems are also taken from a wide range of application fields (e.g., software rejuvenation, performance of access methods in mobile telecommunication systems, resource sharing between different service classes in communication systems, packet switches, multimedia servers, ...) using a wide range of analysis tools (e.g., fluid models, decision processes, matrix geometric methods, reward processes, ...). The classification of these diverse research activities is based on a fuzzy methodological classification of research results. According to this classification the three main directions of applied methodology were: non-Markovian Petri net models, reward models and state space methods. To meet space limitations this dissertation surveys only the last two fields. This choice is motivated by the facts that Petri net models looses importance in applied performance analysis recently and the main focus of [81] was on non-Markovian Petri net models.

The following technical chapters introduce a large number of different measures and provide their analytical description. The applied notations is chosen to indicate the relation of the measures studied in different chapters, but the large number of studied measures do not make possible to apply a unified notation through the whole document without introducing very special notations. E.g., we introduced different

notations for the distribution of the accumulated reward in different partial reward models in Chapter 6 to emphasize the difference of the models, hence we can not use a unified notation for the distribution of the accumulated reward. Anyway, we use a unique notation for the basic quantities of the studied concepts as it is summarized in Appendix A and we define the applied special notations chapter-by-chapter.

Part I

Reward models

Chapter 2

Introduction to stochastic reward processes

This chapter provides a short introduction to the later studied subjects and a summary of the ideas as well as the used notations. Before the concept of reward processes we briefly discuss the considered stochastic processes and their properties based on the pioneer work of Cinlar [24].

2.1 Some basic discrete state stochastic processes

In this work we pay special attention to the stochastic processes $(Z(t))$ defined over a discrete state space (Ω), whose features can be characterized by the existence of (random) time instants, at which the future of the stochastic process depends only on its current state. Theoretically the time instants of this kind cover the past history of the process, thus they are called regenerative time points.

Definition 2.1. *T_n is called a* **regenerative time point**[1] *(RTP) if*

$$E\left\{f(Z(T_n+t_1),\ldots,Z(T_n+t_m),)\,|\,Z(T_n),Z(u),0\le u<T_n\right\}$$

$$=E\left\{f(Z(T_n+t_1),\ldots,Z(T_n+t_m),)\,|\,Z(T_n)\right\}$$

for any $0\le t_1\le\ldots\le t_m$, and bounded function defined on Ω^n.

This property is referred to as strong Markov property of the process at T_n taking $m=1$ ([24]).

The sequence of the RTPs plays special role in the study of stochastic processes.

Definition 2.2. *The sequence of the random variables $\{X_n,T_n;n\ge 0\}$ is said to be*

[1] It is referred as regeneration time in [24] p. 298 for renewal processes.

a (time homogeneous) **Markov renewal sequence**[2] *provided that*

$$Pr\{X_{n+1} = x, T_{n+1} - T_n \leq t \,|\, X_0, \ldots, X_n, T_0, \ldots, T_n\}$$

$$= Pr\{X_{n+1} = x, T_{n+1} - T_n \leq t \,|\, X_n\}$$

$$= Pr\{X_1 = x, T_1 - T_0 \leq t \,|\, X_0\}$$

for all $n \geq 0$, $x \in \Omega$ *and* $t \geq 0$.

It follows that the series of states $\{X_n; n \geq 0\}$ forms a Markov chain ([24]). It is commonly referred as embedded Markov chain (EMC).
In the following we restrict the considered Markov renewal sequences to the ones whose RTPs compose a strictly monotone increasing series ($T_0 < T_1 < T_2 < \ldots$). We generally suppose that the studied process starts at $T_0 = 0$.

In the sequel, we assume the considered stochastic processes to be right continuous, i.e., $Z(t) = Z(t^+), \quad \forall t \geq 0$.

2.1.1 Semi-Markov process

The time continuous stochastic process defined as the continuous time extension of a Markov renewal sequence is called semi-Markov process.

Definition 2.3. *$Z(t)$ is a (homogeneous)* **semi-Markov process** *(SMP) if a $\{X_n, T_n; n \geq 0\}$ Markov renewal sequence exists and*

$$Z(t) = X_n \ , \qquad \text{if } \ T_n \leq t < T_{n+1} \ .$$

There are some obvious consequences of this definition:

- $T_n, \ n \geq 0$ are RTPs of the process,
- there is no state transition between two consecutive RTPs,
- there can be RTP without real state transition (this case is considered as a virtual state transition from state i to state i [42]).

From the definition of the time homogeneous Markov renewal sequence one can argue that the probability

$$Pr\{X_1 = j, T_1 - T_0 \leq t \,|\, X_0 = i\} \ , \qquad i, j \in \Omega$$

plays a central role in the description of the Markov renewal sequences and the corresponding probability

$$Q_{ij}(t) = Pr\{Z(T_1) = j, T_1 \leq t \,|\, Z(0) = i\} \qquad i, j \in \Omega$$

in the description of the SMPs as well. The matrix $\mathbf{Q}(t) = \{Q_{ij}(t)\}$ is called the **kernel** of the SMP and summarizes all the information on the process that is necessary

[2]This sequence of random variables is called Markov renewal process in [24], but it is referred to as Markov renewal sequence in some later works ([21, 22]).

for evaluating its probabilistic behaviour. However, the $\mathbf{Q}(t)$ kernel representation of a SMP is not unique since there can be different kernels describing the same SMP, a canonical (unique and "minimal") representation can be obtained with 0 diagonal elements in the kernel matrix [81].

The Markov chain $\{X_n; n \geq 0\}$ is called the **embedded Markov chain** (EMC) of the SMP. According to this approach the time points T_n are called embedded time points, since the embedded Markov chain is formed by sampling the SMP at these time instants. The $\{X_n, T_n; n \geq 0\}$ Markov renewal sequence is also called the embedded Markov renewal sequence.

The one step state transition matrix of the EMC ($\mathbf{\Pi} = \{p_{ij}\}$) is derived from the kernel as:

$$p_{ij} = Pr\{Z(T_1) = j \,|\, Z(0) = i\} = \lim_{t\to\infty} Q_{ij}(t) \quad . \tag{2.1}$$

There are two possible interpretations of the evolution of a SMP:

- being in a given RTP, first, the next state is chosen from a discrete distribution (independent of the waiting time) and then the waiting time is sampled considering the next state from a (generally) continuous distribution,
- being in a given RTP, first, the waiting time is sampled from a (generally) continuous distribution (independent of the next state), then the next state is chosen considering the waiting time.

The quantities associated with these interpretations are as follows. The (unconditional) distribution of the next state p_{ij}, which is sometimes referred as switching probability, has been already introduced (2.1). The probability distribution of the waiting time conditioned on the next state is written as

$$H_{ij}(t) = Pr\{T_1 \leq t \,|\, Z(T_1) = j, Z(0) = i\} = \frac{Q_{ij}(t)}{p_{ij}} \quad ,$$

the (unconditional) distribution of the waiting time is obtained as

$$Q_i(t) = Pr\{T_1 \leq t \,|\, Z(0) = i\} = \sum_{j\in\Omega} Q_{ij}(t) \quad ,$$

and finally the switching probability conditioned on the holding time is given by

$$p_{ij}(t) = Pr\{Z(T_1) = j \,|\, T_1 = t, Z(0) = i\} =$$

$$\lim_{\Delta\to 0} \frac{Pr\{Z(T_1) = j, t < T_1 \leq t + \Delta \,|\, Z(0) = i\}}{Pr\{t < T_1 \leq t + \Delta \,|\, Z(0) = i\}} =$$

$$\lim_{\Delta\to 0} \frac{Q_{ij}(t+\Delta) - Q_{ij}(t)}{Q_i(t+\Delta) - Q_i(t)} = \frac{dQ_{ij}(t)}{dQ_i(t)} \quad .$$

p_{ij} and $H_{ij}(t)$ are the functions for the description of the SMP according to the first interpretation, while $Q_i(t)$ and $p_{ij}(t)$ defines the distributions according to the second one.

Stationary analysis

The simplest stationary analysis of SMPs is based on the Markov chain embedded into state transitions instances. The transition probabilities of the EMC are $\mathbf{\Pi} = \{p_{ij}\}$. The stationary distribution of the EMC is obtained from the linear system of equations: $\hat{P} = \hat{P}\Pi, \sum_i \hat{P}_i = 1$. The stationary distribution of the SMP is obtained by weighting with the mean sojourn times of states $\hat{\tau}_i$:

$$P_i = \frac{\hat{P}_i \hat{\tau}_i}{\sum_j \hat{P}_j \hat{\tau}_j} \tag{2.2}$$

where $\hat{\tau}_i = \int_0^\infty 1 - Q_i(t)dt$.

Transient analysis

The state transition matrix is denoted by $\mathbf{V}(t)$, whose elements are

$$V_{ij}(t) = Pr\left\{Z(t) = j \,|\, Z(0) = i\right\} \quad .$$

Theorem 2.4. *The transition probability ($V_{ij}(t)$) satisfies the following equation [42]:*

$$V_{ij}(t) \;=\; \delta_{ij}\left[1 - Q_i(t)\right] \;+\; \sum_{k\in\Omega} \int_{h=0}^{t} V_{kj}(t-h)\, dQ_{ik}(h) \tag{2.3}$$

Proof. Based on the above defined properties of the SMP in the RTPs and by conditioning on the time to the next RTP ($T_1 = h$) we have:

$$V_{ij}(t \,|\, T_1 = h) \;=\; \begin{cases} \delta_{ij} & \text{if}: h \;>\; t \\ \displaystyle\sum_{k\in\Omega} \frac{dQ_{ik}(h)}{dQ_i(h)} \;\cdot\; V_{kj}(t-h) & \text{if}: h \;\leq\; t \end{cases} \tag{2.4}$$

where δ_{ij} is the Kronecker delta[3]. In (2.4) two mutually exclusive events are defined. If there is no RTP up to t the value of the state transition probability can be 1 (if $i = j$) or 0 (if $i \neq j$). If the first RTP occurs before t a state transition (real or virtual) happens and the state transition probability can be evaluated independently from that time.

Based on the distribution of T_1, $Q_i(t)$, the unconditional state transition probabilities are:

$$V_{ij}(t) \;=\; \int_{h=t}^{\infty} \delta_{ij}\, dQ_i(h) + \int_{h=0}^{t} \sum_{k\in\Omega} V_{kj}(t-h)\, dQ_{ik}(h) \tag{2.5}$$

[3] $\delta_{ij} = \begin{cases} 1 & \text{if}: i = j \\ 0 & \text{if}: i \neq j \end{cases}$

Equation 2.3 is obtained from Equation 2.5.

□

By solving this integral equation set we have the transient behaviour of a SMP in time domain. The convolution in (2.3) suggests us to look for the solution also in transform domain.

Let us denote the Laplace transform *(LT)* and the Laplace-Stieltjes transform *(LST)* of $F(t), t \geq 0$ as $F^*(s)$ and $F^{\sim}(s)$ respectively, where:

$$F^*(s) \;=\; \int_0^\infty e^{-st}\, F(t)\, dt \qquad \text{and} \qquad F^{\sim}(s) \;=\; \int_0^\infty e^{-st}\, d\, F(t) \;\; .$$

The introduction of the second one is useful for the cases in which $F_X(t)$ is the cumulated distribution function of a positive random variable X, because

$$F_X^{\sim}(s) \;=\; E\left\{e^{-sX}\right\} \;\; .$$

By transforming (2.3) into LST domain we have:

$$V_{ij}^{\sim}(s) \;=\; \delta_{ij}\left[1 - Q_i^{\sim}(s)\right] + \sum_{k\in\Omega} Q_{ik}^{\sim}(s)\, V_{kj}^{\sim}(s) \tag{2.6}$$

By rearranging (2.6) into matrix form we obtain:

$$\mathbf{V}^{\sim}(s) \;=\; \mathbf{Q}_{\mathbf{D}}^{\sim}(s) \;+\; \mathbf{Q}^{\sim}(s)\, \mathbf{V}^{\sim}(s) \tag{2.7}$$

where $\mathbf{Q}_{\mathbf{D}}^{\sim}(s)$ is a diagonal matrix with elements $\{1 - Q_i^{\sim}(s)\}$. Finally the solution of (2.7) can be easily derived as:

$$\mathbf{V}^{\sim}(s) \;=\; \left[\mathbf{I} - \mathbf{Q}^{\sim}(s)\right]^{-1} \mathbf{Q}_{\mathbf{D}}^{\sim}(s) \tag{2.8}$$

The $\left[\mathbf{I} - \mathbf{Q}^{\sim}(s)\right]^{-1}$ matrix is called the Markov renewal kernel, and its elements are called the Markov renewal functions in [24].

Note that the steady state results can be obtained from the transient ones taking the $t \to \infty$ limit in the time domain description or the $s \to 0$ limit in the transform domain one.

SMP representation of Continuous Time Markov Chains

We can introduce Continuous Time Markov Chains (CTMC) as a special SMP, whose every time instant $t \geq 0$ is a RTP.

A homogeneous CTMC can be described by its (time independent) infinitesimal generator matrix $\mathbf{A}$, whose $a_{ij};\ i \neq j$ elements are the transition rates from state i to state j ($a_{ij} \geq 0;\ i \neq j$) and $a_{ii} = -\sum_{i\in\Omega, i\neq j} a_{ij}$ $(a_{ii} \leq 0)$ [42]. The following (canonical) kernel provides the SMP description of a CTMC with infinitesimal generator $\mathbf{A}$:

$$Q_{ij}(t) \;=\; \begin{cases} \dfrac{a_{ij}}{-a_{ii}}\left(1 \;-\; e^{a_{ii}\, t}\right) & \text{if}: i \neq j \\ \\ 0 & \text{if}: i \;=\; j \end{cases} \tag{2.9}$$

2.1.2 Markov regenerative process

Markov regenerative processes (MRP) form a more general class of stochastic processes. Similar to SMPs, MRPs exhibit RTPs, but state transitions are allowed between any two consecutive RTPs. It is the key property by which the class of MRPs is more general than the class of SMPs.

Definition 2.5. *$Z(t)$ is a (homogeneous)* **Markov regenerative process** *if there exists a Markov renewal sequence $\{X_n, T_n; n \geq 0\}$ that*

$$Pr\{Z(T_n + t_1) = x_1, \ldots, Z(T_n + t_m) = x_m \,|\, Z(T_n), Z(u), 0 \leq u < T_n\} =$$

$$Pr\{Z(T_n + t_1) = x_1, \ldots, Z(T_n + t_m) = x_m \,|\, Z(T_n)\}$$

for all $m \geq 1$, $0 < t_1 < \ldots < t_m$ and $x_1, \ldots, x_m \in \Omega$.

This definition can be expressed in words as, $Z(t)$ is a MRP if there exists a Markov renewal sequence $\{X_n, T_n;\ n \geq 0\}$ of random variables such that all the finite dimensional distributions of $\{Z(T_n + t);\ t \geq 0\}$ given $\{Z(u),\ 0 \leq u < T_n,\ X_n = i\}$ are the same as those of $\{Z(t);\ t \geq 0\}$ given $X_0 = i$.

Due to the homogenity of the process, Definition 2.5 states that a MRP process viewed from two RTPs with the same states (for example $Z(t - T_n)$ and $Z(t - T_m)$ if $X_n = X_m$) forms the probabilistic replica of each other. The Markov renewal sequence $\{X_n, T_n;\ n \geq 0\}$ is also referred to as the embedded Markov renewal sequence of the MRP.

Define the state transition probabilities of the process before the next RTP

$$G_{ij}(t) = Pr\{Z(t) = j \,|\, T_1 > t,\ Z(0) = i\} \quad ,$$

and the probabilities which describe the occurrence of the next RTP

$$K_{ij}(t) = Pr\{Z(T_1) = j, T_1 \leq t \,|\, Z(0) = i\} \quad .$$

The matrix $\mathbf{K}(t)$ is the kernel of the embedded Markov regenerative sequence ($\{X_n, T_n; n \geq 0\}$) and plays similar role as $\mathbf{Q}(t)$ for SMPs. The switching probability conditioned on the time to the next RTP is:

$$p_{ij}(t) = Pr\{Z(T_1) = j \,|\, T_1 = t, Z(0) = i\} = \frac{dK_{ij}(t)}{dK_i(t)} \quad .$$

The evolution of MRPs can be divided into independent parts by the RTPs.

Definition 2.6. *The stochastic process (denoted by $Z^i(t)$) subordinated to a MRP starting from state i in a RTP up to the next RTP is the restriction of the MRP $Z(t)$ for $t \leq T_1$ given $Z(T_0) = i; T_0 = 0$:*

$$Z^i(t) \;=\; [Z(t) \;:\; 0 \leq t \leq T_1,\ Z(0) = i]$$

referred to as the subordinated process starting from state i.

Stationary analysis

Similar to the SMP case the stationary analysis of MRPs is based on the Markov chain embedded into RTPs of the process. The stationary distribution of the EMC is obtained from the linear system of equations: $\hat{P} = \hat{P}\Pi, \sum_i \hat{P}_i = 1$. The stationary distribution of the MRP is obtained by weighting with respect to the time spent in different system states between consecutive RTPs. $\check{\tau}_{ij}$ denotes the mean time spent in state j during the subordinated process starting in state i. The stationary probabilities are obtained as:

$$P_i = \sum_j \hat{P}_j \frac{\check{\tau}_{ji}}{\sum_k \check{\tau}_{jk}}, \tag{2.10}$$

where $\check{\tau}_{ij} = \int_0^\infty E_{ij}(t)dt$.

Transient analysis

For notation convenience the following quantity is introduced ([24]):

$$E_{ij}(t) = G_{ij}(t)\left[1 - K_i(t)\right] = Pr\left\{Z(t) = j \,|\, T_1 > t,\, Z(0) = i\right\} Pr\left\{T_1 > t\right\}$$

$$= Pr\left\{Z(t) = j,\, T_1 > t,\, |\, Z(0) = i\right\} \quad ,$$

Theorem 2.7. *The state transition probability ($V_{ij}(t)$) satisfies the following equation [24]:*

$$V_{ij}(t) \;=\; E_{ij}(t) \;+\; \sum_{k\in\Omega} \int_{h=0}^{t} V_{kj}(t-h)\, dK_{ik}(h) \tag{2.11}$$

Proof. Let us define the state transition probabilities conditioning on $T_1 = h$:

$$V_{ij}(t \,|\, T_1 = h) \;=\; \begin{cases} G_{ij}(t) & \text{if}: h \;>\; t \\ \displaystyle\sum_{k\in\Omega} \frac{dK_{ik}(h)}{dK_i(h)} \;\cdot\; V_{kj}(t-h) & \text{if}: h \;\leq\; t \end{cases} \tag{2.12}$$

In (2.12), similarly to (2.4) two mutually exclusive events are defined. If there is no RTP up to t, $G_{ij}(t)$ is the probability of the state transition by its definition. If there is at least one RTP before t the process jumps to the next regeneration state (which can be i as well in general) according to the switching probabilities and due to the property of the RTPs, the state transition probability is evaluated from that time.

By evaluating the unconditional state transition probability based on the distribution of T_1 ($K_i(t)$) (2.12) becomes:

$$\begin{aligned} V_{ij}(t) \;&=\; \int_{h=t}^{\infty} G_{ij}(t)\, dK_i(h) + \int_{h=0}^{t} \sum_{k\in\Omega} V_{kj}(t-h)\, dK_{ik}(h) \\ &=\; G_{ij}(t)\left[1 - K_i(t)\right] \;+\; \sum_{k\in\Omega} \int_{h=0}^{t} V_{kj}(t-h)\, dK_{ik}(h) \end{aligned} \tag{2.13}$$

Equation 2.13 yields Equation 2.11 by substituting $E_{ij}(t)$ for $G_{ij}(t)\left[1 - K_i(t)\right]$. □

The solution of (2.11) can be performed in the same manner as (2.3). The transformation of (2.11) into LST domain results in:

$$V_{ij}^{\sim}(s) \;=\; E_{ij}^{\sim}(s) \;+\sum_{k\in\Omega} K_{ik}^{\sim}(s)\, V_{kj}^{\sim}(s) \tag{2.14}$$

whose matrix form is:

$$\mathbf{V}^{\sim}(s) \;=\; \mathbf{E}^{\sim}(s) \;+\; \mathbf{K}^{\sim}(s)\, \mathbf{V}^{\sim}(s) \tag{2.15}$$

and the matrix form solution can be written as:

$$\mathbf{V}^{\sim}(s) \;=\; [\mathbf{I} - \mathbf{K}^{\sim}(s)]^{-1}\, \mathbf{E}^{\sim}(s) \tag{2.16}$$

Equations (2.11) and (2.16) are the commonly applied equations for the analysis of MRPs, and matrices $\mathbf{K}$(t) (called **global kernel**) and $\mathbf{E}$(t) (called **local kernel**) are the usual descriptors of a MRP. The $\mathbf{K}(t)$, $\mathbf{E}(t)$ representation of MRPs is much weaker than the $\mathbf{Q}(t)$ representation of SMPs because the $\mathbf{K}(t)$, $\mathbf{E}(t)$ representation does not defines all finite dimensional joint distribution of the MRP process.

Basically, Markov renewal theory allows as to partition the analysis of MRPs into independent sub-problems. The evolution of a MRP is composed by the "local" evolution inside subordinated processes (described by $\mathbf{E}(t)$) and a "global" evolution of the occurrence of the RTPs and the associated states (described by $\mathbf{K}(t)$).

2.2 Stochastic Reward Models

Adding a continuous variable to discrete state system models enlarges the *modeling power* of system description and allows to evaluate more performance parameters. The adopted modeling framework consists in describing the behaviour of the system configuration (system state) in time by means of a stochastic process, called the *structure-state process*, and by associating a (non-negative) reward function to the structure-state process which describes the "reward" accumulated by the system during its evolution. The interpretation of "reward" might be taken from a wide range of engineering problems, e.g.: amount of good produced by a machine, amount of stress accumulated by a system, amount of data transmitted by a communication system, revenue, etc.

In applied performance modeling the reward function is restricted to some special simple functions. The amount of reward might increase due to "rate" and "impulse" reward accumulation and it might decrease due to partial or complete "reward loss" (Figure 2.1). Rate reward is continuously accumulated at rate r_i ($r_i \geq 0$) during the sojourn in state i (Figure 2.1.a) and ϱ_{ij} ($\varrho_{ij} \geq 0$, possibly random) amount of impulse reward is instantaneously gained at a state transition from state i to j (Figure 2.1.c). Instantaneous partial or complete reward loss might also occur at state transitions. Complete reward loss sets the amount of accumulated reward to 0 (Figure 2.1.b), while a partial reward loss sets it value somewhere between the so far accumulated value and 0 (Figure 2.1.d). The structure-state process together with the reward function forms the Stochastic Reward Model (SRM).

Let the *structure-state process* $Z(t)$ ($t \geq 0$) be a (right continuous) stochastic process defined over a discrete and finite state space Ω of cardinality n.

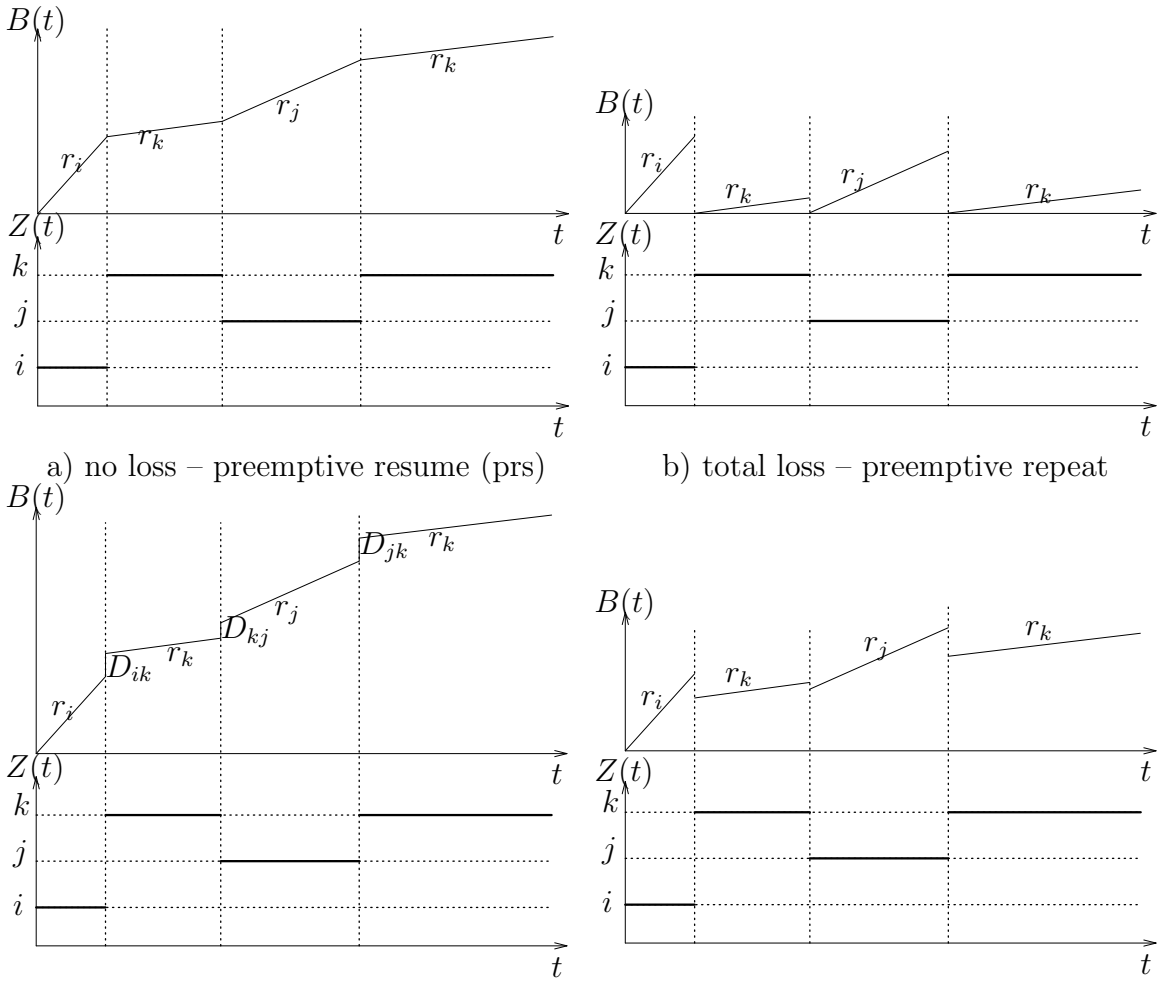


Figure 2.1: Change of accumulated reward at state transitions

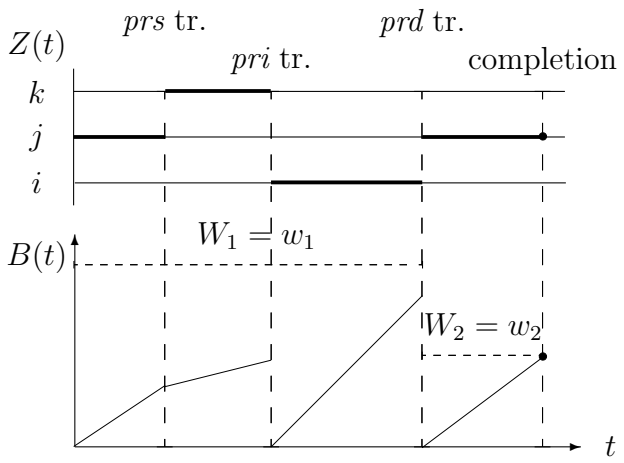


Figure 2.2: The behaviour of the functional $B(t)$ versus time.

Definition 2.8. *The* **accumulated reward** $B(t)$ *is a random variable which represents the accumulation of reward in time.*

During the sojourn of $Z(t)$ in state i between t and $t+\delta$, $B(t)$ increases by $r_i\,\delta$. $B(t)$ is a stochastic process that depends on $Z(u)$ for $0 \leq u \leq t$ [24]. However, a transition in $Z(t)$ may induce a modification in the accumulation process depending whether the transition entails a *loss of accumulated reward* , or *no loss of accumulated reward*. A transition which does not entail any loss of reward already accumulated by the system is called *preemptive resume* (Figure 2.1.a and c), and its effect on the model is that the functional $B(t)$ resumes the previous value in the new state. A transition which entails the total loss of reward accumulated by the system is called *preemptive repeat* (Figure 2.1.a), and its effect on the model is that the functional $B(t)$ is reset to 0 in the new state.

A state whose outgoing transitions are all of preemptive resume type is called a *preemptive resume (prs)* state, while a state whose outgoing transitions are all of preemptive repeat type is called a *preemptive repeat (prt)* state.

A possible realization of the accumulation process $B(t)$ with only rate reward and complete reward loss is shown in Figure 2.1.b.

The complementary problem concerning the reward accumulation of SRMs is the amount of time for completing a given (possibly random) work requirement (i.e. time to accumulate the required amount of reward).

Definition 2.9. *The* **completion time** C *is a random variable representing the time to accumulate a reward requirement equal to a random variable* W*:*

$$C \;=\; min\,[t \geq 0 \;:\; B(t) \;=\; W] \;\;.$$

C is the time at which the work accumulated by the system reaches the value W for the first time. With reference to Figure 2.2, the completion time is the time at which $B(t)$ hits the barrier W for the first time.

We assume, in general, that W is a random variable with distribution $W(w)$ with support on $(0,\ \infty)$. The degenerate case, in which W is deterministic and the distribution $W(w)$ becomes the unit step function $U(w - w_d)$, can be considered as well. When W is a random variable and there is a complete reward loss at a state transition (*prt* policy), two cases arise depending whether the repeated task has the identical work requirement as the original task (*preemptive repeat identical (pri)* - policy) (second transition on Figure 2.2), or a different work requirement is sampled from the same distribution (*preemptive repeat different (prd)* - policy) (third transition on Figure 2.2). In the latter case, each time when the functional $B(t)$ goes to zero, the barrier height W is resampled from the same distribution $W(w)$, while in the former case W maintains an identical value.
For a barrier height $W\ =\ w$, the completion time $C(w)$ is defined as:

$$C(w)\ =\ \min\ [t \geq 0\ :\ B(t)\ =\ w]\ . \tag{2.17}$$

Let $C(t, w)$ be the *Cdf* of the completion time when the barrier height is w:

$$C(t, w)\ =\ Pr\,\{C(w)\ \leq\ t\} \tag{2.18}$$

The completion time C of a SRM with *prs* and *pri* transitions is characterized by the following distribution:

$$\hat{C}(t)\ =\ Pr\,\{\,C \leq t\}\ =\ \int_0^\infty C(t, w)\, dW(w) \tag{2.19}$$

The distribution of the completion time $C(t, w)$ incorporates the effect of random variation of the reward rate, instantaneous reward accumulation and reward loss during the evolution of the structure state process.

The following relationships between the different preemption policies can be easily established. If the work requirement W is an exponential random variable, the two policies *prs* and *prd* give rise to the same completion time (due to the memoryless property of the exponential distribution, the residual task requirement under the *prs* policy coincides with the resampled requirement under the *prd* policy). On the other hand, if W is deterministic, the two policies *pri* and *prd* are coincident (resampling a step function provides always the same constant value).

Moreover, assuming that the structure-states are all of *prs* type, so that no loss of reward occurs, the distribution of the completion time is closely related to the distribution of the accumulated reward by means of the following relation:

$$Pr\,\{B(t)\ \leq\ w\}\ =\ Pr\,\{C(w)\ \geq\ t\} \tag{2.20}$$

2.3 Classification of the SRM problems

To characterize the SRM problems we introduce a structure of the considered parameters.

Stochastic process The stochastic behaviour of the structure-state process gains a significant importance at the first sight. SRMs of simple (CTMC) or less

complex (SMP) stochastic processes are considered since a long time [49, 50, 67]. The analysis of SRM with MRP structure-state process is an original result in this work which was first reported in [78].

Reward accumulation The two considered ways of reward accumulation are rate and impulse reward accumulation. The majority of SRMs applied in practice contain only rate reward accumulation. SRMs with only impulse reward accumulation can be used to "count" events in stochastic processes. A unit impulse reward associated with a set of state transitions representing a particular event of the system counts the occurrence of that event. This work focuses on SRMs with rate reward accumulation. Research results about SRMs with rate and impulse reward accumulation are provided in [68, 69].

This work is restricted to singe reward accumulation. A more general class of SRMs is obtained when multiple type of rewards are accumulated by the system [7].

Preemption policy (Reward loss) The effect of the state transitions, is also referred to as preemption policy. The preemption policy defines which portion of the accumulated reward is lost at a state transition. Traditionally lossless, *prs*, and total loss, *prd* and *pri*, policies were applied. The extension to partial loss reward models is a new contribution which was originally presented in [11]. The existence of the different policies in a single model increases its modeling power, but it also increases the complexity of the model description and analysis. This work is restricted to SRMs with unique preemption policy, but a wide range of possible preemption policies are studied.

Evaluated measure The analysis of SRMs means indeed two analysis problems: the evaluation of the distribution of the accumulated reward and of the completion time. Both problems are considered in this work. It turns out that particular preemption policies result in qualitative differences in the accumulated reward and the completion time measures (e.g., the distribution of accumulated reward exhibit a closed form transform domain expression while the distribution of the completion time does not).

Absorbing subset of states There are practically important modeling problems in which the entrance of the structure state process in a special subset of states stops the accumulation of further reward independent of the later life of the model. For the purpose of the analysis a subset of this kind can be considered as an absorbing one.

State dependent measures The analysis of complex discrete state models often requires to evaluate state dependent measures, where the initial and final state of the underlying process are given. This state dependent measures, which are commonly presented in form of matrix function, provide the joint distribution of the system state and the studied reward measure. Examples of state dependent reward measures are:

- the probability of completion in a given state before time t,

- the probability of being in a given state at time t suppose that $C > t$

The numerical analysis of state dependent reward measures is usually similar to the analysis of the equivalent global (state independent) measures, but computationally it is far more expensive in case of large state spaces, since matrices needs to be stored in memory instead of vectors and matrix-matrix multiplications are executed instead of matrix-vector ones.

Chapter 3

Semi-Markov reward models with PH work requirement

Various numerical techniques have been investigated for the evaluation of reward models defined over a CTMC, but the numerical analysis of SRMs with underlying Semi-Markov Process (referred to as Semi-Markov Reward Process, SMRP) is still a challenge. This chapter introduce an effective algorithm for the analysis of SMRPs when the work requirement is a Phase type (PH) random variable. Bobbio and Trivedi [17] studied this problem when the structure state process is a CTMC, but the analysis of SMRPs with PH work requirement requires a completely different approach.

3.1 Reward Semi-Markov Process

Let Ω be the set of structure states and $Z(t)$ $(t \geq 0)$ be the semi-Markov process defined over Ω. r_i is the reward rate associated with state $i \in \Omega$ and the initial state probability vector is $\underline{P}(0)$ $(Pr\{Z(0) = i\} = P_i(0))$.

Let $\mathbf{Q}(t) = [Q_{ij}(t)]$ be the kernel of the semi-Markov process $Z(t)$. We denote by H the time duration until the first embedded time point in the semi-Markov process starting from state i at time 0 ($Z(0) = i$).

We introduce the following matrix functions to describe the distribution of the completion time:

$$F_{ij}(t,w) = Pr\{Z(C(w)) = j\,,\, C(w) \leq t \,|\, Z(0) = i\}, \tag{3.1}$$

$$\hat{C}_{ij}(t) = Pr\{Z(C) = j\,,\, C \leq t \,|\, Z(0) = i\}, \tag{3.2}$$

where $C(w)$ (C) is the completion time r.v. of w unit of work (of the random work requirement W with distribution $G_W(w) = Pr\{W \leq w\}$). $F_{ij}(t,w)$ $(\hat{C}_{ij}(t))$ is the probability that the completion of w (W) unit of work happens in state j before time t, starting in state i at $t = 0$. The LST of $\hat{C}_{ij}(t)$ is denoted by $C^{\sim}_{ij}(s)$. The distribution of the completion time is determined from $F_{ij}(t,w)$ by the mean of the following equation:

$$C(t) = \sum_{j\in\Omega}\sum_{i\in\Omega} P_i(0)C_{ij}(t). = \int_{w=0}^{\infty} \sum_{j\in\Omega}\sum_{i\in\Omega} P_i(0)F_{ij}(t,w)\; dG_W(w) \tag{3.3}$$

The derivation of $F_{ij}(t,w)$ based on the kernel matrix $Q_{ij}(t)$ can be inferred from [50, 13]:

$$F_{ij}^{\sim *}(s,v) \;=\; \delta_{ij}\,\frac{r_i\,[1\,-\,Q_i^{\sim}(s\,+\,v\,r_i)\,]}{s\,+\,v\,r_i}\,+\,\sum_{k\in\Omega}\,Q_{ik}^{\sim}(s\,+\,v\,r_i)\,F_{kj}^{\sim *}(s,v). \qquad (3.4)$$

3.1.1 Evaluation of the completion time

The evaluation of the completion time requires the execution of the following steps:

- Derivation of the matrix function $F_{ij}^{\sim *}(s,v)$ in double transform domain according to Equation (3.4).
- Evaluation of the LST transform $F_{ij}^{\sim}(s,w)$ by symbolic inverse Laplace transformation with respect to the work requirement variable v.
- Evaluation of the LST transform of the completion time $\mathbf{C}^{\sim}(s)$ by unconditioning the results of the previous step with respect to the distribution of the work requirement $G_W(w)$ (see Equation (3.3)).
- Time domain solution obtained by a numerical inversion of $\mathbf{C}^{\sim}(s)$, for example by resorting to the Jagerman's method [44].

Due to the required symbolic and numerical steps, the outlined procedure is not applicable for SRMs with more than ∼10 states. When the work requirement ($G_W(w)$) is a *PH* random variable, steps 2 and 3, can be evaluated by an effective computational method.

3.1.2 PH distributed work requirement

Let us define a Phase type *(PH)* distribution as the time to absorbtion in a CTMC with N transient and one absorbing state. The probability distribution of a *(PH)* r.v. has a rational Laplace transform, hence its probability density function can be expressed as:

$$g(w) = \sum_{p=1}^{n}\sum_{r=0}^{m_p-1}\; c_{pr}\; w^r\; e^{-\lambda_p w}, \qquad (3.5)$$

where n is the number of distinct poles (of the rational function in Laplace domain), m_p is the multiplicity of pole λ_p, and c_{pr} is a constant coefficient. $N = \sum_{p=1}^{n} m_p$.

When the work requirement is a *PH* random variable the following effective computational procedure can be used to speed up the computation and to handle larger models. The proposed procedure completes the inverse transformation with respect to v and the integration with respect to $g(w)$ in one step with low computational cost.

Theorem 3.1. *The distribution of the completion time of a* PH *the work requirement with probability density function $g(w)$ (3.5), can be evaluated as follows:*

$$C_{ij}^{\sim}(s) \;=\; \sum_{p=1}^{n}\sum_{r=0}^{m-1}(-1)^r c_{pr}\frac{d^r F_{ij}^{\sim *}(s,v)}{dv^r}\bigg|_{v=\lambda_p} \tag{3.6}$$

where the derivative of order $r=0$ is the original function. This way for $r=0$ the $F_{ij}^{\sim *}(s,v)$ function is evaluated at $v=\lambda_p$.

Proof. When γ is a *PH* r.v. Equation (3.3) becomes:

$$C_{ij}^{\sim}(s) \;=\; \int_{w=0}^{\infty} F_{ij}^{\sim}(s,w)\,dG(w) \;=\; \int_{w=0}^{\infty} g(w)\,F_{ij}^{\sim}(s,w)\,dw \;=$$

$$\sum_{p=1}^{n}\sum_{r=0}^{m-1} c_{pr}\int_{w=0}^{\infty} w^r e^{-\lambda_p w}\,F_{ij}^{\sim}(s,w)\,dw = \sum_{p=1}^{n}\sum_{r=0}^{m-1}(-1)^r c_{pr}\int_{w=0}^{\infty}\frac{d^r}{d\lambda_p^r}e^{-\lambda_p w}\,F_{ij}^{\sim}(s,w)\,dw =$$

$$\sum_{p=1}^{n}\sum_{r=0}^{m-1}(-1)^r c_{pr}\frac{d^r}{d\lambda_p^r}\int_{w=0}^{\infty} e^{-\lambda_p w}\,F_{ij}^{\sim}(s,w)\,dw = \sum_{p=1}^{n}\sum_{r=0}^{m-1}(-1)^r c_{pr}\frac{d^r\,F_{ij}^{\sim *}(s,\lambda_p)}{d\lambda_p^r} \tag{3.7}$$

from which the theorem (Equation (3.6)) follows. □

This approach is very effective when the multiplicity of the poles is equal to 1, since the inverse Laplace transformation and integration in (3.7) reduces to a simple substitution; otherwise the symbolic derivation of $F_{ij}^{\sim *}(s,v)$ is required.

3.2 Numerical example: Series System with Repair

Consider a series system of two machines, **a** and **b**, with constant failure rates λ_a and λ_b, respectively. If any of the machines fails, both machines are switched off, and the faulty machine is repaired with a generally distributed random repair time, according to distribution functions $G_a(t)$ or $G_b(t)$. We assume that no machine can fail while the system is down, and that the two machines are independent.

The system behaviour is described on Figure 3.1(a) by a Stochastic Petri Net. Place p_1 contains a token, when machine **a** is in up state. Transition t_1 represents the failure of machine **a**. When a failure happens, a token is placed to p_3, and the repair is immediately started. Transition t_3 represents the repair of machine **a**. The firing time of t_1 is exponentially distributed with parameter λ_a, while the firing time of t_3 is generally distributed, according to $G_a(t)$. The same description applies for machine **b**, with the appropriate indices. The inhibitor arcs represent the restriction that no machine can fail when the system is down, i.e., when there is a token in place p_3 or p_4.

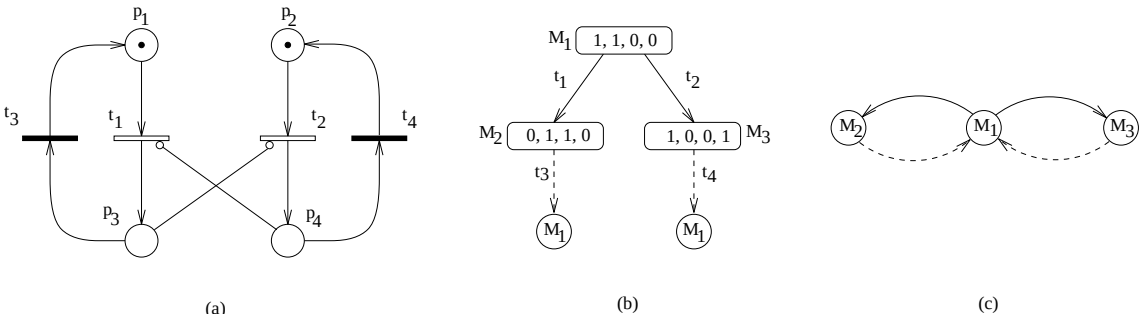


Figure 3.1: Series System with Repair

The reachability tree and the reachability graph of this Petri net is provided on Figure 3.1(b) and (c), respectively. Each marking is a 4-tuple counting the number of tokens in places p_1 to p_4. Solid arcs represent transitions according to exponential distribution, while dashed arcs represent transitions according to general distributions.

3.2.1 Evaluation of the Completion Time

Since the only up state of the system is when both p_1 and p_2 contain a token, the reward rate vector is $\underline{r} = \{1, 0, 0\}$. Let us suppose that the system starts from state 1 at time $t = 0$, i.e. $P(0) = \{1, 0, 0\}$. The $Q^{\sim}(s)$ matrix can be written as

$$Q^{\sim}(s) = \begin{bmatrix} 0 & \dfrac{\lambda_a}{s+\lambda_a+\lambda_b} & \dfrac{\lambda_b}{s+\lambda_a+\lambda_b} \\ G_a^{\sim}(s) & 0 & 0 \\ G_b^{\sim}(s) & 0 & 0 \end{bmatrix} . \tag{3.8}$$

Since the procedure starts in state 1, only the first column of matrix $\mathbf{F}^{\sim *}(s, v)$ plays role in the evaluation of the completion time. Furthermore, since the reward vector in our example is $\underline{r} = \{1, 0, 0\}$, $F_{11}^{\sim *}(s, v)$ is the only entry of $\mathbf{F}^{\sim *}(s, v)$ that has an affect on $C^{\sim}(s)$:

$$C^{\sim}(s) = \int_{w=0}^{\infty} F_{11}^{\sim}(s, w)\ dG_g(w), \tag{3.9}$$

where

$$F_{11}^{\sim *}(s, v) = \frac{1}{s + v + \lambda_a(1 - G_a^{\sim}(s)) + \lambda_b(1 - G_b^{\sim}(s))} . \tag{3.10}$$

There are two ways to derive $C^{\sim}(s)$:

- Symbolic inverse Laplace transformation of $F_{11}^{\sim *}(s, v)$, and integration according to Equation 3.9.
- Application of Theorem 3.1 where phase type approximation of the work requirement is applied when it is not a *PH* random variable.

	Exact calc.	Erlang(2)			Erlang(4)			Erlang(8)		
		Intensity diff.			Intensity diff.			Intensity diff.		
		0%	1%	10%	0%	1%	10%	0%	1%	10%
m_1	33	33	33	33	33	33	33	33	33	33
m_2	1239	1784	1784	1785	1511	1511	1515	1405	1375	1382
m_3	51537	130836	130839	131123	86694	86700	87247	72234	68003	68996

Table 3.1: First three moments of the distributions derived in different ways

In the latter case, the multiple poles of the *PH* random variable causes an other computationally intensive step, i.e. the symbolic evaluation of the first or higher order derivatives of $C_{ij}^{\sim *}(s,v)$ according to v. An alternative solution to avoid this time consuming method is to approximate the random work requirement with a *PH* random variable, whose poles are distinct.

In the following numerical example we introduce all of these cases.

3.2.2 Numerical Results

To indicate the limits of this modeling approach we analyze the case of constant work requirement and deterministic repair times. The model parameters are as follows: the failure rates are $\lambda_a = \lambda_b = 1$, the deterministic repair times are $\mu_a = \mu_b = 5$, and the deterministic work requirement is $W = 3$.

The best kth order *PH* approximation of the deterministic work requirement is the Erlang(k) structure, where k is the number of phases. However, this model results in equal poles for the approximate *PH* distribution, i.e., $n = k$ and $m = 1$ in Theorem 3.1, so the evaluation of the order n derivative is necessary. The calculation can be simplified if we enforce the poles to be different in the approximating structure. The *PH* approximation of the deterministic work requirement with different poles was obtained by slightly modifying the intensities of the Erlang(k) structure and by maintaining the mean value.

The constant work requirement was approximated by phase type distributions of order 2, 4 and 8. The figures below show the distribution of the completion time derived by the exact calculation method (introduced in Section 3.1.1), by approximating the work requirement with Erlang(k) distributions and with modified Erlang distributions, where the intensities were separated by 1% and 10%, in percent of the original Erlang(k) intensity.

Figures 3.3, 3.4 and 3.5 show, as it is expected, that the higher is the order of the *PH* approximation, the more accurate is the approximation of the completion time distribution. More interesting conclusion is that the separation of the poles of the approximating *PH* distribution made no significant effect on the resulting curves, i.e., the proposed effective numerical algorithm practically does not decrease the accuracy compared to the Erlang(k) approximation.

The approximation with 10% separated intensities (poles) has better numerical properties than the corresponding ones with 1% separated intensities.

The Laplace-Stieltjes transform can be used to get the moments of the completion

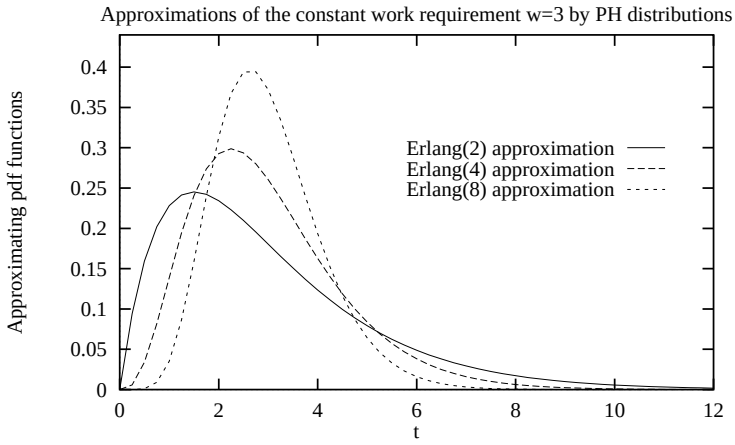


Figure 3.2: Approximations of the constant work requirement $W = 3$ by PH distributions

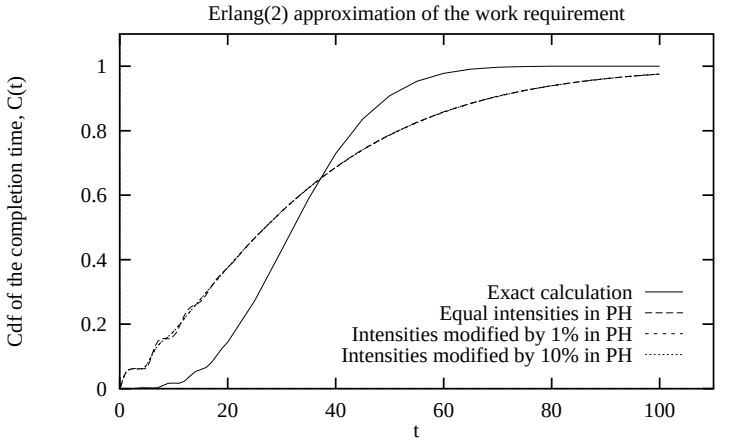


Figure 3.3: Erlang(2) approximation of the work requirement

time distributions. This way the numerical problems caused by the Jagerman method are avoided. The first three moments of the resulting probability density functions are included in Table 3.1. The first moments are the same for all the distributions, while the higher is the degree of the PH approximation, the closer are the higher moments of the distributions. The separation of the poles does not result in significant difference of the higher order moments.

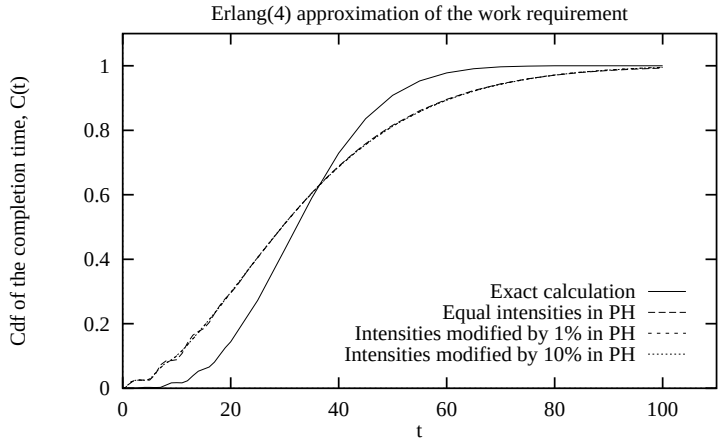


Figure 3.4: Erlang(4) approximation of the work requirement

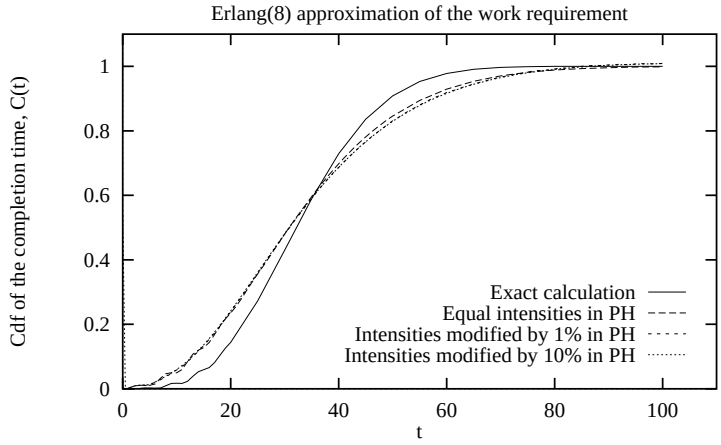


Figure 3.5: Erlang(8) approximation of the work requirement

Chapter 4

Performance Analysis of Markov Regenerative Reward Models

The modeling framework applied in the analysis of complex computer/communication systems depends on the behaviour of the analyzed system and the aim of the analysis. The most frequently applied stochastic modeling technique is the Markovian approach, which is based on the memoryless (Markov) property of the system behaviour. Nevertheless, this property and its consequence, the exponentially distributed event times have been recognized as one of the main restrictions in the application of Markovian models [29]. An alternative non-Markovian modeling approach which allows to relax this restriction is based on Markov renewal theory [24]. This way the application of *MRP*s received an increasing attention in stochastic modeling of computer and communication systems [57, 30]. The automated generation of such models by non-Markovian Stochastic Petri Nets [22, 12, 32] increase the applicability of this modeling framework. A more detailed insight of stochastic models can be obtained by associating a reward variable to the analyzed stochastic process

The former studied stochastic reward models are based on underlying Continuous Time Markov Chains (*CTMC*) or Semi-Markov Processes (*SMP*), and various techniques have been published for the evaluation of the accumulated reward, the completion time and other related reward measures of these models [60, 49, 50].

The reward accumulation of *MRP*s (referred to as Markov regenerative reward models *MRRM*) was first considered in [57]. [57] provided the analysis of a limited set of reward measures: the mean accumulated reward up to time t, the mean instantaneous reward and its limiting behaviour. These are the reward measures which can be evaluated based on the transient behaviour of the underlying *MRP*. The main limitation of these works comes from the fact that they are based on the widely used kernels representation of *MRP*s.

The global ($\mathbf{K}(t)$) and the local ($\mathbf{E}(t)$) kernels of *MRP*s do not define the stochastic process properly in the sense that they do not contain enough information for the analysis of standard reward measures like the distribution of accumulated reward up to time t. The global and local kernel representation of MRPs provide

- a proper description of the embedded Markov renewal sequence and
- the state transition probabilities of the subordinated process between the re-

generative epochs.

Reward measures which require a more detailed knowledge on the process, such as the higher moments of the accumulated reward or the completion time, can not be evaluated based on the global and the local kernels of *MRP*s.

The majority of the former literature on *MRP*s follows the approach summarized in [29]:

".. solving problems using Markov renewal theory is a two step process:

- *First, we need to construct both kernel matrices* $\mathbf{K}(t)$ *and* $\mathbf{E}(t)$*.*
- *We then solve one set of Volterra integral equations for the conditional transition probabilities or for some measures of interest."*

The first step, of course, should be based on the "complete" knowledge of the evolution of the process and it results in a dense description which can be used for the transient and steady state analysis, but which does not contain the "complete" description of the process any more. This way we loose information about the process at the first step of the analysis. The approach adopted in this chapter is similar, but instead of the local and the global kernels we introduce a proper pair of "kernels" (referred to as *reward kernels*), that contain all the required information for the purposes of the analysis of reward measures.

4.1 Analysis of *MRRM*s

A reward rate (r_i) is assigned to each state and an impulse reward (ϱ_{ij}) to each pair of states of an *MRP* ($Z(t) \in \Omega$). The reward accumulated up to time t is defined as

$$B(t) = \int_{\tau=0}^{t} r_{Z(\tau)} d\tau + \sum_i \sum_j N_{ij}(t)\ \varrho_{ij}\ ,$$

where $N_{ij}(t)$ is the number of state transitions from state i to state j up to time t[1].

To utilize the Markov regenerative property of the analyzed reward measures we define the following random variables:

$$\begin{aligned} \mathcal{R}_{ij}(t) &= \{B(t) \mid Z(t) = j,\ Z(0) = i\}, \\ \mathcal{G}_{ij}(t) &= \{B(t) \mid Z(t) = j,\ Z(0) = i,\ T_1 > t\}, \\ \mathcal{S}_{ij}(t) &= \{B(t) \mid Z(t) = j,\ Z(0) = i,\ T_1 = t\}. \end{aligned}$$

- $\mathcal{R}_{ij}(t)$ is the accumulated reward given that the process started in state i and it stays in state j at time t.
- $\mathcal{G}_{ij}(t)$ is the accumulated reward supposed that the process started in state i, it stays in state j at time t and t is inside the first regeneration period.

[1]The framework presented in Section 4.1 and 4.2 is general enough to evaluate models with rate and impulse rewards, but it can not capture the effect of reward loss. The subordinated processes analyzed in Section 4.3 accumulate only rate reward without reward loss.

- $\mathcal{S}_{ij}(t)$ is the accumulated reward supposed that the process started in state i, it stays in state j at time t and t is the first regeneration instance.

Furthermore we define the local and the global *reward kernels*, respectively:

$$G_{ij}(t,w) = \Pr\{B(t) \le w,\ Z(t) = j,\ T_1 > t \mid Z(0) = i\},$$
$$S_{ij}(t,w) = \Pr\{B(T_1) \le w,\ Z(T_1) = j,\ T_1 \le t \mid Z(0) = i\},$$

and the state dependent distribution of the accumulated reward:

$$R_{ij}(t,w) = \Pr\{B(t) \le w,\ Z(t) = j \mid Z(0) = i\}\ .$$

The matrices composed by these elements are denoted as $\mathbf{R}(t,w) = [R_{ij}(t,w)]$, $\mathbf{G}(t,w) = [G_{ij}(t,w)]$ and $\mathbf{S}(t,w) = [S_{ij}(t,w)]$. The following theorem provides the fundamental relation of these quantities.

Theorem 4.1. *The distribution of the accumulated reward of an* MRP *is characterized by the following double LST domain equation:*

$$\mathbf{R}^{\sim\sim}(s,v) = [\mathbf{I} - \mathbf{S}^{\sim\sim}(s,v)]^{-1}\ \mathbf{G}^{\sim\sim}(s,v) \tag{4.1}$$

Proof. Conditioning on the occurrence of the first regeneration instance (T_1) and unconditioning based on its distribution ($K_i(t) = \sum_{j\in\Omega} K_{ij}(t)$) we have[2]:

$$\mathcal{R}_{ij}(t) = E_{ij}(t)\ \mathcal{G}_{ij}(t)\ +\ \sum_k \int_{\tau=0}^{t} \mathcal{S}_{ik}(\tau) + \mathcal{R}_{kj}(t-\tau)\ dK_{ik}(\tau),$$

from which

$$R_{ij}(t,w) = G_{ij}(t,w)\ +\ \sum_k \int_{\tau=0}^{t}\int_{\alpha=0}^{w} R_{kj}(t-\tau, w-\alpha)\ dS1_{ik}(\tau,\alpha)\ dK_{ik}(\tau),$$

where $S1_{ij}(t,w) = \Pr\{B(T_1) \le w \mid Z(T_1) = j,\ T_1 = t,\ Z(0) = i\}$.
An LST with respect to w, denoting the transform variable by v, results in:

$$R^{\sim}_{ij}(t,v) = G^{\sim}_{ij}(t,v) + \sum_k \int_{\tau=0}^{t} R^{\sim}_{kj}(t-\tau,v)\ S1^{\sim}_{ik}(\tau,v)\ dK_{ik}(\tau) =$$
$$G^{\sim}_{ij}(t,v) + \sum_k \int_{\tau=0}^{t} R^{\sim}_{kj}(t-\tau,v)\ dS^{\sim}_{ik}(\tau,v).$$

A second LST with respect to t, denoting the transform variable by s, results in

$$R^{\sim\sim}_{ij}(s,v) = G^{\sim\sim}_{ij}(s,v) + \sum_k S^{\sim\sim}_{ik}(s,v)\ R^{\sim\sim}_{kj}(s,v). \tag{4.2}$$

In matrix form $\mathbf{R}^{\sim\sim}(s,v) = \mathbf{G}^{\sim\sim}(s,v) + \mathbf{S}^{\sim\sim}(s,v)\ \mathbf{R}^{\sim\sim}(s,v)$ □

[2]The proof is based on Markov Renewal Theory, i.e., it is similar to the one applied for reward analysis of *CTMC*s and *SMP*s (see for example [60, 49, 50]), but in this case the stochastic process can experience state transitions up to T_1 which makes our analysis problem rather complex.

4.2 Evaluation of reward measures based on $\mathbf{R}^{\sim\sim}(s,v)$

4.2.1 Accumulated reward

The distribution of the accumulated reward is given by:

$$\begin{aligned} B(t,w) &= \Pr\{B(t) \le w\} = \sum_{i\in\Omega}\sum_{i\in\Omega} P_i(0) R_{ij}(t,w) = \underline{P}(0)\ \mathbf{R}(t,w)\ \underline{h} \\ &= LST^{-1}_{s\to t}\ LST^{-1}_{v\to w}\ \underline{P}(0)\ \mathbf{R}^{\sim\sim}(s,v)\ \underline{h} \end{aligned} \tag{4.3}$$

where $\underline{P}(0) = \{P_i(0)\}$ is the row vector of the initial state probabilities and $\underline{h}$ is the column vector with all the entries equal to 1.

For the numerical evaluation of the distribution of the accumulated reward based on (4.3) two inverse transformations are necessary according to the time ($s \to t$) and the reward variables ($v \to w$). As it can be seen in the subsequent numerical example a symbolic inverse transformation can be very hard even for a simple model.

Instead, the evaluation of the moments of the accumulated reward at time t is based on a single inverse transformation according to the time variable ($s \to t$) by applying the following equation:

$$\mathbf{E}\big[B(t)^k\big] = LST^{-1}_{s\to t}\left\{\ (-1)^k\ \frac{\partial^k}{\partial\ v^k}\ \underline{P}^T(0)\ \mathbf{R}^{\sim\sim}(s,v)\ \underline{h}\Big|_{v\to 0}\right\} \tag{4.4}$$

A symbolic evaluation of the k-th derivative of $\underline{P}^T(0)\ \mathbf{R}^{\sim\sim}(s,v)\ \underline{h}$ and a numerical inverse transformation of the result can be performed in a reasonable respond time.

4.2.2 Completion time

The completion time, $C = \min[t \ge 0 :\ B(t) = W]$, is the time at which the work accumulated by the system reaches the random value W for the first time. Let $W(w) = Pr\{W \le w\}$ be the distribution of the random work requirement and $C(t,w)$ be the *cdf* of the completion time when the reward requirement is $W = w$: $C(t,w) = Pr\{C \le t \mid W = w\}$. The distribution of the completion time (C) of the random reward requirement W is characterized by the following distribution:

$$\widehat{C}(t) = Pr\{C \le t\} = \int_0^\infty C(t,w)\, dW(w) \tag{4.5}$$

In case of prs (no reward loss) reward accumulation the distribution of the completion time is closely related to the distribution of the accumulated reward by the following relation:

$$B(t,w) = Pr\{B(t) \le w\} = Pr\{C(w) \ge t\} = 1 - C(t,w) \tag{4.6}$$

from which

$$C^{\sim\sim}(s,v) = 1 - B^{\sim\sim}(s,v) = 1 - \underline{P}^T(0)\ \mathbf{R}^{\sim\sim}(s,v)\ \underline{h} \tag{4.7}$$

The kth moments of the completion time of the reward requirement w can be evaluated as follows:

$$\mathbf{E}\big[C(w)^k\big] = LST^{-1}_{v\to w}\left\{ (-1)^{k+1} \frac{\partial^k}{\partial\, s^k}\underline{P}^T(0)\ \mathbf{R}^{\sim\sim}(s,v)\ \underline{h}\Big|_{s\to 0}\right\} \tag{4.8}$$

When the reward requirement is a phase type (PH) random variable the moments of the completion time can be evaluated by applying the results of the previous chapter. The following simple example demonstrates the simplicity of the computation. When the reward requirement W is an exponentially distributed r.v. with parameter μ the moments of the completion time are given as follows:

$$\mathbf{E}\big[C^k\big] = \mu \quad (-1)^{k+1} \frac{\partial^k}{\partial\, s^k}\underline{P}^T(0)\ \mathbf{R}^{\sim\sim}(s,\mu)\ \underline{h}\Big|_{s\to 0} . \tag{4.9}$$

4.3 Analysis of subordinated processes

*MRRM*s can be analyzed based on Theorem 4.1, when the reward kernels are known. This section provides results for $\mathbf{S}^{\sim\sim}(s,v)$ and $\mathbf{G}^{\sim\sim}(s,v)$ in case of some simple subordinated processes with loss-less rate reward accumulation.

4.3.1 Semi-Markov subordinated process with random delay

Consider a subordinated *SMP* with state space Ω, kernel $\mathbf{Q}(t)$ and reward rates $r_i, i \in \Omega$. The regenerative period is concluded by the expiration of the random delay θ which is distributed according to $T(\tau) = Pr\{\theta \le \tau\}$ (independent of the subordinated process). At the end of the regeneration period a state transition from state i to state j can take place with probability Δ_{ij}. $\mathbf{\Delta} = \{\Delta_{ij}\}$ is called the branching probability matrix.

To analyze an *MRRM* with this kind of subordinated processes one has to evaluate $\mathbf{S}^{\sim\sim}(s,v)$ and $\mathbf{G}^{\sim\sim}(s,v)$. Since they depend on θ we introduce

$$G_{ij}(t,w,\tau) = \Pr\{B(t) \le w,\ Z(t) = j,\ T_1 > t \mid Z(0) = i, \theta = \tau\}, \tag{4.10}$$
$$S_{ij}(t,w,\tau) = \Pr\{B(T_1) \le w,\ Z(T_1) = j,\ T_1 \le t \mid Z(0) = i, \theta = \tau\}, \tag{4.11}$$

from which $S^{\sim\sim}_{ij}(s,v)$ and $G^{\sim\sim}_{ij}(s,v)$ can be obtained as:

$$G_{ij}(t,w) = \int_{\tau=0}^{\infty} G_{ij}(t,w,\tau)\ dT(\tau), \qquad S_{ij}(t,w) = \int_{\tau=0}^{\infty} S_{ij}(t,w,\tau)\ dT(\tau).$$

Theorem 4.2. *The distribution of the accumulated reward of a complete regenerative period, $S_{ij}(t,w,\tau)$, satisfies the following transform domain equation:*

$$S^{\sim **}_{ij}(s,v,\chi) = \Delta_{ij}\ \frac{1 - Q^{\sim}_i(s + r_i v + \chi)}{v(s + r_i v + \chi)} + \sum_{k\in\Omega} Q^{\sim}_{ik}(s + r_i v + \chi)\ S^{\sim **}_{kj}(s,v,\chi) \tag{4.12}$$

Proof. Conditioning on the sojourn time h in state i we have:

$$S_{ij}(t,w,\tau \mid h) = \begin{cases} 0 & \text{if } r_i\tau > w \text{ and } h \geq \tau \\ \Delta_{ij}\ U(t-\tau) & \text{if } r_i\tau \leq w \text{ and } h \geq \tau \\ 0 & \text{if } r_i h > w \text{ and } h < \tau \\ \sum\limits_{k\in\Omega} \dfrac{dQ_{ik}(h)}{dQ_i(h)}\ S_{kj}(t-h, w-r_i h, \tau-h) & \text{if } r_i h \leq w \text{ and } h < \tau \end{cases} \tag{4.13}$$

where $U(.)$ denotes the unit step function.

In Equation (4.13) the condition $h \geq \tau$ means that there is no state transition before τ (the actual value of the random delay). In this case the relation of the accumulated reward $r_i\tau$ and the reward bound w determine the probability defined in (4.11). When the accumulated reward exceeds the reward bound, i.e., $r_i\tau > w$, $S_{ij}(t,w,\tau)$ equals to 0. Otherwise it depends on state j and time t. $S_{ij}(t,w,\tau)$ equals to the probability that the next regeneration period start from state j, i.e., Δ_{ij}, if $t > \tau$ and it is 0 for $t < \tau$.

When a state transition takes place before τ ($h < \tau$) the following cases have to be considered. If the accumulated reward up to the state transition exceeds the reward bound ($r_i h > w$) then $S_{ij}(t,w,\tau)$ equals to 0, otherwise a state transition from state i to state k takes place at time h with probability $dQ_{ik}(h)/dQ_i(h)$, and a similar analysis problem arises from that point on.

Unconditioning according to the distribution of the sojourn time ($Q_i(t) = \sum_j Q_{ij}(t)$) yields:

$$\begin{aligned} S_{ij}(t,w,\tau) &= \Delta_{ij}\ [1-Q_i(\tau)]\ U(t-\tau)\ U(w-r_i\tau) \\ &+ \sum_{k\in\Omega} \int_{h=0}^{\tau} S_{kj}(t-h, w-r_i h, \tau-h)\ U(w-r_i\tau)\ dQ_{ik}(h) \end{aligned} \tag{4.14}$$

An LST with respect to t, denoting the transform variable by s, results in:

$$\begin{aligned} S^{\sim}_{ij}(s,w,\tau) &= \Delta_{ij}\ [1-Q_i(\tau)]\ e^{-s\tau}\ U(w-r_i\tau) \\ &+ \sum_{k\in\Omega} \int_{h=0}^{\tau} e^{-sh}\ S^{\sim}_{kj}(s, w-r_i h, \tau-h)\ U(w-r_i\tau)\ dQ_{ik}(h) \end{aligned} \tag{4.15}$$

An LT with respect to w, denoting the transform variable by v, results in:

$$\begin{aligned} S^{\sim *}_{ij}(s,v,\tau) &= \Delta_{ij}\ [1-Q_i(\tau)]\ \tfrac{1}{v}\ e^{-(s+r_i v)\tau} \\ &+ \sum_{k\in\Omega} \int_{h=0}^{\tau} e^{-(s+r_i v)h}\ S^{\sim *}_{kj}(s,v,\tau-h)\ dQ_{ik}(h) \end{aligned} \tag{4.16}$$

And finally another LT with respect to τ, denoting the transform variable by χ provides the theorem. □

Theorem 4.3. *The distribution of the accumulated reward inside a regenerative period, $G_{ij}(t,w,\tau)$, satisfies the following transform domain equation:*

$$\begin{aligned} G^{\sim **}_{ij}(s,v,\chi) &= \delta_{ij}\ \frac{s[1-Q^{\sim}_i(s+r_i v+\chi)]}{v\chi(s+r_i v+\chi)} \\ &+ \textstyle\sum_{k\in\Omega} Q^{\sim}_{ik}(s+r_i v+\chi)\ G^{\sim **}_{kj}(s,v,\chi) \end{aligned} \tag{4.17}$$

where δ_{ij} is the Kronecker delta.

Proof. Conditioning on the sojourn time h in state i we have:

$$G_{ij}(t,w,\tau \mid h) = \begin{cases} \delta_{ij}[U(t) - U(t - w/r_i)] & \text{if } r_i\tau > w \text{ and } h \geq \tau \\ \delta_{ij}[U(t) - U(t - \tau)] & \text{if } r_i\tau \leq w \text{ and } h \geq \tau \\ \delta_{ij}[U(t) - U(t - w/r_i)] & \text{if } r_i h > w \text{ and } h < \tau \\ \delta_{ij}[U(t) - U(t - h)] + & \\ \quad \displaystyle\sum_{k\in\Omega} \frac{dQ_{ik}(h)}{dQ_i(h)}\, G_{kj}(t-h, w - r_i h, \tau - h) & \text{if } r_i h \leq w \text{ and } h < \tau \end{cases} \tag{4.18}$$

Similar to Equation (4.13) in Equation (4.18) the condition $h \geq \tau$ means that there is no state transition before τ. In these cases the probability defined in (4.10) equals to 1 if $t < \tau$ and the accumulated reward is less than the reward bound, i.e., $r_i t < w$ and it equals to 0 otherwise.

When we have a state transition before τ and the accumulated reward exceeds the reward bound before ($r_i h > w$) $G_{ij}(t,w,\tau)$ equals to 1 up to time $t = w/r_i$ and it equals to 0 after that. When we have a state transition before τ but the accumulated reward does not exceed the reward bound before ($r_i h < w$) $G_{ij}(t,w,\tau)$ equals to 1 up to the state transition $t = h$ and a similar analysis problem arises from that point on.

To simplify the notation we introduce $\rho = min(\tau, w/r_i)$. Unconditioning yields:

$$\begin{aligned} G_{ij}(t,w,\tau) = {} & \delta_{ij}\ [1 - Q_i(\rho)]\ [U(t) - U(t-\rho)] \\ & + \int_{h=0}^{\rho} \delta_{ij}\ [U(t) - U(t-h)]\ dQ_i(h) \\ & + \sum_{k\in\Omega} \int_{h=0}^{\rho} G_{kj}(t-h, w - r_i h, \tau - h)\ dQ_{ik}(h) \end{aligned} \tag{4.19}$$

An LST with respect to t, denoting the transform variable by s, results in:

$$\begin{aligned} G^{\sim}_{ij}(s,w,\tau) = {} & \delta_{ij}[1 - e^{-s\rho} + e^{-s\rho}\ Q_i(\rho) - \int_{h=0}^{\rho} e^{-sh}\ dQ_i(h)] \\ & + \sum_{k\in\Omega} \int_{h=0}^{\rho} e^{-sh}\ G^{\sim}_{kj}(s, w - r_i h, \tau - h)\ dQ_{ik}(h) \end{aligned} \tag{4.20}$$

An LT with respect to w, denoting the transform variable by v, and taking care of the dependence of ρ on w and τ, results in:

$$\begin{aligned} G^{\sim *}_{ij}(s,v,\tau) = {} & \delta_{ij} \left[\frac{s}{v(s + r_i v)}\ (1 - e^{-(s+r_i v)\tau}) + \frac{1}{v}\ e^{-(s+r_i v)\tau}\ Q_i(\tau) \right. \\ & \left. + \int_{w=0}^{r_i\tau} e^{-(s/r_i + v)w}\ Q_i(w/r_i)\ dw - \frac{1}{v} \int_{h=0}^{\tau} e^{-(s+r_i v)h}\ dQ_i(h) \right] \\ & + \sum_{k\in\Omega} \int_{h=0}^{\tau} e^{-(s+r_i v)h}\ G^{\sim *}_{kj}(s, v, \tau - h)\ dQ_{ik}(h) \end{aligned} \tag{4.21}$$

And finally an other LT with respect to τ, denoting the transform variable by χ provides the theorem. □

To evaluate the accumulated reward of an *MRRM* based on Equation (4.1) an inverse Laplace transformation of $S_{ij}^{\sim **}(s,v,\chi)$ and $G_{ij}^{\sim **}(s,v,\chi)$ is necessary with respect to χ.

The execution of the inverse transformation depends on the particular *SMP* described by $Q_{ij}(t)$. Below we consider the special case when the subordinated process is a *CTMC*.

4.3.2 *CTMC* subordinated process with random delay

Suppose the subordinated process is a *CTMC* with infinitesimal generator $\mathbf{A} = \{a_{ij}\}$ and the diagonal matrix of the reward rates is denoted by $\widehat{\mathbf{R}} = \langle r_i \rangle$ the reward measures are characterized by the following theorems.

Theorem 4.4. *The distribution of the accumulated reward of a complete regenerative period can be evaluated as follows:*

$$\mathbf{S}^{\sim\sim}(s,v) = \int_{\tau=0}^{\infty} e^{-\tau(s\mathbf{I}+v\widehat{\mathbf{R}}-\mathbf{A})} \ \mathbf{\Delta} \ dT(\tau) \tag{4.22}$$

Proof. Substituting $Q_i^{\sim}(s + r_i v + \chi)$ by $\dfrac{-a_{ii}}{-a_{ii} + s + r_i v + \chi}$, $Q_{ik}^{\sim}(s + r_i v + \chi)$ by $\dfrac{a_{ik}}{-a_{ii} + s + r_i v + \chi}$ if $k \neq i$ and $Q_{ii}^{\sim}(s + r_i v + \chi)$ by 0 in Equation (4.12) results in:

$$\begin{aligned} S_{ij}^{\sim **}(s,v,\chi) = \Delta_{ij} \ & \frac{1}{v(-a_{ii} + s + r_i v + \chi)} \\ & + \sum_{k \in R, k \neq i} \frac{a_{ik}}{-a_{ii} + s + r_i v + \chi} \ S_{kj}^{\sim **}(s,v,\chi) \end{aligned} \tag{4.23}$$

Which can be organized into matrix form as:

$$\mathbf{S}^{\sim **}(s,v,\chi) = \frac{1}{v} \ \Big((s+\chi)\mathbf{I} + v\widehat{\mathbf{R}} - \mathbf{A}) \Big)^{-1} \ \mathbf{\Delta} \tag{4.24}$$

From which an inverse Laplace transformation with respect to χ, a multiplication with v (to reach the LST with respect to v) and the integral according to the distribution of θ provides the theorem. □

Theorem 4.5. *The accumulated reward inside of a regenerative period can be evaluated as follows:*

$$\mathbf{G}^{\sim\sim}(s,v) = \int_{\tau=0}^{\infty} s \ \Big(s\mathbf{I} + v\widehat{\mathbf{R}} - \mathbf{A}) \Big)^{-1} \ \Big[\mathbf{I} - e^{-\tau(s\mathbf{I}+v\widehat{\mathbf{R}}-\mathbf{A})} \Big] \ dT(\tau) \tag{4.25}$$

Proof. The same substitution of the kernel elements in Equation (4.17), and the same series of steps provide the theorem. □

4.3.3 Semi-Markov subordinated process with random delay and concluding state transitions

Consider a subordinated *SMP* over Ω with kernel $\mathbf{Q}(t)$. The regenerative period starts in $\Psi \subset \Omega$ and is concluded by the expiration of the random delay θ which is distributed according to $T(\tau)$ (independent of the subordinated process) or a preceding state transition to $\Psi^c = \Omega - \Psi$.

Theorem 4.6. *The distribution of the accumulated reward of a complete regenerative period, $S_{ij}(t, w, \tau)$ satisfies the following transform domain equation:*

$$\begin{aligned} S_{ij}^{\sim **}(s, v, \chi) = \Delta_{ij} \; & \frac{1 - Q_i^{\sim}(s + r_i v + \chi)}{v(s + r_i v + \chi)} \\ & + \sum_{k \in \Psi} Q_{ik}^{\sim}(s + r_i v + \chi) \; S_{kj}^{\sim **}(s, v, \chi) + I_{j \in \Psi^c} \frac{1}{v\chi} \; Q_{ij}^{\sim}(s + \chi) \end{aligned} \tag{4.26}$$

where $I_{j \in \Psi^c}$ is the indicator that state j is in Ψ^c.

Proof. Conditioning on the sojourn time h in state i we have:

$$S_{ij}(t, w, \tau \mid h) = \begin{cases} 0 & \text{if } r_i \tau > w \text{ and } h \geq \tau \\ \Delta_{ij} \; U(t - \tau) & \text{if } r_i \tau \leq w \text{ and } h \geq \tau \\ 0 & \text{if } r_i h > w \text{ and } h < \tau \\ I_{j \in \Psi^c} \; \dfrac{dQ_{ij}(h)}{dQ_i(h)} \; U(t - h) + & \\ \displaystyle\sum_{k \in \Psi} \frac{dQ_{ik}(h)}{dQ_i(h)} \; S_{kj}(t - h, w - r_i h, \tau - h) & \text{if } r_i h \leq w \text{ and } h < \tau \end{cases} \tag{4.27}$$

The effect of a concluding state transition is captured by the last condition, where different cases arise for state transitions out of Ψ and inside Ψ.

The same series of steps as in Theorem 4.2 results in the theorem. □

Theorem 4.7. *The distribution of the accumulated reward inside a regenerative period, $G_{ij}(t, w, \tau)$ satisfies the following transform domain equation:*

$$\begin{aligned} G_{ij}^{\sim **}(s, v, \chi) = \delta_{ij} \; & \frac{s[1 - Q_i^{\sim}(s + r_i v + \chi)]}{v\chi(s + r_i v + \chi)} \\ & + \sum_{k \in \Psi} Q_{ik}^{\sim}(s + r_i v + \chi) \; G_{kj}^{\sim **}(s, v, \chi) \end{aligned} \tag{4.28}$$

Proof. Conditioning on the sojourn time h in state i we have:

$$G_{ij}(t, w, \tau \mid h) = \begin{cases} \delta_{ij}[U(t) - U(t - w/r_i)] & \text{if } r_i \tau > w \text{ and } h \geq \tau \\ \delta_{ij}[U(t) - U(t - \tau)] & \text{if } r_i \tau \leq w \text{ and } h \geq \tau \\ \delta_{ij}[U(t) - U(t - w/r_i)] & \text{if } r_i h > w \text{ and } h < \tau \\ \delta_{ij}[U(t) - U(t - h)] + & \\ \displaystyle\sum_{k \in \Psi} \frac{dQ_{ik}(h)}{dQ_i(h)} \; G_{kj}(t - h, w - r_i h, \tau - h) & \text{if } r_i h \leq w \text{ and } h < \tau \end{cases} \tag{4.29}$$

Note that state transitions only inside Ψ are summed up in the last case. The same series of steps as in Theorem 4.3 result in the theorem. □

4.3.4 *CTMC* subordinated process with random delay and concluding state transitions

Consider a subordinated *CTMC* with infinitesimal generator $\mathbf{A}$. The regenerative period starts in $\Psi \subset \Omega$ and is concluded by the expiration of the random delay θ which is distributed according to $T(\tau)$ or a preceding state transition to $\Psi^c = \Omega - \Psi$, so $\mathbf{A}$ can be partitioned as $\mathbf{A} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array}\right]$, where $\mathbf{A}_{11}$ describes the transitions inside Ψ, $\mathbf{A}_{12}$ contains the intensity of the transitions from Ψ to Ψ^c, $\mathbf{A}_{21}$ the transitions from Ψ^c to Ψ, and $\mathbf{A}_{22}$ the transitions inside Ψ^c, however $\mathbf{A}_{21}$ and $\mathbf{A}_{22}$ are irrelevant since the subordinated process is concluded by the state transition out of Ψ.

Theorem 4.8. *The distribution of the accumulated reward of a complete regenerative period, $S_{ij}(t, w, \tau)$ satisfies the following transform domain equation:*

$$\begin{aligned} \mathbf{S}^{\sim\sim}(s, v) = & \int_{\tau=0}^{\infty} e^{-\tau(s\mathbf{I}_\Psi + v\widehat{\mathbf{R}}_\Psi - \mathbf{A}_{11})} \left[\mathbf{I}_\Psi \;\; \mathbf{0}\right] \;\; \mathbf{\Delta} \\ & + (s\mathbf{I}_\Psi + v\widehat{\mathbf{R}}_\Psi - \mathbf{A}_{11})^{-1}(\mathbf{I}_\Psi - e^{-\tau(s\mathbf{I}_\Psi + v\widehat{\mathbf{R}}_\Psi - \mathbf{A}_{11})})\mathbf{A}_{12}\left[\mathbf{0} \;\; \mathbf{I}_{\Psi^c}\right] \; dT(\tau) \end{aligned} \tag{4.30}$$

where $\mathbf{I}_\Psi$ and $\mathbf{I}_{\Psi^c}$ are identity matrices of dimension $\#\Psi$ and $\#\Psi^c$, respectively, $\mathbf{0}$ is the matrix of zeroes of the proper size and $\widehat{\mathbf{R}}_\Psi = \langle r_i \rangle$, $i \in \Psi$ is the diagonal matrix of the reward rates associated with the states in Ψ.

Proof. Substituting the entries of the kernel in Equation (4.26) following the way described in Theorem 4.4 yields an equation, from which an inverse Laplace transformation with respect to χ, a multiplication with v (to reach the LST with respect to v) and the integral according to the distribution of θ provides the theorem. □

Theorem 4.9. *The distribution of the accumulated reward inside a regenerative period, $G_{ij}(t, w, \tau)$ satisfies the following transform domain equation:*

$$\mathbf{G}^{\sim\sim}(s, v) = \left[(s\mathbf{I}_\Psi + v\widehat{\mathbf{R}}_\Psi - \mathbf{A}_{11})^{-1} \; (\mathbf{I}_\Psi - e^{-\tau(s\mathbf{I}_\Psi + v\widehat{\mathbf{R}}_\Psi - \mathbf{A}_{11})}) \quad \mathbf{0}\right]. \tag{4.31}$$

Proof. Starting from Equation (4.28) and repeating the algebraic transformations used to prove Theorem 4.8 yield the theorem. □

This kind of subordinated processes arises in various practical cases when Markovian system evolution goes on during a general activity (with any generally distributed delay) and the activity can be concluded by its completion or by a particular event of the Markovian system (e.g., a transition to a set of states). Non-markovian stochastic Petri nets with non-overlapping activity cycle results in this kind of subordinated processes as well [15].

4.3.5 Subordinated process without internal state transition

An *MRP* often has a simple subordinated process without internal state transition. This special case is considered below.

Theorem 4.10. *When there is no state transition during the subordinated process and the distribution of the time to the next regeneration epoch is $T(\tau)$, $S_{ij}^{\sim *}(s,v)$ and $G_{ij}^{\sim *}(s,v)$ satisfy the following equations:*

$$S_{ij}^{\sim *}(s,v) = \int_{\tau=0}^{\infty} \Delta_{ij} \ \frac{1}{v} e^{-\tau(s+r_i v)} \ dT(\tau) \ , \tag{4.32}$$

$$G_{ij}^{\sim *}(s,v) = \int_{\tau=0}^{\infty} \delta_{ij} \ \frac{s}{v(s+r_i v)} \left[1 - e^{-\tau(s+r_i v)}\right] \ dT(\tau) \ . \tag{4.33}$$

Proof. Substituting $Q_i^{\sim}(s+r_i v+\chi)$ and $Q_{ik}^{\sim}(s+r_i v+\chi)$ by 0, in Equation (4.12) and (4.17), inverse Laplace transforming the results with respect to χ, and integrating according to the distribution of θ gives the theorem. □

Two often applied special cases are the exponentially distributed and the deterministic delay of the subordinated process. In the first case when θ has an exponential distribution with parameter λ:

$$S_{ij}^{\sim\sim}(s,v) = \ \Delta_{ij} \ \frac{\lambda}{\lambda + s + r_i v} \ , \quad G_{ij}^{\sim\sim}(s,v) = \ \delta_{ij} \ \frac{s}{\lambda + s + r_i v} \ ; \tag{4.34}$$

in the second case when θ is deterministic, i.e., $\theta = \tau$:

$$\begin{aligned} S_{ij}^{\sim\sim}(s,v) &= \ \Delta_{ij} e^{-\tau(s+r_i v)} \ , \\ G_{ij}^{\sim\sim}(s,v) &= \ \delta_{ij} \ \frac{s}{s + r_i v} \left[1 - e^{-\tau(s+r_i v)}\right] \ . \end{aligned} \tag{4.35}$$

The multiplication of Equation (4.32) and Equation (4.33) by v results in the Laplace-Stieltjes transform from the Laplace transform.

4.4 Numerical Example

As a simple example to illustrate the analysis steps of the proposed method we consider an M/D/1/2/2 queueing system. This is a finite queueing system with at most two customers in it and with a FIFO service mechanism. The steady state behaviour of this system was studied in [2], while the transient analysis was accomplished in [21].

The Petri net description of the system, is reported in Figure 4.1. Place p_1 contains the "thinking" customers, i.e., the customers waiting to submit a job, and transition t_1 represents the submission of a job. Tokens in place p_2 represent the jobs queueing for service. A token in p_3 means that the server is busy while a token in p_4 means that the server is idle. Transition t_2 represents the service of a job; when the job is completed the customer returns to its thinking state. Transition t_3 is an immediate transition modeling the start of service, i.e., the transfer of the job from the queue to the server, this transfer becomes possible when the service unit gets free.

The firing time of t_1 is exponentially distributed with rate $m_1 \cdot \lambda$ being m_1 is the number of tokens in p_1 and $\lambda = 0.5$ *job/hour*. t_2 is a *DET* transition modeling a

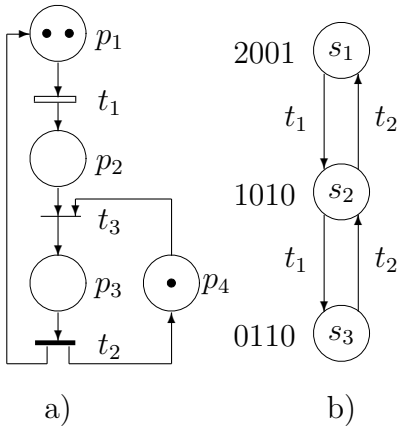


Figure 4.1: a) PN modeling of a M/D/1/2/2; b) corresponding reduced reachability graph.

constant service time of duration $d = 1.0$ *hour*. We augment this description by the reward rates, $r_1 = 0$, $r_2 = 1$, $r_3 = 0.8$, representing the idle server, the busy server and the busy server with some penalty charged because a job is waiting for service, respectively.

The reduced state space of the system eliminating the vanishing markings arising from the immediate transition, t_3, is composed of three states, called s_1, s_2 and s_3 (Figure 4.1b).

There are three regenerative transitions (state transitions result in a regeneration epoch): $s_1 \to s_2$, $s_2 \to s_1$, $s_3 \to s_2$. Hence s_3 can never be a regeneration state when the process is started from state s_1 or s_2. $Z^i(t)$ denotes the subordinated process started in state i. To determine the $\mathbf{S}^{\sim\sim}(s,v)$ and $\mathbf{G}^{\sim\sim}(s,v)$ matrices we analyze the subordinated processes one by one. Each subordinated process determines one row of the $\mathbf{S}^{\sim\sim}(s,v)$, $\mathbf{G}^{\sim\sim}(s,v)$ matrices.

1. $Z^1(t)$: The is no internal state transitions in the subordinated process starting from state s_1, since the transition to state s_2 terminates the process. The distribution of the delay in s_1 is exponentially distributed, hence Equation (4.34) is directly applicable:

$$S_{11}^{\sim\sim}(s,v) = 0,\ S_{12}^{\sim\sim}(s,v) = \frac{\lambda}{\lambda + s + vr_1},\ S_{13}^{\sim\sim}(s,v) = 0,$$
$$G_{11}^{\sim\sim}(s,v) = \frac{s}{\lambda + s + vr_1},\ G_{12}^{\sim\sim}(s,v) = 0,\ G_{13}^{\sim\sim}(s,v) = 0.$$

2. $Z^2(t)$: The subordinated process starting from state s_2 is a one-step *CTMC*, i.e., the only possible state transition in the subordinated process is a transition from state s_2 to state s_3. The time of the subordinated process is deterministic (τ). Thus we can apply Theorem (4.4) and Theorem (4.5) to determine the

second row of the reward kernel matrices. The $\widehat{\mathbf{R}}$, $\mathbf{A}$ and $\mathbf{\Delta}$ matrices of the subordinated process are:

$$\widehat{\mathbf{R}} = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{\Delta} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.36}$$

Using these matrices, Equation (4.22) and (4.25) results in:

$$\begin{aligned}
S_{21}^{\sim\sim}(s,v) &= e^{-\tau(\lambda+s+r_2 v)}, \\
S_{22}^{\sim\sim}(s,v) &= \frac{\lambda}{(r_2-r_3)v+\lambda}(e^{-\tau(s+r_3 v)} - e^{-\tau(\lambda+s+r_2 v)}), \\
S_{23}^{\sim\sim}(s,v) &= 0, \\
G_{21}^{\sim\sim}(s,v) &= 0, \\
G_{22}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_2}(1-e^{-\tau(\lambda+s+r_2 v)}), \\
G_{23}^{\sim\sim}(s,v) &= \frac{\lambda s}{(s+r_3 v)} + \frac{\lambda s e^{-\tau(\lambda+s+r_2 v)}}{(\lambda+s+r_3 v)(\lambda+(r_2-r_3)v)} \\
&\quad - \frac{\lambda s e^{-\tau(s+r_3 v)}}{(s+r_3 v)(\lambda+(r_2-r_3)v)}.
\end{aligned}$$

3. $Z^3(t)$: Finally, when the process starts from state s_3, the subordinated process does not contain internal state transitions, since the only possible state transition to state s_2 terminates the subordinated process. The distribution of the delay is deterministic. Based on Equation (4.35) the third row of the reward kernel matrices are:

$$\begin{aligned}
&S_{31}^{\sim\sim}(s,v) = 0,\ S_{32}^{\sim\sim}(s,v) = e^{-\tau(s+vr_2)},\ S_{33}^{\sim\sim}(s,v) = 0, \\
&G_{31}^{\sim\sim}(s,v) = 0,\ G_{32}^{\sim\sim}(s,v) = 0,\ G_{33}^{\sim\sim}(s,v) = \frac{s}{s+r_3 v}(1-e^{-\tau(s+vr_3)}),
\end{aligned}$$

since in this case the only relevant non-zero entry of $\mathbf{\Delta}$ is $\Delta_{23} = 1$.

Equation (4.1) provides the $\mathbf{R}^{\sim\sim}(s,v)$ matrix of the accumulated reward form the reward kernel matrices:

$$\mathbf{R}^{\sim\sim}(s,v) = \frac{1}{c}\begin{bmatrix} R_{11}^{\sim\sim}(s,v) & R_{12}^{\sim\sim}(s,v) & R_{13}^{\sim\sim}(s,v) \\ R_{21}^{\sim\sim}(s,v) & R_{22}^{\sim\sim}(s,v) & R_{23}^{\sim\sim}(s,v) \\ R_{31}^{\sim\sim}(s,v) & R_{32}^{\sim\sim}(s,v) & R_{33}^{\sim\sim}(s,v) \end{bmatrix},$$

where

$$\begin{aligned}
c &= 1 - \frac{\lambda}{\lambda+s+r_1 v} e^{-\tau(\lambda+s+r_2 v)} \\
&\quad - \frac{\lambda}{\lambda+(r_2-r_3)v}\left(e^{-\tau(s+r_3 v)} - e^{-\tau(\lambda+s+r_2 v)}\right),
\end{aligned}$$

$$
\begin{aligned}
R_{11}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_1}\left(1-\frac{\lambda}{(r_2-r_3)v+\lambda}\left(e^{-\tau(s+r_3v)}-e^{-\tau(\lambda+s+r_2v)}\right)\right), \\
R_{12}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_2}\,\frac{\lambda}{\lambda+s+vr_1}\,(1-e^{-\tau(\lambda+s+r_2v)}), \\
R_{13}^{\sim\sim}(s,v) &= \frac{\lambda}{\lambda+s+vr_1}\left(\frac{\lambda s}{(s+r_3v)}+\frac{\lambda s e^{-\tau(\lambda+s+r_2v)}}{(\lambda+s+r_3v)(\lambda+(r_2-r_3)v)}\right. \\
&\qquad \left. -\frac{\lambda s e^{-\tau(s+r_3v)}}{(s+r_3v)(\lambda+(r_2-r_3)v)}\right), \\
R_{21}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_1}e^{-\tau(\lambda+s+r_2v)}, \\
R_{22}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_2}(1-e^{-\tau(\lambda+s+r_2v)}), \\
R_{23}^{\sim\sim}(s,v) &= \frac{\lambda s}{(s+r_3v)}+\frac{\lambda s e^{-\tau(\lambda+s+r_2v)}}{(\lambda+s+r_3v)(\lambda+(r_2-r_3)v)} \\
&\quad -\frac{\lambda s e^{-\tau(s+r_3v)}}{(s+r_3v)(\lambda+(r_2-r_3)v)}, \\
R_{31}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_1}e^{-\tau(\lambda+s+r_2v)}e^{-\tau(s+vr_2)}, \\
R_{32}^{\sim\sim}(s,v) &= \frac{s}{\lambda+s+vr_2}(1-e^{-\tau(\lambda+s+r_2v)})e^{-\tau(s+vr_2)}, \\
R_{33}^{\sim\sim}(s,v) &= c\,\frac{s}{s+r_3v}\left(1-e^{-\tau(s+vr_3)}\right)+\frac{\lambda s e^{-\tau(\lambda+s+r_2v)}}{(\lambda+s+r_3v)(\lambda+(r_2-r_3)v)} \\
&\quad +\frac{\lambda s}{(s+r_3v)}-\frac{\lambda s e^{-\tau(s+r_3v)}}{(s+r_3v)(\lambda+(r_2-r_3)v)}e^{-\tau(s+vr_2)}.
\end{aligned}
$$

To obtain the distribution in original (time and reward) domain, multi-dimensional inverse transformation is necessary [23]. However, the moments of the corresponding distribution can be calculated based on Equation (4.4) or Equation (4.8) using the transform domain expression and a one-dimensional numerical inverse transformation method.

In Figures 4.2, 4.3, 4.4, 4.5 the numerical results are depicted for the mean and the standard deviation of the accumulated reward and the completion time, respectively, when the system was started in state s_1. We emphasize that any moments can be calculated using the proposed analytical method, however we chose to depict the most frequently used quantities, the mean and the standard deviation. The results were obtained by a numerical inverse transformation method written in Mathematica by resorting to the Jagerman method [44]. Some numerical uncertainties were experienced in the values close to zero especially when calculating the standard deviation of the completion time (Figure 4.5). The mean completion time tends to 2 as the work requirement goes to 0 (Figure 4.4), since the mean holding time in state s_1 is $1/\lambda = 2$, and the reward accumulation starts in state s_2.

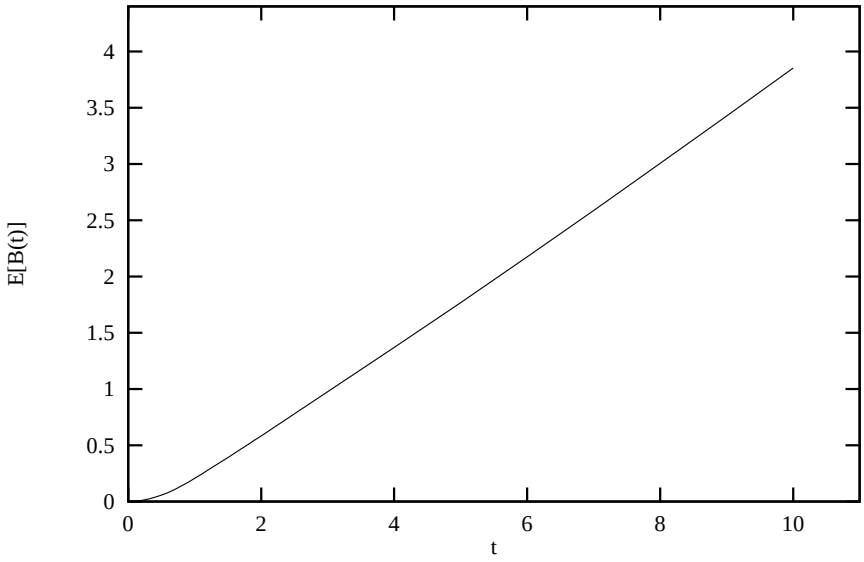


Figure 4.2: Mean of the accumulated reward

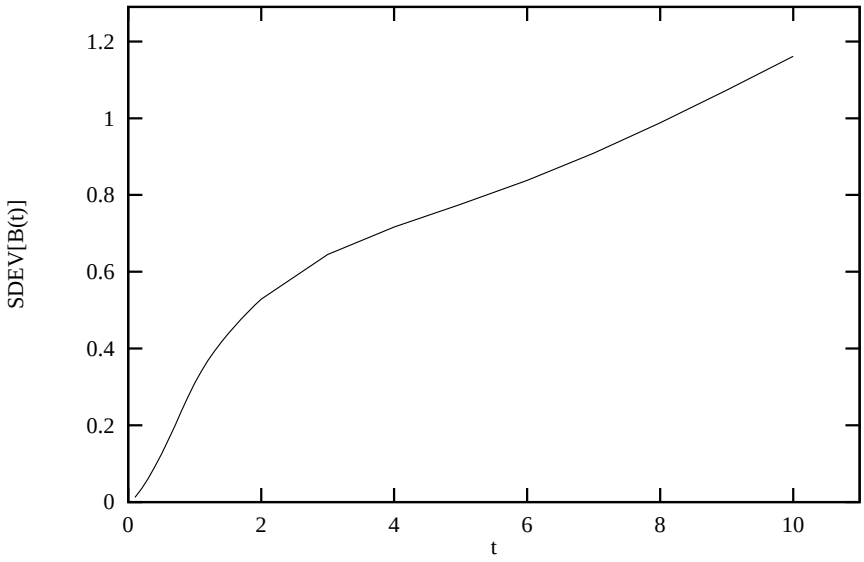


Figure 4.3: Standard deviation of the accumulated reward

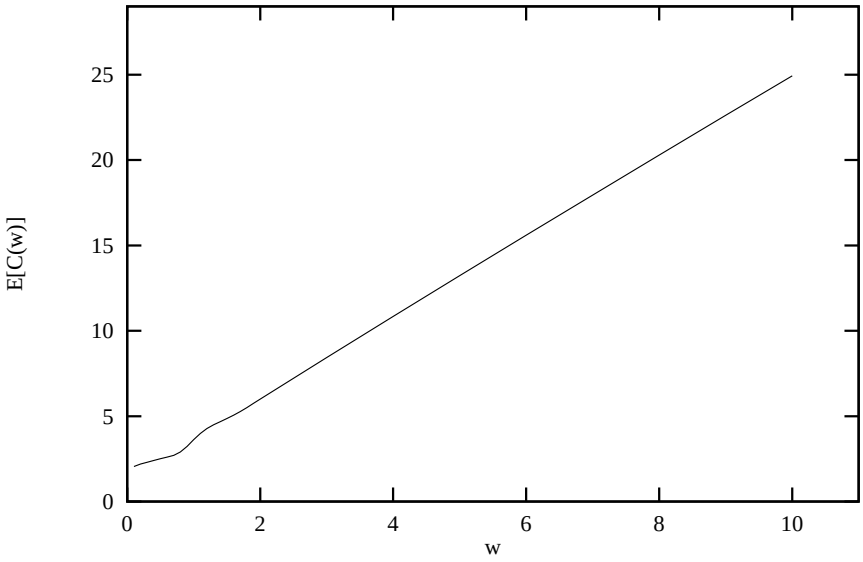


Figure 4.4: Mean of the completion time

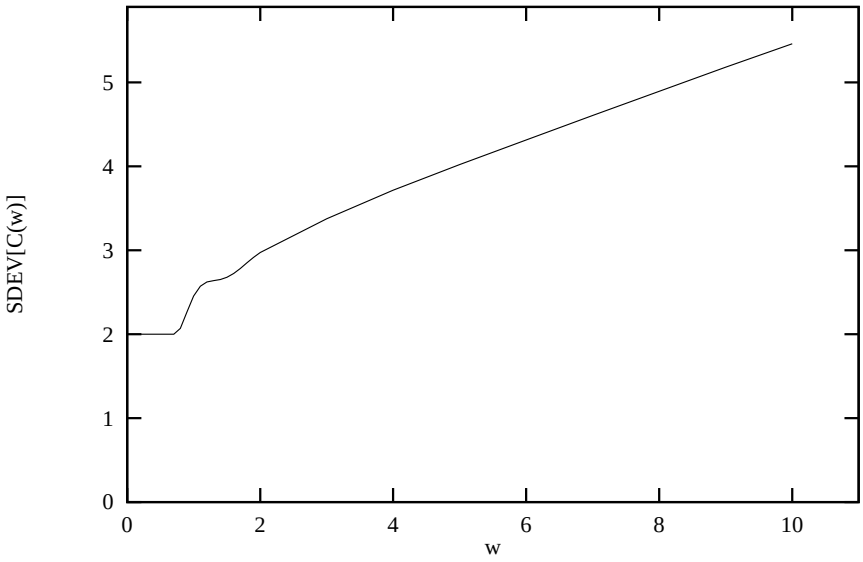


Figure 4.5: Standard deviation of the completion time

Chapter 5

Numerical analysis of large MRMs

Various numerical techniques were proposed for the evaluation of the accumulated reward and completion time measures of *MRM*s. Some of these methods calculate the distribution of reward measures some other evaluate the moments of those measures. The distribution, in double transform domain, can be obtained by a symbolic matrix inversion. If the size of the state space allows to obtain the solution of the symbolic matrix inversion then multi-dimensional numerical inverse transform methods [23] can provide time domain results, but due to the computational complexity of the symbolic inversion of matrices and the multi-dimensional numerical inverse transformation, this approach is not applicable for models with more than $\sim$10 states.

In time domain, reward measures can be described either by a set of Volterra integral equations, or by a set of partial differential equations. The numerical methods compute the distribution in time domain are usually based on the evaluation of a double summation, where both summations go to infinity. The discrete summations are obtained by adopting the randomization technique [76] (or Jensen's method). The randomization technique usually provides nice numerical properties and an overall error bound for finite truncation of infinite sums. The numerical methods based on this approach [26, 28, 62] differ in the complexity and memory requirement of one iteration step. The methods in [28, 62] are with polynomial complexity with respect to the size of the state space.

The numerical analysis of the distribution of reward measures is, in general, more complex than the computation of the moments of those measures. The mean accumulated reward can be obtained by the transient analysis of the underlying *CTMC*. A numerical convolution approach is proposed in [43] to evaluated the $(n+1)$-th moment of accumulated reward based on its n-th moment. A similar approach is followed in [79] to calculate the moments of the completion time measures, but the high computational complexity of the numerical convolution does not allow to apply this approach for the analysis of *MRM* with large (> 100) state spaces.

To the best of our knowledge the applicability of numerical methods for the reward analysis of *MRM*s is restricted to with *MRM*s with less than 10^4 states, while there are effective numerical methods to compute the steady state, the transient and the cumulative transient measures of *CTMC*s with 10^6–10^7 states [76, 70].

In this chapter, we provide a method based on the transform domain description of MRMs which allows the reward analysis of large models. Indeed, the proposed

method evaluates each required moments of reward measures on the same computational cost as the transient analysis of the underlying CMTC, hence, it outperforms all the above mentioned general methods, regarding the size of the models for which the numerical analysis is feasible.

In spite of the above statements on general methods *MRM*s with special features allow special, effective numerical approaches. In the case when the underlying *CTMC* has an absorbing state, in which no useful work is performed, it is easy to evaluate the limiting distribution of performability [5]. The numerical method in [34] also makes use of a special structure of the underlying *CTMC*.

Other *direct* methods make use of a spectral or partial fraction decomposition, which is relatively easy for acyclic *CTMC*s, since the eigenvalues of the generator matrix are available in its diagonal [75]. The subclass of *MRM*s where the user has an associated Phase-type distributed random work requirement was studied in [17]. In this case the completion time is Phase type distributed, i.e., an "extended" *CTMC* characterizes the distribution of the completion time.

5.1 Markov Reward Models

In this section we provide the definitions and the well known results about MRMs with rate reward accumulation, but following a simpler way of reasoning than the one in the original papers.

Let $\{\mathcal{Z}(t),\ t \geq 0\}$ be a *CTMC* over the finite state space $\Omega = \{1, 2, \ldots, M\}$ with generator $\mathbf{A} = [a_{ij}]$ and reward matrix $\mathbf{R} = diag\langle r_i \rangle$. The distribution of the accumulated reward and the completion time are: $B_i(t,w) = Pr\{B(t) \leq w | Z(0) = i\}$ and $C_i(t,w) = Pr\{C(w) \leq t | Z(0) = i\}$.

Theorem 5.1. *The column vector of the distribution of the accumulated reward ($\underline{B}(t,w) = [B_i(t,w)]$) is defined as follows:*

$$\underline{B}^{\sim}(t,v) = e^{(\mathbf{A}-v\mathbf{R})t} \cdot \underline{h} \tag{5.1}$$

where $^{\sim}$ denotes the Laplace-Stieltjes transform with respect to $w(\rightarrow v)$, and $\underline{h}$ is the column vector with all the entries equal to 1.

Proof: Consider an exponentially distributed work requirement ($\mathcal{W}$) with parameter m. On the one hand, the completion time is characterized by the following distribution function

$$\begin{aligned} C_i(t) &= \int_0^{\infty} C_i(t,w)\ dG(w) = \int_0^{\infty} \Big(1 - B_i(t,w)\Big)\ dG(w) \qquad (5.2) \\ &= m \int_0^{\infty} \Big(1 - B_i(t,x)\Big) e^{-mx}\ dx = 1 - B_i^{\sim}(t,v)\Big|_{v=m} \end{aligned}$$

which, in vector form, is

$$\underline{C}(t) = \underline{h} - \underline{B}^{\sim}(t,v)\Big|_{v=m}. \tag{5.3}$$

The second equality in (5.2) is due to (2.20).

On the other hand, $C_i(t)$ is phase type distributed and its distribution can be obtained by the representation of the phase type distribution (the original *CTMC* plus an absorbing state to which transition from state $i \in \Omega$ is at rate $m\ r_i$) [17]:

$$\underline{C}(t) = \underline{h} - e^{(\mathbf{A}-m\mathbf{R})t} \cdot \underline{h}\ . \tag{5.4}$$

And since (5.1) is analytical for $\Re(v) \geq 0$ the theorem is given. □

A further Laplace-Stieltjes transform of (5.1) with respect to t results:

$$\underline{B}^{\sim\sim}(s,v) = s(s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1} \cdot \underline{h}\ . \tag{5.5}$$

In order to simplify the transform domain expressions, in the rest of the paper, we apply the most convenient version of them using the $F^{\sim}(a) = aF^{*}(a)$ rule[1]. Detailed derivations in [43] resulted in the same expression for distribution of the accumulated reward based on different approaches. From (2.20), (5.5), using $\mathbf{A} \cdot \underline{h} = \underline{0}$, we have:

$$\begin{aligned}
\underline{C}^{\sim\sim}(s,v) &= \underline{h} - \underline{B}^{\sim\sim}(s,v) \\
&= [\mathbf{I} - s(s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1}] \cdot \underline{h} \\
&= [(s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1} \cdot (s\mathbf{I} + v\mathbf{R} - \mathbf{A}) - s(s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1}] \cdot \underline{h} \\
&= (s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1} \cdot (v\mathbf{R} - \mathbf{A}) \cdot \underline{h} \\
&= v(s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1} \cdot \mathbf{R} \cdot \underline{h}\ ,
\end{aligned} \tag{5.6}$$

which was obtained with a different way of reasoning in [49]. Suppose $\mathbf{R}^{-1}$ exists, i.e., $r_i > 0, \forall i \in \Omega$, (5.6) can be inverse transformed with respect to the reward variable as follows:

$$\begin{aligned}
\underline{C}^{\sim *}(s,v) &= (s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1} \cdot (\mathbf{R}^{-1})^{-1} \cdot \underline{h} \\
&= (s\mathbf{R}^{-1} + v\mathbf{I} - \mathbf{R}^{-1}\mathbf{A})^{-1} \cdot \underline{h}\ ,
\end{aligned} \tag{5.7}$$

from which

$$\underline{C}^{\sim}(s,w) = e^{(\mathbf{R}^{-1}\mathbf{A} - s\mathbf{R}^{-1})w} \cdot \underline{h}\ . \tag{5.8}$$

Note that, we did not restrict the class of *MRM*s till the first row of (5.7), hence the results are valid for any reducible and irreducible underlying *CTMC* and any non-negative reward rates. In (5.7) – (5.8), the only restriction is that $\mathbf{R}$ must be invertable, i.e., strictly positive reward rates are only allowed.

5.2 Moments of the accumulated reward

Let $m_i^{(n)}(t) = E\{\mathcal{B}_i(t)^n\}$ be the n-th moment of the reward accumulated in $[0,t)$. The column vector $\underline{m}^{(n)}(t) = [m_i^{(n)}(t)]$ can be evaluated based on $\underline{B}^{\sim}(t,v)$ as

$$\underline{m}^{(n)}(t) = (-1)^n \frac{\partial^n \underline{B}^{\sim}(t,v)}{\partial v^n}\bigg|_{v=0}\ . \tag{5.9}$$

[1] E.g., $\underline{B}^{*\sim}(s,v) = (s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1} \cdot \underline{h}$ and $\underline{B}^{\sim *}(s,v) = \frac{s}{v}\ (s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1} \cdot \underline{h}$

The following theorem provides a computationally effective, recursive method for the numerical analysis of the moments of accumulated reward.

Theorem 5.2. *The n-th moment ($n \geq 1$) of the accumulated reward is*

$$\underline{m}^{(n)}(t) = (-1)^n \sum_{i=0}^{\infty} \frac{t^i}{i!} \mathbf{N}^{(n)}(i) \cdot \underline{h} \tag{5.10}$$

where $\mathbf{N}^{(n)}(i)$ *is defined as*

$$\mathbf{N}^{(n)}(i) = \begin{cases} \mathbf{I} \ , & \text{if} \quad i = n = 0 \ , \\ \mathbf{0} \ , & \text{if} \quad i = 0, n \geq 1 \ , \\ \mathbf{A}^i \ , & \text{if} \quad i \geq 1, n = 0 \ , \\ \mathbf{A} \cdot \mathbf{N}^{(n)}(i-1) - n \ \mathbf{R} \cdot \mathbf{N}^{(n-1)}(i-1) \ , & \text{if} \quad i \geq 1, n \geq 1 \ . \end{cases} \tag{5.11}$$

To prove the theorem we need the following results.

Lemma 1 If $\mathbf{F}(t)$ and $\mathbf{G}(t)$ are real-valued, n times derivable matrix functions and $\mathbf{F}''(t) = \mathbf{0}$, then

$$(\mathbf{F}(t) \cdot \mathbf{G}(t))^{(n)} = \mathbf{F}(t) \cdot \mathbf{G}^{(n)}(t) + n \ \mathbf{F}'(t) \cdot \mathbf{G}^{(n-1)}(t), \quad n \geq 1 \ . \tag{5.12}$$

Proof of Lemma 1

1. For $n = 1$

$$(\mathbf{F}(t) \cdot \mathbf{G}(t))' = \mathbf{F}(t) \cdot \mathbf{G}'(t) + \mathbf{F}'(t) \cdot \mathbf{G}(t) \tag{5.13}$$

holds.

2. Assuming (5.12) holds for $n = k$, it follows

$$(\mathbf{F}(t) \cdot \mathbf{G}(t))^{(k+1)} \ = \ \mathbf{F}(t) \cdot \mathbf{G}^{(k+1)}(t) + (k+1) \ \mathbf{F}'(t) \cdot \mathbf{G}^{(k)}(t) \tag{5.14}$$

where the assumption for $n = k$ and $\mathbf{F}''(t) = \mathbf{0}$ is used. □

Lemma 2 If $i, n \geq 1$ then

$$\begin{aligned} &\frac{\partial^n}{\partial v^n}(\mathbf{A} - v\mathbf{R})^i \Big|_{v=0} = \\ &\quad \mathbf{A} \cdot \frac{\partial^n}{\partial v^n}(\mathbf{A} - v\mathbf{R})^{i-1} \Big|_{v=0} - n \ \mathbf{R} \cdot \frac{\partial^{n-1}}{\partial v^{n-1}}(\mathbf{A} - v\mathbf{R})^{i-1} \Big|_{v=0} \end{aligned} \tag{5.15}$$

Proof of Lemma 2 Let $\mathbf{F}(v) = (\mathbf{A} - v\mathbf{R})$ and $\mathbf{G}(v) = (\mathbf{A} - v\mathbf{R})^{i-1}$. From Lemma 1

$$\begin{aligned} &\frac{\partial^n}{\partial v^n}(\mathbf{A} - v\mathbf{R})^i = \\ &\quad (\mathbf{A} - v\mathbf{R}) \cdot \frac{\partial^n}{\partial v^n}(\mathbf{A} - v\mathbf{R})^{i-1} - n \ \mathbf{R} \cdot \frac{\partial^{n-1}}{\partial v^{n-1}}(\mathbf{A} - v\mathbf{R})^{i-1} \end{aligned} \tag{5.16}$$

which implies the Lemma. □

Proof of Theorem 5.2 From (5.9) and (5.1)

$$\begin{aligned} \underline{m}^{(n)}(t) &= (-1)^n \frac{\partial^n e^{(\mathbf{A} - v\mathbf{R})t}}{\partial v^n} \Big|_{v=0} \cdot \underline{h} \\ &= (-1)^n \frac{\partial^n}{\partial v^n} \sum_{i=0}^{\infty} \frac{t^i}{i!} (\mathbf{A} - v\mathbf{R})^i \Big|_{v=0} \cdot \underline{h} \\ &= (-1)^n \sum_{i=0}^{\infty} \frac{t^i}{i!} \ \frac{\partial^n}{\partial v^n} (\mathbf{A} - v\mathbf{R})^i \Big|_{v=0} \cdot \underline{h} \ . \end{aligned} \tag{5.17}$$

Let

$$\mathbf{N}^{(n)}(i) = \frac{\partial^n}{\partial v^n}(\mathbf{A} - v\mathbf{R})^i\Big|_{v=0}, \quad \text{for } \forall n, i \geq 1. \tag{5.18}$$

From Lemma 2 it follows

$$\mathbf{N}^{(n)}(i) = \mathbf{A} \cdot \mathbf{N}^{(n)}(i-1) - n\ \mathbf{R} \cdot \mathbf{N}^{(n-1)}(i-1), \tag{5.19}$$

with the initial conditions $\mathbf{N}^{(0)}(0) = \mathbf{I}$, $\mathbf{N}^{(0)}(i) = \mathbf{A}^i$ and $\mathbf{N}^{(n)}(0) = \mathbf{0}$. By this recursion $\mathbf{N}^{(n)}(i) = \mathbf{0}$, if $i < n$. This completes the proof of Theorem 5.2. □
The iterative procedure to evaluate $\mathbf{N}^{(n)}(i)$ has the following properties:

- it is not possible to evaluate the nth moment itself, but to obtain the nth moment all the previous moments (or at least the associated $\mathbf{N}^{(n)}(i)$ terms) must be computed;
- matrix-matrix multiplications are computed in each iteration steps;
- numerical problems can arise due to the repeated multiplication with $\mathbf{A}$, which contains both positive and negative elements, hence Theorem 5.2 is not directly applicable for numerical analysis.

5.3 Moments of the completion time

Let $s_i^{(n)}(w) = E\{\mathcal{C}_i(w)^n\}$ be the n-th moment of the time to accumulate w amount of reward. The column vector $\underline{s}^{(n)}(w) = [s_i^{(n)}(w)]$ can be evaluated based on $\underline{C}^{\sim}(s, w)$ as

$$\underline{s}^{(n)}(w) = (-1)^n \frac{\partial^n \underline{C}^{\sim}(s, w)}{\partial s^n}\Big|_{s=0}. \tag{5.20}$$

Theorem 5.3. *The n-th moment of completion time, $\underline{s}^{(n)}(w)$, satisfies the following equation*

$$\underline{s}^{(n)}(w) = (-1)^n \sum_{i=n}^{\infty} \frac{w^i}{i!} \mathbf{M}^{(n)}(i) \cdot \underline{h} \tag{5.21}$$

where $\mathbf{M}^{(n)}(i)$ is defined as

$$\mathbf{M}^{(n)}(i) = \begin{cases} \mathbf{I}, & i = n = 0\ , \\ \mathbf{0}, & i = 0, n \geq 1\ , \\ (\mathbf{R}^{-1} \cdot \mathbf{A})^i, & i \geq 1, n = 0\ , \\ \mathbf{R}^{-1}\left(\mathbf{A} \cdot \mathbf{M}^{(n)}(i-1) - n\ \mathbf{M}^{(n-1)}(i-1)\right), & i, n \geq 1\ . \end{cases} \tag{5.22}$$

Proof of Theorem 5.3 Using

$$\underline{s}^{(n)}(w) = (-1)^n \frac{\partial^n}{\partial s^n} e^{(\mathbf{R}^{-1} \cdot \mathbf{A} - s\mathbf{R}^{-1})w}\Big|_{s=0} \cdot \underline{h} \tag{5.23}$$

the proof follows the same pattern as the proof of Theorem 5.2. □

The numerical method based on Theorem 5.3 has the same properties as the one based on Theorem 5.2. In contrast with Theorem 5.2, the application of Theorem 5.3 is restricted to *MRM*s with strictly positive reward rates, while, as in Theorem 5.2, we do not have restriction on the underlying *CTMC*.

5.3.1 System with zero reward rates

Theorem 5.3 can not be applied for computing the moments of completion time when some of the reward rates are zero. In this section we give a method to handle this case.

Let us partition the state space, Ω, into two disjoint sets Ω_+ and Ω_0. Ω_+ (Ω_0) contains the states with associated positive (0) reward rate, i.e., $r_i > 0; \forall i \in \Omega_+$ and $r_i = 0; \forall i \in \Omega_0$. The accumulated reward does not increase during the sojourn in Ω_0. If Ω_0 has got an absorbing subset then the distribution of the completion time is defective, i.e., there is a positive probability that $C_i(w) = \infty$. In the subsequent analysis we do not allow this case.

Without loss of generality, we number the states in Ω such that $i < j$, for $\forall i \in \Omega_+$ and $\forall j \in \Omega_0$. By this partitioning of the state space the reward rate and the generator matrix have the following sub-block structure:

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} , \qquad \mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{pmatrix} . \tag{5.24}$$

Note that $\mathbf{A}_4$ is invertable as a consequence of the assumption that Ω_0 has no absorbing subset. The partitioned form of the performance vectors are:

$$\underline{C}^{\sim\sim}(s,v) = \begin{pmatrix} \underline{C}_+^{\sim\sim}(s,v) \\ \underline{C}_0^{\sim\sim}(s,v) \end{pmatrix} , \qquad \underline{s}^{(n)}(w) = \begin{pmatrix} \underline{s}_+^{(n)}(w) \\ \underline{s}_0^{(n)}(w) \end{pmatrix} . \tag{5.25}$$

Theorem 5.4. *The n-th moment of completion time, $\underline{s}^{(n)}(w)$, can be computed as follows:*

$$\underline{s}_+^{(n)}(w) = (-1)^n \sum_{i=0}^{\infty} \frac{w^i}{i!} \mathbf{L}^{(n)}(i) \cdot \underline{h} \tag{5.26}$$

$$\underline{s}_0^{(n)}(w) = (-1)^n \sum_{i=0}^{\infty} \frac{w^i}{i!} \mathbf{H}^{(n)}(i) \cdot \underline{h} \tag{5.27}$$

where

$$\mathbf{L}^{(n)}(i) = \begin{cases} \mathbf{0} , & i = 0, n > 0 , \\ (\mathbf{R}_1^{-1} \cdot \mathbf{A}_1 - \mathbf{R}_1^{-1} \cdot \mathbf{A}_2 \cdot \mathbf{A}_4^{-1} \cdot \mathbf{A}_3)^i , & i \geq 0, n = 0 , \\ -\mathbf{R}_1^{-1} \cdot \mathbf{A}_2 \cdot \mathbf{A}_4^{-2} \cdot \mathbf{A}_3 - \mathbf{R}_1^{-1} , & i = 1, n = 1 , \\ (-1)^{n+1}\ n!\ \mathbf{R}_1^{-1} \cdot \mathbf{A}_2 \cdot \mathbf{A}_4^{-n-1} \cdot \mathbf{A}_3 , & i = 1, n \geq 2 , \\ \sum_{l=0}^{n} \binom{n}{l} \mathbf{L}^{(l)}(1) \cdot \mathbf{L}^{(n-l)}(i-1) , & i \geq 2, n \geq 1 , \end{cases} \tag{5.28}$$

$$\mathbf{H}^{(n)}(i) = \begin{cases} (-1)^n \mathbf{A}_4^{-(n+1)} \cdot \mathbf{A}_3 \ , & i = 0, n \geq 0 \ , \\ \mathbf{A}_3 \cdot \mathbf{A}_4^{-1} \left(\mathbf{R}_1^{-1} \left(\mathbf{A}_1 - \mathbf{A}_2 \cdot \mathbf{A}_4^{-1} \cdot \mathbf{A}_3 \right) \right)^i \ , & i \geq 0, n = 0 \ , \\ \sum_{l=0}^{n} \binom{n}{l} \mathbf{H}^{(l)}(i-1) \cdot \mathbf{G}^{(n-l)} \ , & i \geq 1, n \geq 1 \end{cases} \tag{5.29}$$

$$\mathbf{G}^{(n)} = \begin{cases} \mathbf{R}_1^{-1} \cdot (\mathbf{A}_1 - \mathbf{A}_2 \cdot \mathbf{A}_4^{-1} \cdot \mathbf{A}_3) \ , & n = 0 \ , \\ \mathbf{R}_1 \cdot \mathbf{A}_2 \cdot \mathbf{A}_4^{-2} \cdot \mathbf{A}_3 - \mathbf{R}_1 \ , & n = 1 \\ (-1)^{n+1} \ n! \ \mathbf{R}_1 \cdot \mathbf{A}_2 \cdot \mathbf{A}_4^{-n-1} \cdot \mathbf{A}_3 \ , & n \geq 2 \end{cases} \tag{5.30}$$

Proof of Theorem 5.4 Substituting the vectors and matrices in (5.6) with their partitioned form and using the following form of matrix inverse

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C})^{-1} & -(\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C})^{-1}\mathcal{B}\mathcal{D}^{-1} \\ -\mathcal{D}^{-1}\mathcal{C}(\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C})^{-1} & \mathcal{D}^{-1} + \mathcal{D}^{-1}\mathcal{C}(\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C})^{-1}\mathcal{B}\mathcal{D}^{-1} \end{pmatrix}$$

with

$$\begin{aligned} \mathcal{A} &= s\mathbf{I}_1 + v\mathbf{R}_1 - \mathbf{A}_1 \ , & \mathcal{B} &= -\mathbf{A}_2 \ , \\ \mathcal{C} &= -\mathbf{A}_3 \ , & \mathcal{D} &= s\mathbf{I}_4 - \mathbf{A}_4 \end{aligned}$$

for $\underline{C}_+^{\sim\sim}(s, v)$ we have:

$$\underline{C}_+^{\sim\sim}(s, v) = v[s\mathbf{I}_1 + v\mathbf{R}_1 - \mathbf{A}_1 - \mathbf{A}_2 \cdot (s\mathbf{I}_4 - \mathbf{A}_4)^{-1} \cdot \mathbf{A}_3]^{-1} \cdot \mathbf{R}_1 \cdot \underline{h} \ . \tag{5.31}$$

Since ${\mathbf{R}_1}^{-1}$ exists by its definition the inverse Laplace transform of (5.31) with respect to $v \to w$ gives

$$\underline{C}_+^{\sim}(s, w) = e^{\boldsymbol{\alpha}(s)w} \cdot \underline{h} = \sum_{i=0}^{\infty} \frac{\boldsymbol{\alpha}(s)^i}{i!} w^i \cdot \underline{h} \tag{5.32}$$

where

$$\boldsymbol{\alpha}(s) = \mathbf{R}_1^{-1} \cdot \mathbf{A}_1 + \mathbf{R}_1^{-1} \cdot \mathbf{A}_2 \cdot (s\mathbf{I}_4 - \mathbf{A}_4)^{-1} \cdot \mathbf{A}_3 - s\mathbf{R}_1^{-1} \ . \tag{5.33}$$

The n-th moment of completion time is

$$\underline{s}_+^{(n)}(w) = (-1)^n \frac{\partial^n}{\partial s^n} \underline{C}_+^{\sim}(s, w) \Big|_{s=0} = (-1)^n \sum_{i=0}^{\infty} \frac{w^i}{i!} \frac{\partial^n}{\partial s^n} \boldsymbol{\alpha}(s)^i \Big|_{s=0} \cdot \underline{h} \tag{5.34}$$

where the n-th deviate of $\boldsymbol{\alpha}(s)^i$ can be evaluated using the Leibniz rule

$$(\boldsymbol{\alpha}(s) \cdot \boldsymbol{\alpha}(s)^{i-1})^{(n)} = \sum_{l=0}^{n} \binom{n}{l} \boldsymbol{\alpha}(s)^{(l)} \cdot \left(\boldsymbol{\alpha} \, (s)^{i-1} \right)^{(n-l)} \ . \tag{5.35}$$

Now $\mathbf{L}^{(n)}(i) = \dfrac{\partial^n}{\partial s^n} \boldsymbol{\alpha}(s)^i \Big|_{s=0}$, completes the proof for $\underline{s}_+^{(n)}(w)$.

The same partitioning of (5.6) gives

$$\begin{aligned}
&\underline{C}_0^{\sim}(s,w) \\
&\quad = (s\mathbf{I}_4 + \mathbf{A}_4)^{-1} \cdot \mathbf{A}_3 \cdot \underline{C}_+(s,w) \\
&\quad = \sum_{i=0}^{\infty} \frac{w^i}{i!} (s\mathbf{I}_4 + \mathbf{A}_4)^{-1} \cdot \mathbf{A}_3 \cdot \boldsymbol{\alpha}(s)^i \cdot \underline{h} \\
&\quad = \sum_{i=0}^{\infty} \frac{w^i}{i!} \mathbf{A}_3 \cdot {\mathbf{A}_4}^{-1} \left({\mathbf{R}_1}^{-1} \left(\mathbf{A}_1 - s\mathbf{I}_1 + \mathbf{A}_2 \left(s\mathbf{I}_1 - \mathbf{A}_4 \right)^{-1} \mathbf{A}_3 - s\mathbf{I}_4 \right) \right)^i \cdot \underline{h}
\end{aligned} \tag{5.36}$$

and applying the Leibniz-rule as before:

$$\underline{s}_0^{(n)}(x) = (-1)^n \cdot \frac{\partial^n}{\partial s^n} \underline{C}_0^{\sim}(s,x) \Big|_{s=0} = (-1)^n \sum_{i=0}^{\infty} \frac{w^i}{i!} \mathbf{H}^{(n)}(i) \cdot \underline{h} \tag{5.37}$$

gives the theorem. □

5.4 Numerical methods based on randomization

In the previous sections iterative procedures were provided to compute the moments of reward measures, but due to the properties of digital computers using floating point numbers a direct application of those methods would result in numerical problems such as instabilities, "ringing" (negative probabilities), etc. The main reason of these problems is that matrices with positive and negative elements (like $\mathbf{A}$) are multiplied several times. To avoid these problems a modified procedure is proposed. Let

$$\mathbf{D} = \frac{\mathbf{A}}{q} + \mathbf{I} \,, \qquad \mathbf{S} = \frac{\mathbf{R}}{qd} \tag{5.38}$$

where $q = \max_{i,j \in \Omega} (|a_{ij}|)$ and $d = \max_{i \in \Omega}(r_i)/q$. By this definition $\mathbf{D}$ is a stochastic matrix ($0 \leq d_{ij} \leq 1, \forall i,j \in \Omega$ and $\sum_{j \in \Omega} d_{ij} = 1, \forall i \in \Omega$) and $\mathbf{S}$ is a diagonal matrix such that $0 \leq s_{ii} \leq 1, \forall i \in \Omega$. The dimension of d is unit of reward. d can be considered as a scaling factor of the accumulated reward. Using these matrices

$$\underline{B}^{\sim}(t,v) = e^{(\mathbf{A} - v\mathbf{R})t} \cdot \underline{h} = e^{(\mathbf{D} - vd\mathbf{S})qt} \cdot \underline{h} e^{-qt} \,. \tag{5.39}$$

Theorem 5.5. *The moments of accumulated reward can be computed using only matrix-vector multiplications and saving only vectors of size $\#\Omega$ in each step of the iteration as*

$$\underline{m}^{(n)}(t) = n!\ d^n \ \sum_{i=0}^{\infty} \underline{U}^{(n)}(i) \frac{(qt)^i}{i!} e^{-qt} \tag{5.40}$$

where

$$\underline{U}^{(n)}(i) = \begin{cases} \underline{0} \,, & \text{if} \quad i = 0, n \geq 1 \,, \\ \underline{h} \,, & \text{if} \quad i \geq 0, n = 0 \,, \\ \mathbf{D} \cdot \underline{U}^{(n)}(i-1) + \mathbf{S} \cdot \underline{U}^{(n-1)}(i-1) \,, & \text{if} \quad i \geq 1, n \geq 1 \,. \end{cases} \tag{5.41}$$

$\underline{U}^{(n)}(i)$	i=0	i=1	i=2	i=3
n=0	$\underline{h}$	$\underline{h}$	$\underline{h}$	$\underline{h}$
n=1	$\underline{0}$	$\mathbf{S}\underline{h}$	$\mathbf{DS}\underline{h} + \mathbf{S}\underline{h}$	$\mathbf{DDS}\underline{h} + \mathbf{DS}\underline{h} + \mathbf{S}\underline{h}$
n=2	$\underline{0}$	$\underline{0}$	$\mathbf{SS}\underline{h}$	$\mathbf{DSS}\underline{h} + \mathbf{SDS}\underline{h} + \mathbf{SS}\underline{h}$
n=3	$\underline{0}$	$\underline{0}$	$\underline{0}$	$\mathbf{SSS}\underline{h}$

Table 1.

Proof of Theorem 5.5 Starting from (5.39) the proof of Theorem 5.5 follows the same pattern as the proof of Theorem 5.2. □

To demonstrate the iterative procedure of computing $\underline{U}^{(n)}(i)$ the first elements of $\underline{U}^{(n)}(i)$ evaluated based on (5.41) are provided in Table 1.

Suppose one is interested in the first 3 moments of the accumulated reward. To perform the computation 3 vectors of size $\#\Omega$ needs to store $\underline{U}^{(n)}(i), n = 1, 2, 3$. In each iteration step $i = 1, 2, 3, \ldots$ matrix-vector multiplications and vector summations has to be performed according to (5.41) using the vectors of the previous iteration step and the constant matrices $\mathbf{D}$ and $\mathbf{S}$. Figure 5.1 shows the dependency structure of the computation. One can recognize that only the $(i-1)$-th column (iteration) of $\underline{U}$ is used for calculating the i-th column of $\underline{U}$. Note that $\mathbf{S}$ is a diagonal matrix and $\mathbf{D}$ is as sparse as $\mathbf{A}$ is. Further 3 vectors of the same size need to store

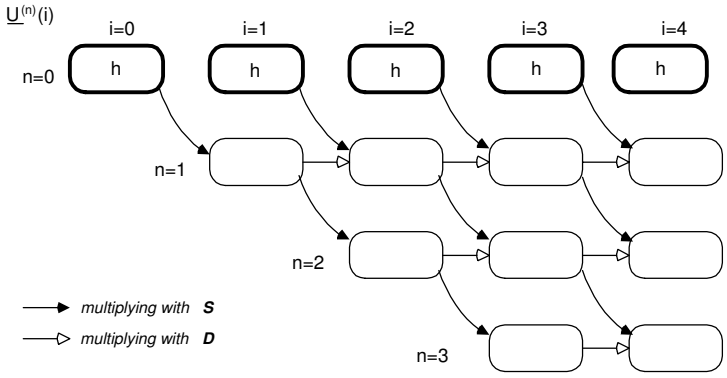


Figure 5.1: The dependency structure of the iteration steps

the "actual value" of $\underline{m}^{(n)}(t), n = 1, 2, 3$ according to (5.40).
The following theorem provides a global error bound of the procedure.

Theorem 5.6. *The n-th moment of accumulated reward can be calculated as a finite sum such that the maximum allowed error is* ε

$$\underline{m}^{(n)}(t) = n!\ d^n \sum_{i=0}^{G-1} \underline{U}^{(n)}(i) \frac{(qt)^i}{i!} e^{-qt} + \underline{\xi}(G) \tag{5.42}$$

where

$$G = \min_{g \in \mathbb{N}} \left((qt)\ n!\ d^n \sum_{i=g-1}^{\infty} \frac{(qt)^i}{i!} e^{-qt} \leq \varepsilon \right) \tag{5.43}$$

and the $\underline{0} \leq \underline{\xi}(G) \leq \underline{h}\ \varepsilon$ *inequality holds for all the elements of the vectors.*

Proof of Theorem 5.6 By the definition of $\mathbf{S}$ and $\mathbf{D}$

$$\underline{0} \leq \mathbf{S} \cdot \underline{h} \ \leq \ \underline{h} \quad \text{and} \quad \underline{0} \leq \mathbf{D} \cdot \mathbf{S} \cdot \underline{h} \ \leq \ \underline{h} \tag{5.44}$$

hold piece-wise (as all the subsequent vector inequalities), hence $\underline{U}^{(n)}(i)$ is bounded by

$$\underline{0} \leq \underline{U}^{(n)}(i) \leq i\ \underline{h}. \tag{5.45}$$

The error $\underline{\xi}(g)$ incurred when eliminating the tale of the infinite sum is also bounded by

$$\begin{aligned} \underline{\xi}(g) &= n!\ d^n\ \sum_{i=g}^{\infty} \underline{U}^{(n)}(i)\ \frac{(qt)^i}{i!} e^{-qt} \leq n!\ d^n\ \sum_{i=g}^{\infty} \underline{h}\ i\ \frac{(qt)^i}{i!} e^{-qt} \\ &\leq (qt)\ n!\ d^n \sum_{i=g-1}^{\infty} \underline{h}\ \frac{(qt)^i}{i!} e^{-qt} \end{aligned} \tag{5.46}$$

which gives the theorem. □

The error bound provided by the theorem is the tail of a Poisson distribution with mean qt multiplied by a constant $(qt)\ n!\ d^n$. A Poisson distribution has a low squared coefficient of variation $(qt)^{-1}$, which decreases as qt increases, and its tail has an exponential decay. Hence, when qt is large (> 100) G is mainly determined by qt and it has only a logarithmic dependence on the constant $(qt)\ n!\ d^n$ and the precision requirement ε. In general, if $qt > 100$ then G and qt are of the same order of magnitude ($G > qt$). A high level description of the proposed method can be found in Section 5.6.

The same approach can be applied for the analysis of completion time, when all the reward rates are positive, i.e., $\mathbf{R}^{-1}$ exists. Let

$$\mathbf{B} = \frac{\mathbf{R}^{-1} \cdot \mathbf{A}}{z} + \mathbf{I}\ , \qquad \mathbf{T} = \frac{\mathbf{R}^{-1}}{zf} \tag{5.47}$$

where $z = \max_{i,j \in \Omega} (|a_{ij}/r_i|)$ and $f = \max_{i \in \Omega}(1/r_i)/z$. By this definition $\mathbf{B}$ is a stochastic matrix ($0 \leq b_{i,j} \leq 1, \forall i,j \in \Omega$ and $\sum_{j \in \Omega} b_{i,j} = 1, \forall i \in \Omega$) and $\mathbf{T}$ is a diagonal matrix such that $0 \leq t_{i,i} \leq 1, \forall i \in \Omega$. f is a number with no dimension.

$$\underline{C}^{\sim}(s,w) = e^{(\mathbf{R}^{-1}\mathbf{A} - s\mathbf{R}^{-1})w} \cdot \underline{h}\ = e^{(\mathbf{B} - sf\mathbf{T})zw} \cdot \underline{h}\ e^{-zw}\ . \tag{5.48}$$

Theorem 5.7. *The moments of the completion time can be computed using only matrix-vector multiplications and saving only vectors of size* $\#\Omega$ *as follows:*

$$\underline{s}^{(n)}(w) = n!\ f^n\ \sum_{i=0}^{\infty} \underline{V}^{(n)}(i) \frac{(zw)^i}{i!} e^{-zw} \tag{5.49}$$

where

$$\underline{V}^{(n)}(i) = \begin{cases} \underline{0} & \text{if} \quad i = 0, n \geq 1\ , \\ \underline{h} & \text{if} \quad i \geq 0, n = 0\ , \\ \mathbf{B} \cdot \underline{V}^{(n)}(i-1) + \mathbf{T} \cdot \underline{V}^{(n-1)}(i-1) & \text{if} \quad i \geq 1, n \geq 1\ . \end{cases} \tag{5.50}$$

Proof of Theorem 5.7 Theorem 5.7 is obtained from (5.48) applying similar steps as in the proof of Theorem 5.4. □

Theorem 5.8. *The n-th moment of completion time can be calculated as a finite sum and an error part, where the maximum allowed error is* ε

$$\underline{s}^{(n)}(w) = n!\ f^n \sum_{i=0}^{G-1} \underline{V}^{(n)}(i) \frac{(zw)^i}{i!} e^{-zw} + \underline{\xi}(G) \tag{5.51}$$

$$\textit{where} \qquad G = \min_{g \in \mathbb{N}} \left((zw)\ n!\ f^n \sum_{i=g-1}^{\infty} \frac{(zw)^i}{i!} e^{-zw} \leq \varepsilon \right) \tag{5.52}$$

$$\textit{and} \qquad \underline{0} \leq \underline{\xi}(G) \leq \underline{h}\ \varepsilon\ . \tag{5.53}$$

Proof of Theorem 5.8 The proof of Theorem 5.8 follows the same pattern as the proof of Theorem 5.6. □.

The numerical analysis of the completion time of large models when states with zero reward rate are present in the system is more complicated. A numerical procedure similar to the one in Theorem 5.8 can be obtained as well, but its applicability is strongly limited by the cardinality of Ω_0. The $\mathbf{A}_4$ matrix of cardinality $\#\Omega_0$ has to be inverted in this case. In general, the complexity of inverting a matrix of cardinality 10^4 has higher computational complexity and memory requirement than the proposed numerical method with 10^6 states.

5.5 Numerical examples

Example 1
Consider a *CTMC* with $n = 1,000,000$ states. The non-zero state transition rates are:

$$a_{ij} = \begin{cases} 5\ , & \text{if} \quad j = i+1\ , \\ 2.5\ , & \text{if} \quad j = i+10,000\ , \\ 2.5\ , & \text{if} \quad j = i-1\ . \end{cases} \tag{5.54}$$

The diagonal matrix of the reward rates, $\mathbf{R}$, has the following structure:

$$r_{i,i} = \begin{cases} 0 & \text{if} \quad i < 800,000\ , \\ 1 & \text{if} \quad i \geq 800,000\ . \end{cases} \tag{5.55}$$

Figure 5.2. shows the structure of the underlying *CTMC*, where $u = 10,000$.

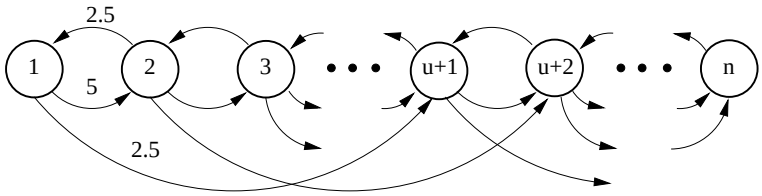


Figure 5.2: The underlying *CTMC* of Example 1.

Mean value	$t = 0.02s$	$t = 0.1s$	$t = 0.2s$	$t = 1s$	$t = 2s$
$\mathcal{Z}(0) = 750,000$	$8.06 \cdot 10^{-12}$	$9.81 \cdot 10^{-8}$	$5.11 \cdot 10^{-6}$	0.022	0.33
$\mathcal{Z}(0) = 790,000$	0.00047	0.010	0.037	0.58	1.54
$\mathcal{Z}(0) = 800,000$	0.019	0.093	0.18	0.94	1.94

Table 2.

Variance	$t = 0.02s$	$t = 0.1s$	$t = 0.2s$	$t = 1s$	$t = 2s$
$\mathcal{Z}(0) = 750,000$	$4.61 \cdot 10^{-14}$	$2.73 \cdot 10^{-9}$	$5.03 \cdot 10^{-7}$	$7.73 \cdot 10{-}3$	0.17
$\mathcal{Z}(0) = 790,000$	$6.07 \cdot 10^{-6}$	$5.85 \cdot 10^{-4}$	$3.62 \cdot 10^{-3}$	0.096	0.16
$\mathcal{Z}(0) = 800,000$	$5.79 \cdot 10^{-6}$	$4.13 \cdot 10^{-4}$	$1.92 \cdot 10^{-3}$	0.018	0.022

Table 3.

Table 2 and 3 contain the mean and the variance of the accumulated reward with different initial state. The accumulated reward represents the total time the system spent in states $800,000, \ldots, 1,000,000$.

Example 2

In the second example, the performance parameters of a Carnegie-Mellon multiprocessor system are evaluated by the proposed method. The system is similar to the one presented in [75]. The system consists of N processors, M memories, and an interconnection network (composed by switches) that allows any processor to access any memory (Figure 5.3). The failure rates are 0.1, 0.05, 0.01 and 0.003 failures per hour for the processors, memories, switches, and general failure, respectively.

Viewing the interconnecting network as S switches and modeling the system at the processor-memory-switch level, the system performance depends on the minimum of the number of operating processors, memories, and switches. Each state is thus specified by a triple (i, j, k) indicating the number of operating processors, memories, and switches, respectively. We augment the states with the nonoperational state F. Events that decrease the number of operational units are the failures and events that increase the number of operational elements are the repairs. We assume that failures do not occur when the system is not operational. When a component fails, a recovery action must be taken (e.g., shutting down the a failed processor, etc.), or the whole system will fail and enter state F.

Two kinds of repair actions are possible, global repair which restores the system to state (N, M, S) with rate $\mu = 0.01$ per hour from state F, and local repair, which can be thought of as a repair person beginning to fix a component of the system as soon as a component failure occurs. We assume that there is only one repair person for each component type. Let the local repair rates be 2.0, 2.0 and 0.1 for the processors, memories and the switch, respectively.

The system starts from the perfect state (N, M, S). The studied system has 32 processors, 64 memories, and 16 switches, thus the state space consists of 36,466 states (247,634 transitions). The performance of the system is proportional to the number of cooperating processors and memories, whose cooperation is provided by one switch. The reward rate is defined as the minimum of the operational processors,

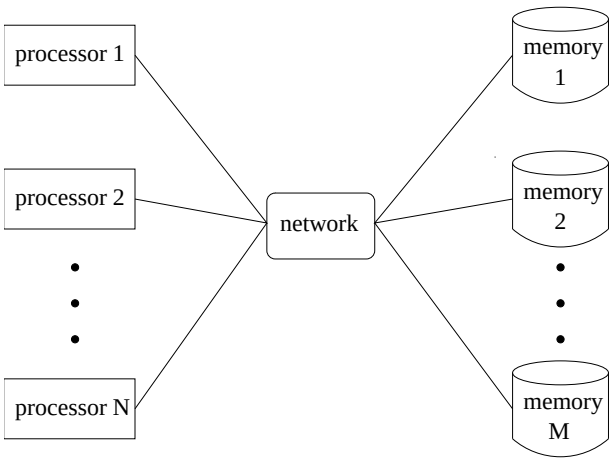


Figure 5.3: Example system structure

t	$E(B(t))$	$E(B(t)^2)$	$E(B(t)^3)$	$E(B(t)^4)$	$E(B(t)^5)$	$E(B(t)^6)$
1	15.89	253.0	4030	$6.41 \cdot 10^4$	$1.02 \cdot 10^6$	$1.63 \cdot 10^7$
2	31.60	1001	$3.14 \cdot 10^4$	$1.00 \cdot 10^6$	$3.19 \cdot 10^7$	$1.01 \cdot 10^9$
5	77.70	6072	$4.75 \cdot 10^5$	$3.72 \cdot 10^7$	$2.92 \cdot 10^9$	$2.30 \cdot 10^{11}$
10	151.5	$2.32 \cdot 10^4$	$3.57 \cdot 10^6$	$5.51 \cdot 10^8$	$8.52 \cdot 10^{10}$	$1.31 \cdot 10^{13}$
20	289.5	$8.57 \cdot 10^4$	$2.55 \cdot 10^7$	$7.67 \cdot 10^9$	$2.30 \cdot 10^{12}$	$6.96 \cdot 10^{14}$
50	648.0	$4.42 \cdot 10^5$	$3.08 \cdot 10^8$	$2.16 \cdot 10^{11}$	$1.53 \cdot 10^{14}$	$1.09 \cdot 10^{17}$

Table 4.

memories, and switches. The minimal operational configuration is supposed to have one processor, one memory and one interconnection switch.

The first 6 moments of the accumulated reward were calculated using Theorem 5.5 in two different cases. In the first case global repair was not possible, hence F was an absorbing state of the system. In the second case global repair was allowed at rate 0.01. Table 4 and 5 contain the results obtained at time $t = 1, 2, 5, 10, 20, 50$ for the case without and with global repair, respectively.

The mean and the variance of the accumulated reward of the two cases are compared in Figures 5.4, and 5.5, respectively. The dashed lines refer to the case when global repair is not possible. As it was expected, the mean accumulated reward of

t	$E(B(t))$	$E(B(t)^2)$	$E(B(t)^3)$	$E(B(t)^4)$	$E(B(t)^5)$	$E(B(t)^6)$
1	15.89	253.0	4030	$6.42 \cdot 10^4$	$1.02 \cdot 10^6$	$1.63 \cdot 10^7$
2	31.60	1001	$3.14 \cdot 10^4$	$1.00 \cdot 10^6$	$3.19 \cdot 10^7$	$1.01 \cdot 10^9$
5	77.70	6073	$4.75 \cdot 10^5$	$3.72 \cdot 10^7$	$2.92 \cdot 10^9$	$2.30 \cdot 10^{11}$
10	151.6	$2.32 \cdot 10^4$	$3.57 \cdot 10^6$	$5.51 \cdot 10^8$	$8.52 \cdot 10^{10}$	$1.31 \cdot 10^{13}$
20	290.1	$8.59 \cdot 10^4$	$2.56 \cdot 10^7$	$7.68 \cdot 10^9$	$2.31 \cdot 10^{12}$	$6.97 \cdot 10^{14}$
50	655.6	$4.48 \cdot 10^5$	$3.11 \cdot 10^8$	$2.19 \cdot 10^{11}$	$1.55 \cdot 10^{14}$	$1.10 \cdot 10^{17}$

Table 5.

the case without global repair is less. The variance curves are misleading for the first sight. The second moment of the case without global repair is still less, but the relation of the variance parameters depend on the difference of the first two moments, and that is why the variance of the case without global repair is higher.

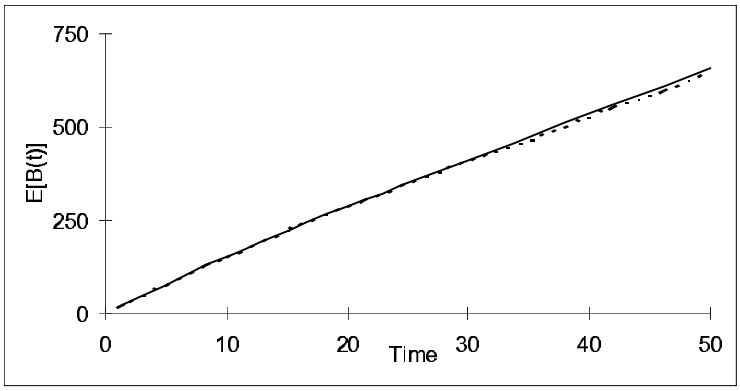


Figure 5.4: Mean accumulated reward

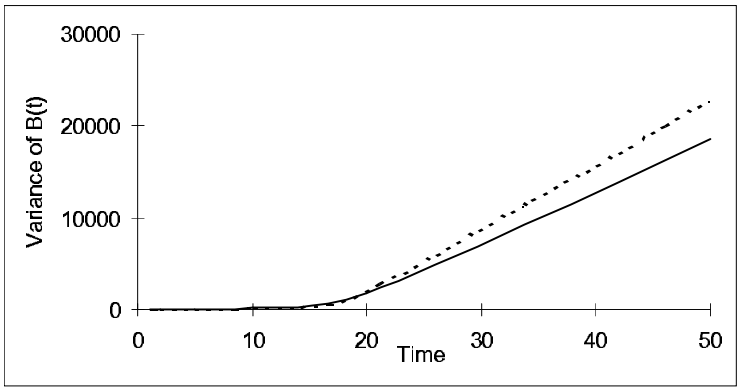


Figure 5.5: Variance of the accumulated reward

5.6 Implementation of the numerical method

A formal description of the program that calculates the moments of accumulated reward according to Theorem 4 is provided. The memory requirement and number of required operations are calculated in advance.

Input	M	CARDINALITY OF THE STATE SPACE
	$\mathbf{A}$	GENERATOR MATRIX OF UNDERLYING CTMC
	$\mathbf{R}$	DIAGONAL MATRIX OF THE REWARD RATES
	$\underline{P}$	INITIAL PROBABILITY VECTOR
	t	TIME OF ACCUMULATION
	n	ORDER OF MOMENT
	G	NUMBER OF ITERATIONS
	z	NUMBER OF NON-ZERO ELEMENTS IN $\mathbf{A}$
Output	m	THE n-TH MOMENT OF ACCUMULATED REWARD
	mem	MEMORY REQUIREMENT
	mul	REQUIRED FLOATING POINT MULTIPLICATION
	add	REQUIRED FLOATING POINT ADDITION

1 $mem_D = z \cdot Size(double)$ *storing elements of* $\mathbf{D}$
$mem_D = mem_D + (z + M) \cdot Size(int)$
$mem_S = M \cdot Size(double)$ *storing* $\mathbf{S}$
$mem_P = M \cdot Size(double)$ *storing* $\underline{P}$
$mem_N = M \cdot (n+1) \cdot Size(double)$ *temporary vectors*
$mem = mem_D + mem_S + mem_P + mem_N$
2 $add = o\left(G \cdot (2 \cdot n \cdot z + (n+1) \cdot M)\right)$ *compute numerical complexity*
$mul = o\left(G \cdot (2 \cdot n \cdot z + M)\right)$
3 $\underline{U}^{(0)} = \underline{h}; \quad \underline{U}^{(i)} = \underline{0}\,, \quad i : 1 \ldots n;$ *compute the n-th moment*
For $i := 1$ **To** G **Do**
Begin
For $j := n$ **DownTo** 0 **Do**
$\underline{U}^{(j)} := \mathbf{S} \cdot \underline{U}^{(j-1)} + \mathbf{D} \cdot \underline{U}^{(j)};$
$\underline{m} := \underline{m} + \underline{U}^{(j)} \cdot Poisson(i; qt);$
End;
$m := m \cdot n! \cdot d^n$

Chapter 6

Partial Loss in Reward Models

Previously applied reward models assumed no reward loss or a complete loss of accumulated reward at state transitions. In this chapter we introduce a new model, the intermediate case when only a portion of the accumulated reward is lost at a state transition of the considered system. Two different partial loss models are considered in the sequel. In the *partial loss of incremental reward* case only a portion of the reward accumulated during the sojourn in the last visited state is lost, while in the *partial loss of total reward* case a portion of the total accumulated reward is lost at a state transition.

The subsequent analysis of these reward models indicate a very important feature of partial loss models: there is dominant qualitative difference between the accumulated reward and the completion time measures. E.g., in some cases the accumulated reward has a closed form description in transform domain, while the completion time has not. The source of this difference is very natural. Due to the fact that the trajectory of the accumulated reward is non-monotone in partial loss models the fact that that accumulated reward is less than an arbitrary limit w at time t does not necessarily imply that the completion time is greater than t.

6.1 Partial incremental work loss in an SMP environment

Let $\{Z(t), t \geq 0\}$ be a semi-Markov process (SMP) on state space $\Omega = \{1, 2, ..., N\}$ with kernel $Q(\cdot) = [Q_{ij}(\cdot)]$. We assume that $Q(t)$ is the canonical kernel of $Z(t)$, i.e., $Q_{ii} = 0, \forall i$. Suppose that whenever the SMP is in state i, reward is accumulated at rate r_i. When the SMP undergoes a transition from state i, a fraction $1 - \mathcal{A}_i$ of the reward obtained during the last sojourn in state i is lost and a fraction $\mathcal{A}_i$ of the reward obtained during the last sojourn remains. $\mathcal{A}_i$ is a r.v. over $(0, 1)$ with distribution $L_i(\alpha) = Pr(\mathcal{A}_i \leq \alpha)$. $B(t)$ denotes the amount of accumulated reward at time t. Let T_n be the time of the n^{th} state transition in the SMP. The dynamics of the right continuous accumulated reward process $\{B(t), t \geq 0\}$ defined as follows (Figure 6.1):

$$\frac{dB(t)}{dt} = r_{Z(t)} \quad \text{for} \quad T_n < t < T_{n+1} \tag{6.1}$$

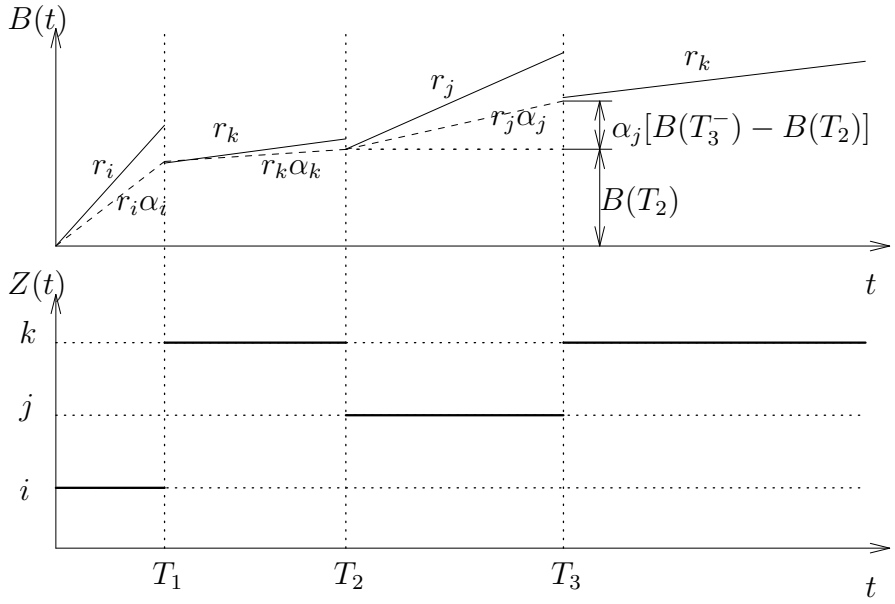


Figure 6.1: Reward accumulation in partial incremental loss model

$$B(T_n) \;=\; B(T_{n-1}) + \mathcal{A}_{Z(T_n^-)}[B(T_n^-) - B(T_{n-1})] \tag{6.2}$$

6.1.1 Limiting distribution of $B(t)$

By (6.2) $B(T_n)$ is an non-decreasing series.

Suppose $Z(t)$ is an ergodic process with steady state distribution $\underline{\pi} = \{\pi_i\}$. Both $lim_{n\to\infty}B(T_n)$ and $lim_{t\to\infty}B(t)$ go to infinity if there is at least one state $i \in \Omega$ such that $\pi_i r_i E[\mathcal{A}_i] > 0$.

6.1.2 Accumulated reward up to time t

Define

$$P_i(t,w) = Pr(B(t) \le w \mid Z(0) = i).$$

The distribution of the accumulated reward is provided in the following theorems.

Theorem 6.1. *In double transform domain the distribution of accumulated reward with deterministic loss portion (i.e., $\mathcal{A}_i = \alpha_i$) satisfies:*

$$P_i^{*\sim}(s,v) \;=\; \frac{1 - Q_i^{\sim}(s + vr_i)}{s + vr_i} + \sum_{k\in\Omega} Q_{ik}^{\sim}(s + vr_i\alpha_i) \;\cdot\; P_k^{*\sim}(s,v) \tag{6.3}$$

Proof. Conditioning on H, the sojourn time in state i, we have:

$$P_i(t,w|\, H=\tau) \;=\; \begin{cases} U_w(w - r_i t) & \text{if} : \tau > t \\ \displaystyle\sum_{k\in\Omega} \frac{dQ_{ik}(\tau)}{dQ_i(\tau)} \;\cdot\; P_k(t-\tau, w - \alpha_i\tau r_i) & \text{if} : \tau < t \end{cases} \tag{6.4}$$

Taking the Laplace-Stieltjes transform with respect to w ($\rightarrow v$)

$$P_i^{\sim}(t,v|\,H=\tau) \;=\; \begin{cases} e^{-vr_it} & \text{if}: \tau > t \\ \displaystyle\sum_{k\in\Omega} \frac{dQ_{ik}(\tau)}{dQ_i(\tau)} \cdot e^{-vr_i\alpha_i\tau} \cdot P_k^{\sim}(t-\tau,v) & \text{if}: \tau < t \end{cases} \tag{6.5}$$

Unconditioning with respect to H (by $Q_i(t)$),

$$P_i^{\sim}(t,v) \;=\; e^{-vr_it}(1-Q_i(t)) + \sum_{k\in\Omega} \int_{\tau=0}^{t} e^{-vr_i\alpha_i\tau} \cdot P_k^{\sim}(t-\tau,v)\; dQ_{ik}(\tau) \tag{6.6}$$

Taking the Laplace transform with respect to t ($\rightarrow s$) results:

$$\begin{aligned}
&P_i^{*\sim}(s,v) \;=\; \int_{t=0}^{\infty} e^{-st} P_i^{\sim}(t,v)dt \;= \\
&\int_{t=0}^{\infty} e^{-st}e^{-vr_it}(1-Q_i(t))\; dt + \sum_{k\in\Omega} \int_{t=0}^{\infty} e^{-st} \int_{\tau=0}^{t} e^{-vr_i\alpha_i\tau}\, P_k^{\sim}(t-\tau,v)\; dQ_{ik}(\tau)\; dt = \\
&\frac{1}{s+vr_i} - Q_i^{*}(s+vr_i) + \sum_{k\in\Omega} \int_{\tau=0}^{\infty} e^{-s\tau}e^{-vr_i\alpha_i\tau} \int_{t=\tau}^{\infty} e^{-s(t-\tau)}\, P_k^{\sim}(t-\tau,v)\; dt\; dQ_{ik}(\tau) = \\
&\frac{1}{s+vr_i} - \frac{Q_i^{\sim}(s+vr_i)}{s+vr_i} + \sum_{k\in\Omega} \int_{\tau=0}^{\infty} e^{-(s+vr_i\alpha_i)\tau}\; dQ_{ik}(\tau)\; P_k^{*\sim}(s,v) \;= \\
&\frac{1-Q_i^{\sim}(s+vr_i)}{s+vr_i} + \sum_{k\in\Omega} Q_{ik}^{\sim}(s+vr_i\alpha_i)\, P_k^{*\sim}(s,v)
\end{aligned} \tag{6.7}$$

□

Theorem 6.2. *With random loss potion, $\mathcal{A}_i$, the following double transform domain equation holds for $P_i(t,w)$:*

$$P_i^{*\sim}(s,v) \;=\; \frac{1-Q_i^{\sim}(s+vr_i)}{s+vr_i} + \sum_{k\in\Omega} \int_{\tau=0}^{\infty} e^{s\tau} L_i^{\sim}(vr_i\tau)\; dQ_{ik}(\tau) \cdot P_k^{*\sim}(s,v) \tag{6.8}$$

where $Q_i(t) = \sum_{j\in R} Q_{ij}(t)$.

Proof. Conditioning on H, the sojourn time in state i, we have:

$$P_i(t,w|\,H=\tau) \;=\; \begin{cases} U_w(w-r_it) & \text{if}: \tau > t \\ \displaystyle\sum_{k\in\Omega} \frac{dQ_{ik}(\tau)}{dQ_i(\tau)} \cdot \int_{\alpha=0}^{1} P_k(t-\tau, w-\alpha\tau r_i) dL_i(\alpha) & \text{if}: \tau < t \end{cases} \tag{6.9}$$

Taking the Laplace-Stieltjes transform with respect to w ($\rightarrow v$)

$$P_i^{\sim}(t,v|\,H=\tau) \;=\; \begin{cases} e^{-vr_it} & \text{if}: \tau > t \\ \displaystyle\sum_{k\in\Omega} \frac{dQ_{ik}(\tau)}{dQ_i(\tau)} \cdot L_i^{\sim}(vr_i\tau) \cdot P_k^{\sim}(t-\tau,v) & \text{if}: \tau < t \end{cases} \tag{6.10}$$

Unconditioning with respect to H (by $Q_i(t)$),

$$P_i^{\sim}(t,v) \;=\; e^{-vr_it}(1-Q_i(t)) + \sum_{k\in\Omega} \int_{\tau=0}^{t} L_i^{\sim}(vr_i\tau) \cdot P_k^{\sim}(t-\tau,v)\; dQ_{ik}(\tau)\;. \tag{6.11}$$

Taking the Laplace transform with respect to t ($\rightarrow s$) results in the theorem. □

Corollary 6.3. *In a CTMC environment with generator $A=[a_{ij}]$ ($a_i=-a_{ii}$)*

$$P_i^{*\sim}(s,v) \;=\; \frac{1}{s+vr_i+a_i} + \sum_{k\in\Omega,k\neq i} \frac{a_{ik}}{s+vr_i\alpha_i+a_i} \cdot P_k^{*\sim}(s,v)\;, \tag{6.12}$$

whose solution, in matrix form, is:

$$P^{*\sim}(s,v) = (sI+vR_\alpha - A)^{-1} D_1(s,v)\;, \tag{6.13}$$

where the diagonal matrices are defined as

$$R_\alpha = \text{diag} < r_i\alpha_i > \quad \textit{and} \quad D_1(s,v) = \text{diag}\left\langle \frac{s+vr_i\alpha_i+a_i}{s+vr_i+a_i} \right\rangle\;.$$

6.1.3 Extreme loss ratio

From the model behaviour it is intuitively clear that the partial loss reward models become equivalent with the corresponding loss-less (prs) models when all the loss variable are set to 1, i.e., $\mathcal{A}_i = \alpha_i = 1, \forall i$ and they become equivalent with the corresponding total loss (prd) models when all the loss variable are set to 0, i.e., $\mathcal{A}_i = \alpha_i = 0, \forall i$. The results provided in this chapter verifies this equivalence for extreme loss ratio.

For example, with $\alpha_i = 1, \forall i$ Equation (6.13) becomes

$$P^{*\sim}(s,v) = (sI+vR-B)^{-1} \tag{6.14}$$

and with $\alpha_i = 0, \forall i$ it is

$$P^{*\sim}(s,v) = (sI-B)^{-1}\text{diag}\left\langle \frac{s+a_i}{s+vr_i+a_i} \right\rangle \tag{6.15}$$

Equation (6.14) is one of the first result on MRMs [59], while Equation (6.15) says that the reward accumulation process till the last state transition does not result in any reward ($(sI-B)^{-1}$) and all the reward is accumulated in the last visited state (say state i) at rate r_i during the sojurn in that state $\frac{s+a_i}{s+vr_i+a_i}$ [16].

6.1.4 Completion time

Let us define the completion time (r.v.) and its conditional distribution as follow

$$C(w) = min[t : B(t) \geq w]$$

and

$$C_i(t, w) = Pr(C(w) \leq t | Z(0) = i).$$

Unfortunately the completion time can not be expressed with such closed form transform domain expression as the accumulated reward.

Theorem 6.4. *The distribution of completion time with deterministic loss portion (i.e., $\mathcal{A}_i = \alpha_i$) satisfies:*

$$\begin{aligned} C_i^{\sim *}(s, v) \;=\; & \frac{r_i\,[1 \,-\, Q_i^{\sim}(s \,+\, v\, r_i)\,]}{s \,+\, v\, r_i} \,+\, \sum_{k \in \Omega} Q_{ik}^{\sim}(s \,+\, v\, r_i\, \alpha_i)\, C_k^{\sim *}(s, v) \,- \\ & \sum_{k \in \Omega} \int_{h=0}^{\infty} e^{-sh} e^{-hvr_i\alpha_i} \int_{w=hr_i\alpha_i}^{hr_i} e^{-v(w - hr_i\alpha_i)}\; C_k^{\sim}(s, w - hr_i\alpha_i)\; dw\; dQ_{ik}(h) \end{aligned} \tag{6.16}$$

Proof. Conditioning on the sojourn time in state i (H), we have:

$$C_i(t, w |\, H = h) \;=\; \begin{cases} U\left(t \,-\, \dfrac{w}{r_i}\right) & \text{if} : h\, r_i \geq w \\ \displaystyle\sum_{k \in \Omega} \frac{dQ_{ik}(h)}{dQ_i(h)} \,\cdot\, C_k(t - h, w - hr_i\alpha_i) & \text{if} : h\, r_i < w \end{cases} \tag{6.17}$$

Taking the Laplace-Stieltjes transform with respect to t results:

$$C_i^{\sim}(s, w |\, H = h) \;=\; \begin{cases} e^{-s\frac{w}{r_i}} & \text{if} : h\, r_i \geq w \\ \displaystyle\sum_{k \in \Omega} \frac{dQ_{ik}(h)}{dQ_i(h)} \,\cdot\, e^{-sh} \,\cdot\, C_k^{\sim}(s, w - hr_i\alpha_i) & \text{if} : h\, r_i < w \end{cases} \tag{6.18}$$

Unconditioning with respect to H, yields

$$\begin{aligned} & C_i^{\sim}(s, w) \;=\; \int_{h=\frac{w}{r_i}}^{\infty} e^{-s\frac{w}{r_i}}\, dQ_i(h) + \sum_{k \in \Omega} \int_{h=0}^{\frac{w}{r_i}} e^{-sh}\; C_k^{\sim}(s, w - hr_i\alpha_i)\; dQ_{ik}(h) \;= \\ & e^{-s\frac{w}{r_i}} \left[1 - Q_i\left(\frac{w}{r_i}\right)\right] + \sum_{k \in \Omega} \int_{h=0}^{\frac{w}{r_i}} e^{-sh}\; C_k^{\sim}(s, w - hr_i\alpha_i)\; dQ_{ik}(h) \end{aligned} \tag{6.19}$$

Now we try to take the Laplace transform with respect to w:

$$
\begin{aligned}
&\widetilde{C_i}^{*}(s,v) \;=\; \int_{w=0}^{\infty} e^{-wv}\; \widetilde{C_i}(s,w)\; dw \;= \\
&\int_{w=0}^{\infty} e^{-wv}\; e^{-s\frac{w}{r_i}} \left[1 - Q_i\left(\frac{w}{r_i}\right)\right]\; dw + \\
&\qquad \sum_{k\in\Omega} \int_{w=0}^{\infty} e^{-wv} \int_{h=0}^{\frac{w}{r_i}} e^{-sh}\; \widetilde{C_k}(s, w - hr_i\alpha_i)\; dQ_{ik}(h)\; dw \;= \\
&\frac{r_i\;[1 \;-\; \widetilde{Q_i}(s \;+\; v\,r_i)\,]}{s \;+\; v\,r_i} + \\
&\qquad \sum_{k\in\Omega} \int_{h=0}^{\infty} e^{-sh} e^{-hvr_i\alpha_i} \int_{w=hr_i}^{\infty} e^{-v(w-hr_i\alpha_i)}\; \widetilde{C_k}(s, w - hr_i\alpha_i)\; dw\; dQ_{ik}(h)
\end{aligned}
\tag{6.20}
$$

Unfortunately the inner integral with respect to w is not a complete Laplace transform, because

$$
\begin{aligned}
&\widetilde{C_i}^{*}(s,v) \;=\; \frac{r_i\;[1 \;-\; \widetilde{Q_i}(s \;+\; v\,r_i)\,]}{s \;+\; v\,r_i} + \\
&\sum_{k\in\Omega} \int_{h=0}^{\infty} e^{-sh} e^{-hvr_i\alpha_i} \int_{w=hr_i\alpha_i}^{\infty} e^{-v(w-hr_i\alpha_i)}\; \widetilde{C_k}(s, w - hr_i\alpha_i)\; dw\; dQ_{ik}(h) - \\
&\sum_{k\in\Omega} \int_{h=0}^{\infty} e^{-sh} e^{-hvr_i\alpha_i} \int_{w=hr_i\alpha_i}^{hr_i} e^{-v(w-hr_i\alpha_i)}\; \widetilde{C_k}(s, w - hr_i\alpha_i)\; dw\; dQ_{ik}(h)
\end{aligned}
\tag{6.21}
$$

□

6.2 Partial loss on the total accumulated reward

The same ergodic semi-Markov environment is considered with a different accumulation process. Whenever the SMP is in state i reward is accumulated at rate r_i. When the SMP undergoes a transition out from state i the fraction $(1 - \mathcal{A}_i)$ of the total accumulated reward is lost and the fraction $\mathcal{A}_i$ of the total reward is resumed in the new state. In the sequel we assume that $\mathcal{A}_i = \alpha_i$ $(0 \le \alpha_i \le 1)$ is deterministic[1]. The dynamics of the right continuous reward process $\{B(t), t \ge 0\}$ is defined as follows (Figure 6.2):

$$
\frac{dB(t)}{dt} \;=\; r_{Z(t)} \quad \text{for } T_n \le t < T_{n+1} \tag{6.22}
$$

$$
B(T_n) \;=\; \mathcal{A}_{Z(T_n^-)} B(T_n^-) \tag{6.23}
$$

[1]The case when $\mathcal{A}_i$ is a r.v. over $(0,1)$ with distribution $L_i(.)$ can be considered following the same approach, but it results in very cumbersome expressions.

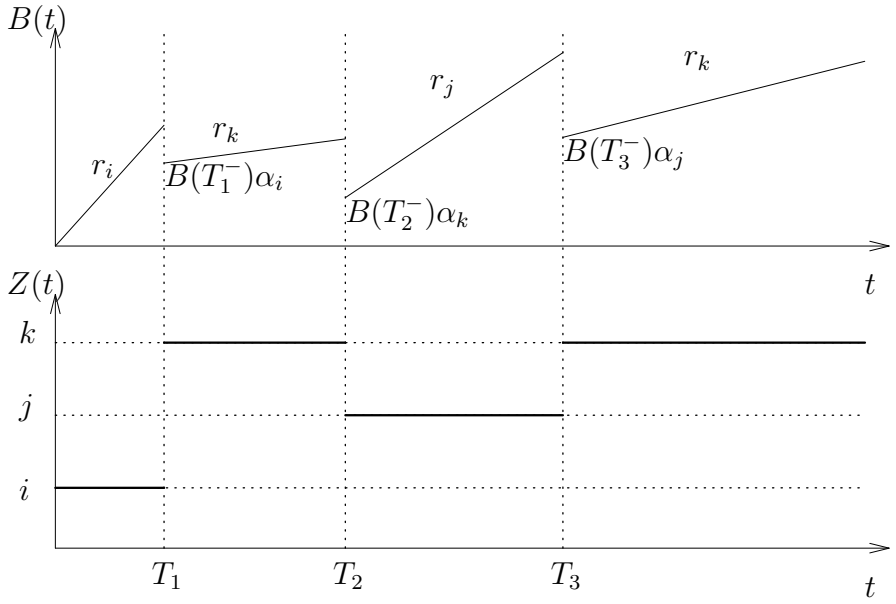


Figure 6.2: Reward accumulation in partial total loss model

6.2.1 Limiting distribution of $B(t)$

In contrast with the partial incremental loss case, $B(T_n)$ does not increase to infinity (if a state $i \in \Omega$ exists such that $\alpha_i < 1$). The average reward accumulated during a sojourn in a state is finite and independent of $B(t)$, while the amount of reward lost at a state transition is proportional to $B(t)$. A larger $B(t)$ results in a larger reward loss which indicates that $B(t)$ has a finite limiting behaviour.

We shall derive the limiting joint distribution of the $\{(B(t), Z(t)), t \geq 0\}$ process, but we first look at the $\{(B(T_n), Z(T_n)), n \geq 0\}$ process. Define

$$\hat{f}_i(n,w)\ dw = Pr\{B(T_n) \in (w, w+dw), Z(T_n) = i\},$$

$$f_i(w) = \lim_{n\to\infty} \hat{f}_i(n,w),$$

$$f_i^*(v) = \lim_{n\to\infty} E(e^{-vB(T_n)}, Z(T_n) = i) = \int_{w=0}^{\infty} e^{-vw} f_i(w)\ dw.$$

Theorem 6.5. *The Laplace transform of the limiting joint distribution of the* $\{(B(T_n), Z(T_n))\}$ *process is given by:*

$$f_j^*(v) = \sum_{i\in\Omega} \alpha_i f_i^*(v\alpha_i) Q_{ij}^{\sim}(v r_i \alpha_i). \tag{6.24}$$

Proof. Suppose a transition to state j occurs after a sojourn time τ in state i, then

$$f_j(w|i,\tau) = f_i(\frac{w}{\alpha_i} - r_i\tau).$$

Unconditioning with respect to the preceding state and the sojourn time gives

$$f_j(w) = \sum_{i\in\Omega} \int_{\tau=0}^{\frac{w}{r_i\alpha_i}} f_i(\frac{w}{\alpha_i} - r_i\tau)\ dQ_{ij}(\tau),$$

finally the Laplace transformation with respect to $w(\rightarrow v)$ results in the theorem. □

Now we define the Laplace transform of the limiting joint distribution of the $\{B(t), Z(t)\}$ process as:

$$\hat{g}_i(t,w)dw = Pr\{B(t) \in (w, w+dw), Z(t) = i\},$$

$$g_i(w) = \lim_{t\to\infty} \hat{g}_i(t,w),$$

$$g_i^*(v) = \lim_{t\to\infty} E(e^{-vB(t)}, Z(t) = i) = \int_{w=0}^{\infty} e^{-vw} g_i(w)\ dw.$$

and let $\gamma_i = \int_0^\infty t\ dQ_i(t)$, be the mean sojourn time in state i.

Theorem 6.6. *The stationary distribution of the accumulated reward is*

$$g_i^*(v) = f_i^*(v)\frac{1 - Q_i^{\sim}(vr_i)}{vr_i\gamma_i}. \tag{6.25}$$

Proof. Suppose the SMP is in steady state at time 0; then the age of the current sojourn is given by the equilibrium distribution of $Q_i(t)$ if the current state is i. Hence B(0) equals the accumulated reward at the last state transition plus the reward accumulated since that. □

Corollary 4: In the special case when $Z(t)$ is a CTMC with generator $A = [a_{ij}]$ and $a_i = -a_{ii}$

$$f_j^*(v) = \sum_{i\in\Omega, i\neq j} \frac{\alpha_i a_{ij}}{vr_i\alpha_i + a_i} f_i^*(v\alpha_i)\ , \tag{6.26}$$

$$g_i^*(v) = \frac{1}{\gamma_i(a_i + vr_i)} f_i^*(v)\ . \tag{6.27}$$

6.2.2 Accumulated reward up to time t

To apply a regenerative approach similar to the one used in Theorem 6.1, a more complicated description has to be used. Indeed, in this case, it is not enough to consider the difference between the present ($B(t)$) and the target value (w) of the reward accumulation process, but we need carry both of these values. Let us define:

$$V_i(t,w,\eta) = Pr(B(t) \le w \mid Z(0) = i, B(0) = \eta)$$

The regenerative description of the process evolution is the following:

$$V_i(t,w,\eta|\ H = \tau) = \begin{cases} U_w(w - \eta - r_i t) & \text{if} : \tau > t \\ \sum_{k\in\Omega} \dfrac{dQ_{ik}(\tau)}{dQ_i(\tau)} \cdot V_k(t-\tau, w, (\eta + \tau r_i)\alpha_i) & \text{if} : \tau < t \end{cases} \tag{6.28}$$

Unfortunately Eq. (6.28) does not exhibit any closed form transform domain expression, hence a different analysis approach is adopted with CTMC background process.

Underlying CTMC: Let $Z(t)$ be a CTMC with generator $A = [a_{ij}]$. Assuming $B(0) = 0$ we define:

$$S_i(t, w) = Pr(B(t) \leq w \mid Z(t) = i).$$

Note that the condition applies for time t.

Theorem 6.7. *$S_i(t, w)$ satisfies the following double transform domain equation:*

$$S_i^{**}(s, v) = \frac{1}{v(s + r_i v + a_i)} + \sum_{k \in \Omega, k \neq i} \alpha_k a_{ik} \; S_k^{**}(s, \alpha_k v) \tag{6.29}$$

Proof. The forward argument describing the evolution of the process is:

$$S_i(t + dt, w) = (1 - a_i dt) S_i(t, w - r_i dt) \; + \sum_{k \in \Omega, k \neq i} a_{ik} \; dt \; S_k(t, \frac{w}{\alpha_k} + \mathcal{O}(dt)) + \sigma(dt),$$

where $\mathcal{O}(dt)$ is such that $\lim_{dt \to 0} \mathcal{O}(dt) = 0$ and $\sigma(dt)$ is such that $\lim_{dt \to 0} \sigma(dt)/dt = 0$. Taking the limit $dt \to 0$, provides:

$$\frac{\partial S_i(t, w)}{\partial t} \; + \; r_i \frac{\partial S_i(t, w)}{\partial w} = -a_i S_i(t, w) \; + \sum_{k \in \Omega, k \neq i} a_{ik} \; S_k(t, \frac{w}{\alpha_k}) \tag{6.30}$$

Taking the Laplace transform with respect to t ($\to s$)

$$s S_i^*(s, w) - S_i(0, w) \; + \; r_i \frac{\partial S_i^*(s, w)}{\partial w} = -a_i S_i^*(s, w) \; + \sum_{k \in \Omega, k \neq i} a_{ik} \; S_k^*(s, \frac{w}{\alpha_k}) \tag{6.31}$$

where $S_i(0, w) = 1$. Taking the Laplace transform with respect to w ($\to v$)

$$\begin{aligned} s S_i^{**}(s, v) \; - \; \frac{1}{v} \; + \; r_i v S_i^{**}(s, v) \; - \; S_i^*(s, 0) = \\ - \, a_i S_i^{**}(s, v) \; + \sum_{k \in \Omega, k \neq i} a_{ik} \alpha_k \; S_k^{**}(s, \alpha_k v) \end{aligned} \tag{6.32}$$

where $S_i^*(s, 0) = 0$, from which the theorem comes. □

6.2.3 Completion time

Define

$$F_i(t, w, x) = Pr(C(w) \leq t \mid Z(0) = i, B(0) = x) \; .$$

Considering an SMP environment the regenerative approach provides

$$F_i(t,w,\eta|\, H=\tau) \;=\; \begin{cases} U_t(t-\frac{w-x}{r_i}) & \text{if}: \tau > \frac{w-x}{r_i} \\ \\ \sum\limits_{k\in\Omega} \dfrac{dQ_{ik}(\tau)}{dQ_i(\tau)} \;\cdot\; F_k(t-\tau,w,(\eta+\tau r_i)\alpha_i) & \text{if}: \tau < \frac{w-x}{r_i} \end{cases} \tag{6.33}$$

Similar to eq. (6.28), Eq. (6.33) does not exhibit any closed form expression in transform domain.

Underlying CTMC:

Theorem 6.8. *When $Z(t)$ is a CTMC with generator $A=[a_{ij}]$, $F_i(t,w,x)$ satisfies the backward differential equation:*

$$\begin{aligned} &\textit{if } x<w: \\ &\frac{\partial F_i(t,w,x)}{\partial t} + r_i\frac{\partial F_i(t,w,x)}{\partial x} = -a_iF_i(t,w,x) + \sum_{k\in\Omega,k\neq i} a_{ik}\; F_k(t,w,\alpha_i x) \\ &\textit{if } x\geq w: \\ &F_i(t,w,x)=1 \end{aligned} \tag{6.34}$$

Proof. The backward argument describing the evolution of the process is:

$$\begin{aligned} F_i(t,w,x) = (1-a_idt)F_i(t-dt,w,x-r_idt)+ \\ \sum_{k\in\Omega,k\neq i} a_{ik}\; dt\; F_k(t-dt,w,\alpha_i x+\mathcal{O}(dt)) + \sigma(dt)\;, \end{aligned} \tag{6.35}$$

which proves the theorem. □

Equation (6.34) can be expresses in transform domain as follows:

$$\begin{aligned} &\text{if } x<w: \\ &r_i\frac{\partial F_i^*(s,w,x)}{\partial x} = -(s+a_i)F_i^*(s,w,x) + \sum_{k\in\Omega,k\neq i} a_{ik}\; F_k^*(s,w,\alpha_i x) \\ &\text{if } x\geq w: \\ &F_i^*(s,w,x)=\frac{1}{s} \end{aligned} \tag{6.36}$$

and

$$F_i^{**}(s,w,v) = \frac{r_i}{s+r_iv+a_i}F_i^*(s,w,0) + \sum_{k\in\Omega,k\neq i} \frac{a_{ik}}{\alpha_i(s+r_iv+a_i)}\; F_k^{**}(s,w,\frac{v}{\alpha_i})\;. \tag{6.37}$$

Unfortunately (6.37) can not be used for numerical analysis since $F_i^*(s,w,0)$ is not known.

The above set of results based on Markov renewal theory can easily be specialized for underlying renewal process and the set of results based on differential equation description can easily be specialized for underlying Poisson process. A telecommunication application of a particular partial reward loss model with underlying Poisson process is presented in [52], where the variation of TCP windows size is modeled and analyzed with the use of a partial loss reward model.

6.3 Numerical analysis techniques

Different numerical techniques can be applied to evaluate reward measures of partial loss reward models. The applicable numerical solution depend on the available analytical description of the considered measure.

Eq. (6.13) can be directly evaluated applying a numerical inverse transform method.

Expressions like (6.26) can be evaluated numerically using the following iterative procedure:

$$f_i^{*(0)}(v) = 1, \quad \forall i \in \Omega$$

and

$$f_j^{*(n+1)}(v) = \sum_{i \in \Omega, i \neq j} \frac{\alpha_i a_{ij}}{v r_i \alpha_i + a_i} f_i^{*(n)}(v\alpha_i) \ .$$

Alternatively, the same iterative approach can be applied for the "time domain" version of (6.26):

$$f_j(w) = \sum_{i \in \Omega, i \neq j} \frac{a_{ij}}{r_i \ \alpha_i} \int_{\tau=0}^{w} e^{-\frac{a_i}{r_i \ \alpha_i}(w-\tau)} f_i(\frac{\tau}{\alpha_i}) d\tau. \tag{6.38}$$

In this case, a convolution integral has to be evaluated numerically at each iteration step.

The moments of reward measures can be obtained based on double transform expressions like Eq. (6.29). E.g., the mean of the accumulated reward at time t, defined as $E_i = \mathrm{E}(B(t)|Z(0) = i)$, can be obtained by a symbolic inverse transform with respect to s, a symbolic derivation with respect to v, evaluating the limit $v \to 0$ and solving the obtained linear system. That is

$$S_i^{\sim}(t, v) = \ e^{-(r_i v + a_i)t} \ + \sum_{k \in \Omega, k \neq i} \alpha_k a_{ik} \ S_k^{\sim}(t, \alpha_k v)$$

$$E_i(t) = r_i \ t \ e^{-a_i t} \ + \sum_{k \in \Omega, k \neq i} \alpha_k^2 a_{ik} \ E_k(t)$$

Equations like (6.30) and (6.34) can be evaluated using numerical differential equation solvers.

Chapter 7

Numerical analysis of partial loss reward models and its application

The aim of this chapter is to present numerical methods to evaluate a particular class of partial loss models and to demonstrate the applicability of partial loss models in the analysis of computer systems executing long running batch programs with checkpointing.

As it is readable from the previous chapters, the analysis of partial reward loss models is more complex than the analysis of prs reward models. Both numerical analysis methods presented in this chapter use the analysis of prs reward models as an elementary step of the procedure. To keep the overall computational cost as low as possible we calculate only the moments of the accumulated reward and apply the effective method presented in Chapter 5 for the embedded calculation of prs models.

7.1 Numerical analysis of partial incremental loss models

The state dependent distribution of the accumulated reward is defined as

$$B_{ij}(t,w) = Pr(B(t) \leq w, Z(t) = j \mid Z(0) = i)$$

and $B(t,w) = \{B_{ij}(t,w)\}$.

The distribution of the accumulated reward is provided in the previous chapter:

$$B^{*\sim}(s,v) = (sI + vR_\alpha - A)^{-1} D(s,v) \tag{7.1}$$

where I is the identity matrix and the diagonal matrices R_α and $D(s,v)$ are defined as $R_\alpha = \text{diag}\langle r_i \alpha_i \rangle$ and $D(s,v) = \text{diag}\left\langle \frac{s + v r_i \alpha_i + a_i}{s + v r_i + a_i} \right\rangle$.

The partial loss models are the transition between the prs (no reward loss) and the prt (complete reward loss) reward models. The numerical methods that are commonly used for the analysis of the prs and the prt reward models utilize the special features of those models and cannot be applied directly for the analysis of partial loss models.

The behavior of the partial incremental loss model can be interpreted as follows. The reward accumulation between 0 and t^* is according to a traditional prs model with reduced reward rates $(\alpha_i r_i)$, and from time t^* the prs reward accumulation goes on with the original reward rates (r_i), where t^* $(0 \leq t^* < t)$ is the instant of the last state transition before t. If there is no state transition till time t, then $t^* = 0$. Unfortunately, t^* is a complex quantity (since it depends on the evolution of the CTMC over the whole $(0, t)$ interval) and it is hard to evaluate the partial loss models with effective numerical methods. The transform domain expression in eq. (7.1) reflects this model interpretation. Matrix $(sI + vR_\alpha - A)^{-1}$ describes the distribution of the reward accumulated by a prs Markov reward model with generator A and reward rates $\alpha_i r_i$ and the $D(s, v)$ diagonal matrix captures the effect of the "different" reward accumulations during the (t^*, t) interval.

As a consequence of this complex behavior the mean accumulated reward at time t cannot be evaluated based on the cumulative transient probabilities of the CTMC, as it was possible for the prs reward models.

To obtain a numerical procedure to evaluate the accumulated reward at time t, we inverse Laplace transform (7.1) with respect to the time variable $(s \to t)$. First we introduce

$$F(s, v) = \text{diag}\left\langle \frac{vr_i(1-\alpha_i)}{s + vr_i + a_i} \right\rangle,$$

and substitute $D(s, v)$ with $I - F(s, v)$ in (7.1). The inverse Laplace transform of $F(s, v)$ with respect to the time variable is

$$F(t, v) = \text{diag}\left\langle vr_i(1-\alpha_i)e^{-(vr_i+a_i)t} \right\rangle \ .$$

Using these matrices we can perform a symbolic inverse Laplace transformation of (7.1) which results in:

$$B^{\sim}(t, v) = e^{(-vR_\alpha + A)t} - \int_{\tau=0}^{t} e^{(-vR_\alpha + A)\tau} F(t-\tau, v) d\tau \tag{7.2}$$

The moments of the accumulated reward is obtained from (7.2) as

$$E(B^n(t)) = \underline{P}(0)\ (-1)^n\ \left.\frac{d^n}{dv^n} B^{\sim}(t, v)\right|_{v=0} \underline{h}\ ,$$

where $\underline{P}(0)$ is the initial probability vector and $\underline{h}$ is the column vector of ones. The nth derivative of $B^{\sim}(t, v)$ at $v = 0$ can be calculated as

$$\begin{aligned} \left.\frac{d^n}{dv^n} B^{\sim}(t, v)\right|_{v=0} &= \left.\frac{d^n}{dv^n}\ e^{(-vR_\alpha + A)t}\right|_{v=0} \\ &- \int_{\tau=0}^{t} \sum_{\ell=0}^{n} \binom{n}{\ell} \left.\frac{d^\ell}{dv^\ell}\ e^{(-vR_\alpha + A)\tau}\right|_{v=0} \left.\frac{d^{n-\ell}}{dv^{n-\ell}} F(t-\tau, v)\right|_{v=0} d\tau \end{aligned} \tag{7.3}$$

where the 0th derivative is the function itself. Since $F(\tau, v)$ is a diagonal matrix the ℓth derivative of $F(\tau, v)$ at $v = 0$ can be calculated in a computationally cheap way as

$$\left.\frac{d^\ell}{dv^\ell} F(\tau, v)\right|_{v=0} = \text{diag}\left\langle r_i(1-\alpha_i)\ \ell(-r_i\tau)^{\ell-1}\ e^{-a_i\tau} \right\rangle \ .$$

Two computationally expensive steps have to be performed to evaluate the nth derivative of $P^{\sim}(t,v)$ at $v=0$ based on (7.3). The first one is the calculation of the first n derivatives of $e^{(-vR_\alpha+A)\tau}$ at $v=0$ and at some time points $\tau \in (0,t]$, and the second one is the numerical integration with respect to τ. The numerical integration is not expensive itself, but it requires the calculation of the first step several times. The numerical method presented in [80] is an effective way of calculating the first n derivatives of $e^{(-vR_\alpha+A)\tau}$ at $v=0$, hence we use it for the calculation of the first step.

The complexity of the proposed numerical procedure is much higher than the analysis of the same Markov reward model without reward loss for two reasons. The first one is the mentioned numerical integration, and the second one is related to the complexity of the elementary steps of the computation of $d^n/dv^n\ e^{(-vR_\alpha+A)t}$. Basically, the first term in (7.3) provides the moments of the Markov reward model of the same CTMC with reduced reward rates $(\alpha_i r_i)$ and without reward loss. For the calculation of the moments it is enough to calculate only the row sum of the first term, $e^{(-vR_\alpha+A)t}$, since it is multiplied by $\underline{h}$ from the right. It is much faster to calculate the row sum of $e^{(-vR_\alpha+A)t}$ instead of the calculation of the whole matrix, because the row sum can be obtained by vector-matrix multiplications, while the calculation of the whole matrix requires matrix-matrix multiplications in each elementary step of the computation [80]. Unfortunately, the second term in (7.3) requires the calculation of the whole matrix (using matrix-matrix multiplications), because of the multiplication by the diagonal matrix $F(t-\tau,v)$ from the right. This is why we defined and calculated $P(t,w)$ as a matrix all along the above derivations.

Finally, we note that the product of two double transform functions in (7.1) results in double convolutions in the original (t,w) domain. In our approach one convolution is avoided due to the calculation of the moments of the accumulated reward. Since the calculation of the distribution of a prs Markov reward model is very expensive itself (it is much more expensive than to calculate its moments), a direct method to calculate the distribution of the accumulated reward by double numerical convolution becomes infeasible even for small models ($\sim$10 states). Instead, the numerical method for the analysis of the moments of the accumulated reward is applicable for models of $\sim$100 states.

7.2 Stationary analysis of accumulated reward

The previous sections provide a numerical method to calculate the moments of the accumulated reward of partial incremental loss models. Using that method the evaluation of partial loss reward models is computationally much more expensive than the calculation of the prs reward models of the same size.

In this section we provide an effective computational approach that makes possible to evaluate much larger partial incremental loss models ($\sim 10^6$ states). This numerical approach allows the analysis of a special class of partial loss models where the background process is in stationary state. Note that the reward accumulation of partial incremental loss models with stationary background process has non-stationary increment on the $(0,t)$ interval (e.g., $E(B(t)) \neq 2E(B(t/2))$), because the reward

accumulated in the last state may have different effects on the overall accumulated reward.

The main idea of the proposed method is to define an equivalent prs reward model, whose accumulated reward equals the reward accumulated by the original partial loss reward model, and to evaluate the accumulated reward of the equivalent model.

The reward accumulation process of a partial loss reward model can be divided into two main parts as it is mentioned above. During the $(0, t^*)$ interval the system accumulates reward at reduced reward rates $(\alpha_i r_i)$ (without reward loss), and during the (t^*, t) interval it accumulates at the original reward rate (r_i). If t^* (and $Z(t^*)$) was known it would be straightforward to calculate the accumulated reward, but t^* depends in a complex way on the CTMC behavior over the whole $(0, t)$ interval. t^* is not a stopping time.

To overcome this difficulty one can interpret the reward accumulation from time t towards time 0. In this case t^* is simply the time instant of the first state transition in the reverse CTMC, and the reverse reward model is such that it accumulates reward at the original rate (r_i) in its first state and it accumulates reward at the reduced rate $(\alpha_i r_i)$ after leaving the first state. To apply this approach we need the generator of the reverse CTMC.

The probability that the process is in state i at time t and in state j $(j \neq i)$ at $t + \Delta$, i.e., $Pr(Z(t) = i, Z(t + \Delta) = j)$, can be calculated as:

$$Pr(Z(t) = i)\ Pr(Z(t + \Delta) = j \mid Z(t) = i) =$$
$$Pr(Z(t + \Delta) = j)\ Pr(Z(t) = i \mid Z(t + \Delta) = j).$$

Dividing both sides by Δ and letting $\Delta \to 0$ we have

$$Pr(Z(t) = i)\ a_{ij} = Pr(Z(t) = j)\ \overleftarrow{a_{ji}}(t)\ ,$$

where $\overleftarrow{a_{ji}}(t)$ is the (j, i) element of the generator of the reverse CTMC. One can see that the generator of the reverse CTMC depends on the transient probabilities of the original CTMC, hence it is time inhomogeneous, in general. In the stationary case the state probabilities are constant and the generator of the reverse CTMC becomes time homogeneous:

$$\overleftarrow{a_{ji}} = \frac{\gamma_i}{\gamma_j} a_{ij}, \tag{7.4}$$

where γ_i is the stationary probability of state i in the original (as well as the reverse) CTMC. The stationary probabilities can be obtained solving $\sum_{i \in \Omega} \gamma_i a_{ij} = 0$ for $\forall j \in \Omega$ with the normalizing condition $\sum_{i \in \Omega} \gamma_i = 1$. The diagonal elements of the generator of the stationary reverse CTMC are the same as the original diagonal elements (since the reverse process spends the same time in each state as the original one). It is easy to check that matrix $\overleftarrow{A} = \{\overleftarrow{a_{ij}}\}$ defined by (7.4) is a proper generator matrix.

In case the original partial loss model starts from the stationary state, we can define an equivalent prs Markov reward model that accumulates the same amount of reward during the $(0, t)$ interval as our original partial loss model using the reverse

interpretation of the reward accumulation. The original partial loss model is defined by $(\underline{\gamma}, A, R, R_\alpha)$ (the initial probability vector – which is the stationary distribution of the CTMC, the generator matrix, the diagonal matrix of the reward rates, the diagonal matrix of the reduced reward rates). Based on this description we define an equivalent prs Markov reward model with state space of $2 \cdot \#\Omega$ states by initial probability vector $\underline{P'}(0)$, generator matrix A', and reward rate matrix R' as follows:

$$\underline{P'}(0) = \{\underline{\gamma}, 0\}, \quad A' = \begin{array}{|c|c|}\hline A_D & \overleftarrow{A} - A_D \\ \hline 0 & \overleftarrow{A} \\ \hline \end{array}, \quad R' = \begin{array}{|c|c|}\hline R & 0 \\ \hline 0 & R_\alpha \\ \hline \end{array}, \tag{7.5}$$

$A_D = \mathrm{diag}\langle a_{ii} \rangle$ is the diagonal matrix composed of the diagonal elements of A. Each state of the original CTMC is represented by two states in the equivalent prs Markov reward model. States 1 to $\#\Omega$ represent the reward accumulation with the original reward rate (r_i). The equivalent model starts from this set of states according to the stationary distribution $\underline{\gamma}$. States $\#\Omega+1$ to $2 \cdot \#\Omega$ represent the reward accumulation after the first state transition with the reduced reward rates. The structure of the A' matrix is such that the equivalent process moves from the first set of states (states 1 to $\#\Omega$) to the second one (states $\#\Omega + 1$ to $2 \cdot \#\Omega$) at the first state transition and remains there. The distribution of the reward accumulated during the $(0, t)$ interval by a prs Markov reward model with initial probability vector $\underline{P'}(0)$, generator matrix A', and reward rate matrix R' is (see e.g., [80])

$$\underline{P'}(0)(sI' + vR' - A')^{-1}\underline{h'} \tag{7.6}$$

where the cardinality of the identity matrix I' and summing vector $\underline{h'}$ is $2 \cdot \#\Omega$.

The formal relation of the original partial loss model and the reverse prs Markov reward model is presented in the following theorem.

Theorem 7.1. *The distribution of reward accumulated by the prs Markov reward model $(\underline{P'}(0), A', R')$ is identical with the distribution of reward accumulated by the partial incremental loss Markov reward model $(\underline{\gamma}, A, R, R_\alpha)$, that is (from eq. (7.1) and (7.6)):*

$$\underline{\gamma}(sI + vR_\alpha - A)^{-1}D(s, v)\underline{h} = \underline{P'}(0)(sI' + vR' - A')^{-1}\underline{h'} \tag{7.7}$$

Proof. The left hand side of eq. (7.7) can be rewritten as

$$\underline{\gamma}(sI + vR_\alpha - A)^{-1}D(s, v)\underline{h} = \underline{\gamma}(sI + vR_\alpha - A)^{-1}(sI + vR_\alpha - A_D)\ (sI + vR - A_D)^{-1}\underline{h} \tag{7.8}$$

For the evaluation of the right hand side of eq. (7.7), we use the partitioned form of matrices I', R', A'. That is

$$(sI' + vR' - A') = \begin{array}{|c|c|}\hline sI + vR - A_D & -\overleftarrow{A} + A_D \\ \hline 0 & sI + vR_\alpha - \overleftarrow{A} \\ \hline \end{array}, \tag{7.9}$$

and

$$(sI' + vR' - A')^{-1} =$$

$$\begin{array}{|c|c|}\hline (sI + vR - A_D)^{-1} & (sI + vR - A_D)^{-1}(\overleftarrow{A} - A_D)(sI + vR_\alpha - \overleftarrow{A})^{-1} \\ \hline 0 & (sI + vR_\alpha - \overleftarrow{A})^{-1} \\ \hline \end{array} . \tag{7.10}$$

Using the special structure of the initial vector $\underline{P'}(0)$ we have:

$$\begin{aligned} &\underline{P'}(0)(sI' + vR' - A')^{-1}\underline{h'} = \\ &\underline{\gamma}(sI + vR - A_D)^{-1}\left[I + (\overleftarrow{A} - A_D)(sI + vR_\alpha - \overleftarrow{A})^{-1}\right]\underline{h} = \\ &\underline{\gamma}(sI + vR - A_D)^{-1} \\ &\quad \left[(sI + vR_\alpha - \overleftarrow{A})(sI + vR_\alpha - \overleftarrow{A})^{-1} + (\overleftarrow{A} - A_D)(sI + vR_\alpha - \overleftarrow{A})^{-1}\right]\underline{h} = \\ &\underline{\gamma}(sI + vR - A_D)^{-1}(sI + vR_\alpha - A_D)\ (sI + vR_\alpha - \overleftarrow{A})^{-1}\underline{h} = \end{aligned} \tag{7.11}$$

Let Γ be the diagonal matrix of the stationary probabilities, i.e., $\Gamma = \mathrm{diag}\langle\gamma_i\rangle$. Using this diagonal matrix $\underline{\gamma} = \underline{h}^T\ \Gamma$ and from eq. (7.4) $\overleftarrow{A} = \Gamma^{-1}A^T\Gamma$, where $\underline{h}^T$ is the row vector of ones. In the following steps the diagonal matrices Γ, R_α, $(sI + vR - A_D)$ and $(sI + vR_\alpha - A_D)$ are commuted if necessary:

$$\begin{aligned} &\underline{h}^T\ \Gamma\ (sI + vR - A_D)^{-1}(sI + vR_\alpha - A_D)\ (sI + vR_\alpha - \Gamma^{-1}A^T\Gamma)^{-1}\underline{h} = \\ &\underline{h}^T\left[\left((sI + vR_\alpha - \Gamma^{-1}A^T\Gamma)^{-1}\right)^T\left(\Gamma\ (sI + vR - A_D)^{-1}(sI + vR_\alpha - A_D)\right)^T\right]^T\underline{h} = \ldots \end{aligned}$$

The external transpose vanishes due to the multiplication by $\underline{h}^T$ from left and $\underline{h}$ from right and the second internal transpose also vanishes because it contains a diagonal matrix. In the first internal transpose we interchange the order of transpose and inversion:

$$\begin{aligned} &\underline{h}^T\left((sI + vR_\alpha - \Gamma^{-1}A^T\Gamma)^T\right)^{-1}\ \Gamma\ (sI + vR - A_D)^{-1}(sI + vR_\alpha - A_D)\ \underline{h} = \\ &\underline{h}^T\left(sI + vR_\alpha - \Gamma\ A\Gamma^{-1}\right)^{-1}\left(\Gamma^{-1}\right)^{-1}\ (sI + vR - A_D)^{-1}(sI + vR_\alpha - A_D)\ \underline{h} = \\ &\underline{h}^T\left(s\Gamma^{-1} + v\Gamma^{-1}R_\alpha - A\Gamma^{-1}\right)^{-1}\ (sI + vR - A_D)^{-1}(sI + vR_\alpha - A_D)\ \underline{h} = \\ &\underline{h}^T\ \Gamma\ (sI + vR_\alpha - A)^{-1}(sI + vR_\alpha - A_D)\ (sI + vR - A_D)^{-1}\ \underline{h} \end{aligned} \tag{7.12}$$

The theorem is given by the equivalence of (7.8) and (7.12).

□

The equivalent reward model is a prs Markov reward model. Its analysis can be performed with effective numerical methods available in the literature. E.g., the distribution of the accumulated reward can be calculated using [62, 27, 28] and its moments using [80].

It is easy to evaluate the limiting behavior of a partial loss model with stationary background CTMC. We use the following notation. $B(t)$ is the reward accumulated by a stationary partial incremental loss model defined by (A, R, R_α). $B'(t)$ and $B"(t)$ are the rewards accumulated by stationary prs reward models defined by (A, R) and (A, R_α), respectively. The stationary distribution of the CTMC with generator A is γ. For short time intervals the loss at the first transition does not play role, hence

$$\lim_{t\to 0} B(t)/t \equiv \lim_{t\to 0} B'(t)/t,$$

and for very long intervals the reward accumulated from the last state transition to the end of the interval is negligible with respect to the total accumulated reward

$$\lim_{t\to \infty} B(t)/t \equiv \lim_{t\to \infty} B"(t)/t$$

E.g., the limiting behavior of the mean accumulated reward can be calculated as

$$\begin{aligned} \lim_{t\to 0} \frac{E(B(t))}{t} &= \lim_{t\to 0} \frac{E(B'(t))}{t} = \sum_{i\in S} \gamma_i r_i \ , \\ \lim_{t\to \infty} \frac{E(B(t))}{t} &= \lim_{t\to \infty} \frac{E(B"(t))}{t} = \sum_{i\in S} \gamma_i \alpha_i r_i \ . \end{aligned} \tag{7.13}$$

7.3 Performance analysis of computer systems with checkpointing

Checkpointing is a widely applied technique to improve the performance of computing servers executing long running batch programs in the presence of failures [20, 19, 31, 51]. Long running batch programs need to be re-executed in case of a system failure. To reduce the extra re-execution work of the system the actual state of the program is saved occasionally during the operational time of the system. This saved program state is used when a failure occurs. After a failure and the subsequent repair the saved program state is reloaded and the program is re-executed from its saved state. The operation of saving the current state of the program is referred to as checkpointing and the reload of the saved program state is called rollback.

It is a common feature of all checkpointing models that a portion of work executed since the last system failure is lost at the next system failure, hence the amount of executed work can be analyzed using partial loss models. To find the relation between the applied checkpointing policy and the parameters of the partial loss reward model depends on the particular system behaviour. Here, we follow a system level approach, which means that the parameters of the partial loss model of the analyzed computing server are assumed to be known. However, some considerations on the behavior of the analyzed system are provided below.

It is important to note that our analysis approach contains a simplifying assumption. The portion of work lost at a system failure is a random quantity. The analysis of partial loss reward models with random loss ratio is studied in [11], but unfortunately, there is no effective numerical method available for their analysis. This is the reason for using (state dependent) deterministic loss ratio in our model.

The overall performance analysis of computing systems with checkpointing is composed of two major steps:

I. Generation of partial loss Markov reward model based on the system behavior:

- characterize the state space of the model based on the system load and the failure process.
- evaluate the failure rate and computing power assigned to the jobs under execution in each system state r_i.
- calculate the (optimal) checkpointing rate in each system state.
- calculate the state dependent loss ratio (the portion of work that needs to be re-executed), based on the failure rate and the checkpointing rate.

II. Solution of the obtained partial loss Markov reward model.

In the following numerical example we utilize the result of step I. and perform step II.

Consider a computing server executing long running batch programs. Jobs of two classes arrive to the server. Class 1 (class 2) jobs arrive according to a Poisson process with rate λ_1 (λ_2). Each of these jobs requires an exponentially distributed execution time with parameter μ_1 (μ_2) with the full computing capacity of the server. The server has finite capacity (N_{MAX}) and the number of class 1 (class 2) jobs cannot exceed N_1 (N_2), i.e., $n_1 \leq N_1$, $n_2 \leq N_2$, $n_1+n_2 \leq N_{MAX}$, where n_1 (n_2) is the number of class 1 (class 2) jobs in the system. The failure rate is load dependent: $\nu(n_1, n_2) = \omega_a + \omega_b(n_1 + n_2)$, where ω_a and ω_b are the parameters of the load independent and load dependent parts of the failure rate, respectively. The repair time, including the rollback time, is exponentially distributed as well. We use state independent repair rate β. (Note that the applied modeling approach can handle state dependent repair rates with the same computational complexity.) Job arrival is also allowed during repair. The computing performance of the server slightly decreases with the number of jobs under execution (e.g., due to the swapping of jobs). r_a ($0 \leq r_a \leq 1$, $r_a \sim 1$) is the portion of the computing power that is utilized for job execution when there is only one job in the server. Suppose the presence of class 1 jobs increases the checkpointing rate, the portion of useful work maintained at a system failure increases with the number of class 1 jobs. α_a and α_b are used to represent the load independent and load dependent part of the useful work ratio, respectively.

Having these Markovian assumptions one can easily model a wide range of service discipline schemes. We consider weighted processor sharing with state dependent weights. Our service discipline assigns a predefined portion of the computing power, ϕ_1 ($0 < \phi_1 < 1$) and $\phi_2 = 1 - \phi_1$, to jobs of class 1 and class 2, respectively. Jobs of the same class are executed at the same speed. If there are only jobs of one class in the system, the whole computing capacity will be utilized by that class. As a special case of this service discipline we obtain the preemptive priority service discipline when ϕ_2 tends to 0 or 1. In the case when $\phi_2 \to 0$ class 1 jobs are executed with the whole computing power of the server as long as there are class 1 jobs in the system.

Based on this system behavior the performance of the considered computing system is analyzed using the partial loss Markov reward model defined in Table 7.1. The

State space description	
$n_1 : 0$ **To** N_1	#class 1 jobs
$n_2 : 0$ **To** N_2	#class 2 jobs
$\{Good,\ To\_fail,\ Repair\}$	operational condition
$n_1 + n_2 \leq N_{MAX}$	

Underlying CTMC	
$(n_1, n_2, Good) \rightarrow (n_1 + 1, n_2, Good) = p\ \lambda_1$	class 1 job arrival
$(n_1, n_2, Good) \rightarrow (n_1 + 1, n_2, To\_fail) = q\ \lambda_1$	
$(n_1, n_2, Repair) \rightarrow (n_1 + 1, n_2, Repair) = \lambda_1$	
$(n_1, n_2, Good) \rightarrow (n_1, n_2 + 1, Good) = p\ \lambda_2$	class 2 job arrival
$(n_1, n_2, Good) \rightarrow (n_1, n_2 + 1, To\_fail) = q\ \lambda_2$	
$(n_1, n_2, Repair) \rightarrow (n_1, n_2 + 1, Repair) = \lambda_2$	
$(n_1, n_2, Good) \rightarrow (n_1 - 1, n_2, Good) = p\ \frac{\phi_1 n_1}{\phi_1 n_1 + \phi_2 n_2}\mu_1$	class 1 job departure
$(n_1, n_2, Good) \rightarrow (n_1 - 1, n_2, To\_fail) = q\ \frac{\phi_1 n_1}{\phi_1 n_1 + \phi_2 n_2}\mu_1$	
$(n_1, n_2, Good) \rightarrow (n_1, n_2 - 1, Good) = p\ \frac{\phi_2 n_2}{\phi_1 n_1 + \phi_2 n_2}\mu_2$	class 2 job departure
$(n_1, n_2, Good) \rightarrow (n_1, n_2 - 1, To\_fail) = q\ \frac{\phi_2 n_2}{\phi_1 n_1 + \phi_2 n_2}\mu_2$	
$(n_1, n_2, To\_fail) \rightarrow (n_1, n_2, Repair) = \omega_a + \omega_b(n_1 + n_2)$	failure
$(n_1, n_2, Repair) \rightarrow (n_1, n_2, Good) = p\ \beta$	repair
$(n_1, n_2, Repair) \rightarrow (n_1, n_2, To\_fail) = q\ \beta$	

Reward and loss structure	
$r(n_1, n_2, Good) = r_a^{n_1+n_2}$ if: $n_1 + n_2 > 0$	reward rate
$r(0, 0, Good) = 0$	
$r(n_1, n_2, To\_fail) = r_a^{n_1+n_2}$ if: $n_1 + n_2 > 0$	
$r(0, 0, To\_fail) = 0$	
$r(n_1, n_2, Repair) = 0$	
$\alpha(n_1, n_2, Good) = 1$	useful work ratio
$\alpha(n_1, n_2, To\_fail) = \alpha_a + \alpha_b \frac{n_1}{n_1 + n_2}$ if: $n_1 + n_2 > 0$	
$\alpha(0, 0, To\_fail) = 0$	
$\alpha(n_1, n_2, Repair) = 0$	

Table 7.1: The partial loss Markov reward model of the computing system

state space of the CTMC is characterized by the number of class 1 and class 2 jobs in the system and the operational condition of the system. The operational condition can be one of the following three: *Good*, *To_fail* and *Repair*. We need to distinguish between the operational states that are followed by another operational state (*Good*) and the operational states that are followed by a failure (*To_fail*), because there is no work loss at the departure from a *Good* state while there is some work loss at the departure from a *To_fail* state. The probability of moving to the *Good* and *To_fail* condition (i.e., p and q, respectively) are calculated based on the number of jobs in the *destination* state. For $0 < n_1 + n_2 < N_{MAX}$ & $n_1 < N_1$ & $n_2 < N_2$:

$$q = 1 - p = \frac{\omega_a + \omega_b(n_1 + n_2)}{\lambda_1 + \lambda_2 + \dfrac{\phi_1 n_1 \mu_1}{\phi_1 n_1 + \phi_2 n_2} + \dfrac{\phi_2 n_2 \mu_2}{\phi_1 n_1 + \phi_2 n_2} + \omega_a + \omega_b(n_1 + n_2)} ,$$

for $n_1 = n_2 = 0$:

$$q = 1 - p = \frac{\omega_a}{\lambda_1 + \lambda_2 + \omega_a} ,$$

for $n_1 + n_2 = N_{MAX}$ or $n_1 < N_1$ & $n_2 < N_2$:

$$q = 1 - p = \frac{\omega_a + \omega_b(n_1 + n_2)}{\dfrac{\phi_1 n_1 \mu_1}{\phi_1 n_1 + \phi_2 n_2} + \dfrac{\phi_2 n_2 \mu_2}{\phi_1 n_1 + \phi_2 n_2} + \omega_a + \omega_b(n_1 + n_2)} ,$$

for $n_1 + n_2 < N_{MAX}$ and $n_1 = N_1$ & $n_2 < N_2$:

$$q = 1 - p = \frac{\omega_a + \omega_b(n_1 + n_2)}{\lambda_2 + \dfrac{\phi_1 n_1 \mu_1}{\phi_1 n_1 + \phi_2 n_2} + \dfrac{\phi_2 n_2 \mu_2}{\phi_1 n_1 + \phi_2 n_2} + \omega_a + \omega_b(n_1 + n_2)} ,$$

and for $n_1 + n_2 < N_{MAX}$ and $n_1 < N_1$ & $n_2 = N_2$:

$$q = 1 - p = \frac{\omega_a + \omega_b(n_1 + n_2)}{\lambda_1 + \dfrac{\phi_1 n_1 \mu_1}{\phi_1 n_1 + \phi_2 n_2} + \dfrac{\phi_2 n_2 \mu_2}{\phi_1 n_1 + \phi_2 n_2} + \omega_a + \omega_b(n_1 + n_2)} .$$

The following set of system parameters were used for the numerical evaluation:

- state space: $N_1 = 3, N_2 = 4, N_{MAX} = 6$;
- job arrival and computing requirement [1/hours]: $\lambda_1 = 0.4, \lambda_2 = 0.4, \mu_1 = 2, \mu_2 = 1$;
- resource sharing between class 1 and class 2 jobs: $\phi_1 = 2/3, \phi_2 = 1/3$;
- failure and repair parameters [1/hours]: $\omega_a = 0.3, \omega_b = 0.03, \beta = 2$;
- overhead parameter: $r_a = 0.98$;
- work loss parameters: $\alpha_a = 0.6, \alpha_b = 0.05$.

The system performance was evaluated with two initial probability distributions (Figure 7.1 and 7.2). In the first case the system starts from stationary state, and in the second case the system starts from state $(0, 0, \mathit{Good})$ with probability 1. The case when the system starts from state $(0, 0, \mathit{Good})$ was evaluated by the method presented in Section 7.1 and the case of stationary background CTMC was evaluated with both methods (Section 7.1 and 7.2). The accuracy of the prs reward analysis method, which is applied in both cases, was 10^{-6}. The numerical integration of the first method was computed over 100 equidistant points. The numerical results obtained for the stationary case were practically identical, hence there are only two curves depicted in Figure 7.1 and 7.2.

Based on the stationary analysis of the prs Markov reward model with reduced reward rates, (A, R_α), (7.13); we have $\lim_{t\to\infty} E(B(t))/t = 0.4718$, and $\lim_{t\to\infty} Var(B(t))/t = 0.0548$. Each pair of mean and variance curves in Figure 7.1 and 7.2 tends to the respective limit. The mean curve associated with the stationary background process starts from the stationary accumulation rate of the prs Markov reward model with original reward rates, (A, R), (7.13).

The detailed analysis of a slightly larger partial loss Markov reward model of the same example with stationary initial distribution and with $N_1 = 10, N_2 = 20, N_{MAX} = \infty, \lambda_1 = 0.5, \lambda_2 = 0.5$ results in the curves in Figure 7.3 and 7.4. It can be seen that the transition from the initial to the final $E(B(t))/t$ value takes place between 0.1 and 10 hours, and the $Var(B(t))/t$ curve has a peak in this range. That is the range where the effect of the reward loss at the first state transition turns up. The peak of the $Var(B(t))/t$ curve is sharper for the small system.

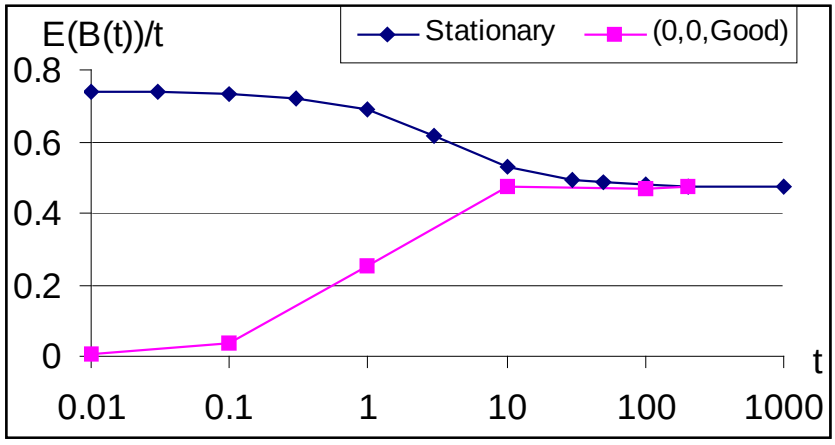


Figure 7.1: Mean of computing system performance (57 state model)

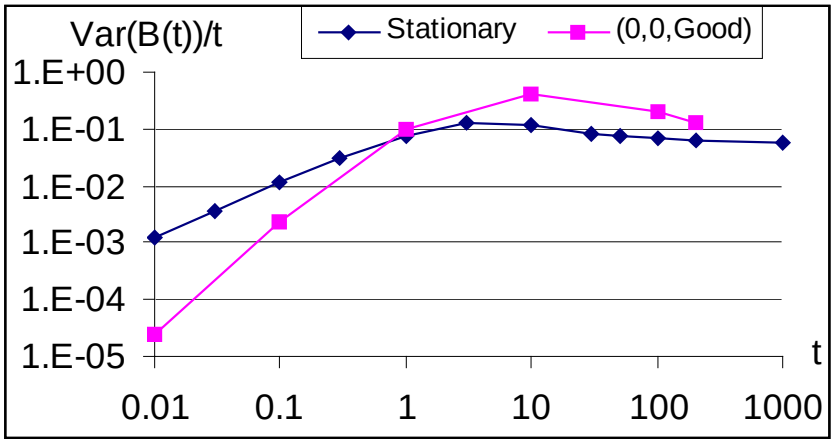


Figure 7.2: Variance of computing system performance (57 state model)

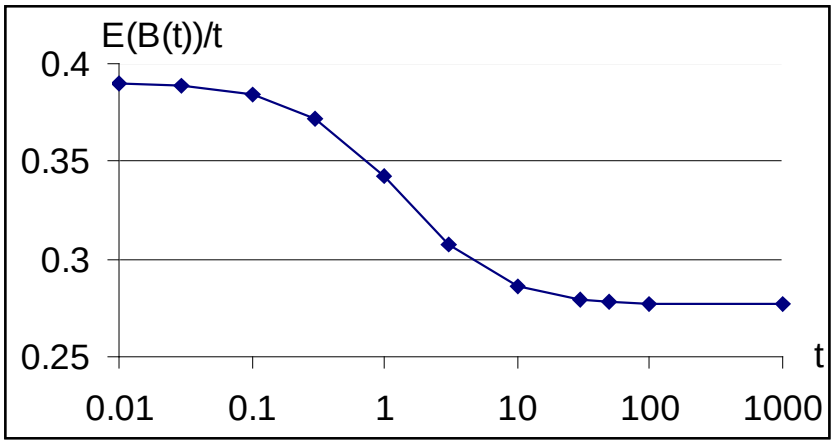


Figure 7.3: Mean of computing system performance (1386 state model)

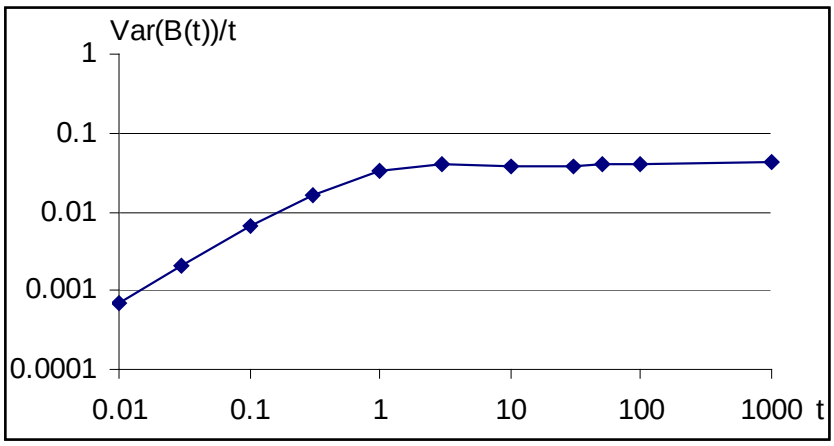


Figure 7.4: Variance of computing system performance (1386 state model)

Part II

State space techniques

Chapter 8

Introduction to state space techniques

State based techniques are developed for extending the applicability of discrete state Markovian methods. For a long period of time the only computable stochastic processes were Markov processes. This fact quite often influenced the system modeling efforts. The random event time of real systems were assumed to be exponentially distributed to obtain a computable stochastic model. A large variety of model description languages and associated analysis tools were developed to assist this "Markovian" modeling and associated analysis effort. Typical examples are generalized stochastic Perti nets [1], stochastic activity network [74], queueing network based descriptions [6, 73], reliability block diagram [73], fault tree [73], task graph [73], stochastic process algebra [41], etc.

The need to extend the applicability of Markovian methods for more complex systems is also old. At the beginning simple structures of, so called, exponential phases were used to represent distributions with low ("series" sequence of identical exponential phases) or high coefficient of variation ("parallel" exponential phases), e.g., in [47], but the lack of effective numerical techniques for the analysis of Markov chains with regular structure prevented the wide use of this approach. In the 70's, this approach was commonly considered to be useless in practice [47].

With this respect the development of matrix geometric methods resulted in a major break through [65]. These methods provide simple numerical procedures for the analysis of infinite or very large finite Markov chains with regular block structure, such as the quasi-birth-death, the M/G/1 type and the G/M/1 type structure. Basically matrix geometric methods make possible the analyzes of queueing systems with "Markovian" input and/or service process. The availability of matrix geometric methods speed up the research for approximate "Markovian" description of non-Markovian stochastic systems. The common root of all Markovian modeling approach is to create a Markov chain over a possible artificial and expanded state space whose state represents all relevant information about the future evolution of the process. This Markov chain is often referred to as background process or modulating process.

Two main Markovian approximation problems were studied intensively: Markovian approximation of non-negative distributions and Markovian approximation of

point processes. The class of distributions created by Markovian models is referred to as *phase type* distributions, and the most flexible class of point processes created by Markovian models is referred to *batch Markovian arrival process* (BMAP).

Due to the fact that the performance models of real systems (like computer systems, and call level behaviour of communication systems) were almost always continuous time models, continuous time Markovian models were applied in practice till the 90's. The introduction of fix size packet or cell based communication systems (like ATM) arose the need for discrete time Markovian models.

The need for discrete time Markovian models also arose in other research fields. Continuous time phase type distributions and associated state space expansion techniques were used for approximate analysis of non-Markovian systems since a long time. The applicability of continuous time phase type (CPH) distributions is limited when stochastic timing with very low coefficient of variation or deterministic timing has to be captured, because the coefficient of variation of the CPH class is lower bounded. The fact that the deterministic distribution is a member of the class of discrete time phase type (DPH) distributions suggested that discrete time Markovian approximation might be more adequate in some of these cases. The next sections present important contributions along this research line.

It was known that the coefficient of variation of a DPH distribution can be as low as 0, but the dependence of the minimal coefficient of variation on the other parameters of the DPH distribution, which characterize the practical applicability of these distributions, was not studied before. The lower bound of the coefficient of variation of DPH distributions is provided in the next section and the consequences of this lower bound on the applicability of approximate discrete time Markovian model for the analysis of continuous time non-Markovian processes is presented in Section 10. The constraints of the PH structure of order 2 on the higher moments is studied in Section 11

Finally, Section 12 presents a Markovian solution to overcome a very old limitation of queueing network models. Queueing network models gain popularity in performance analysis of real computer and communication systems due to the effective exact and approximate analysis techniques (based on product form assumption) available for the evaluation of these models. In traditional queueing network models the inter-node traffic is described with a single traffic intensity parameter (which might be state dependent). This poor traffic description does not allow to capture any detailed inter-node traffic parameter, e.g., the correlation structure of the inter-node traffic, which can be a source of significant error. The use of Markovian inter-node traffic description allows to capture more complex traffic parameters as well. Staring from this main idea the proposed solution combines a wide range of Markovian methods to build a complex analyzes approach out of pieces like phase type fitting, matrix geometric approach, MAP based traffic description and MAP/PH/1(/K) queue analysis.

Chapter 9

The minimal coefficient of variation of discrete phase type distributions

Discrete Phase Type (DPH) distributions have been known since 1975 [64], but they have received less attention than Continuous Phase Type *(CPH)* distributions because continuous time models were more popular in stochastic modeling. Recent attention toward discrete time stochastic models initiated new research on DPH distributions. This chapter provides the lower bound of the coefficient of variation of discrete phase type distributions that is a discrete time counterpart of the essential result by Aldous and Shepp [3].

9.1 Model description and notations

Let $\boldsymbol{X} = \{X_k, k = 0, 1, \ldots\}$ be a time-homogeneous discrete-time Markov chain (DTMC) over $\Omega = \{0, 1, \ldots, N\}$, with N transients states, $\{1, \ldots, N\}$, and an absorbing one, 0. The state transition probability matrix of $\boldsymbol{X}$ is $\boldsymbol{\Pi} = \{\pi_{ij}\}$. The unconditional and the conditional time to absorption is denoted as

$$\tau = \min\{k : X_k = 0\}$$

and

$$\tau_i = \min\{k : X_k = 0 \mid X_0 = i\}.$$

Let $\mu = \mathbf{E}[\tau]$ and $G(i) = \mathbf{E}[\tau_i]$. Without loss of generality we assume that (the states are numbered such that)

$$0 = G(0) < 1 \leq G(1) \leq G(2) \leq \ldots \leq G(N). \tag{9.1}$$

$G(i)$ satisfies

$$G(i) = 1 + \sum_{j \in \Omega} \pi_{ij} G(j) \ . \tag{9.2}$$

The initial distribution of $\boldsymbol{X}$ is given by $\underline{p} = \{p_i\}$ where $p_i = \Pr(X_0 = i)$. τ is a DPH distribution of order N with mean

$$\mu = \sum_{i \in \Omega} p_i G(i). \tag{9.3}$$

Further more, $\lfloor x \rfloor$ and $\langle x \rangle$ denote the integer and fraction part of x, respectively, i.e., $x = \lfloor x \rfloor + \langle x \rangle$, such that $\lfloor x \rfloor$ is an integer and $0 \leq \langle x \rangle < 1$.

9.2 Problem formulation

A significant difference between the minimal coefficient of variation of the DPH and the CPH class can be observed in the following example.

The simplest DPH distribution, the DPH of order 1, i.e., the geometric distribution with pmf $Pr(\tau = k) = (1 - \pi_{10})^{k-1}\pi_{10}$ has the following properties:

$$\mu = G(1) = \mathbf{E}[\tau_1] = 1/\pi_{10},$$

and

$$cv^2(\tau_1) = 1 - \pi_{10} = 1 - 1/G(1) = 1 - 1/\mu.$$

That is, in contrast with the CPH class, the minimal coefficient of variation of DPH distributions is a function of its mean. Hence in the DPH case the following constrained minimizations have to be solved:

$$\min_{\underline{p},\mathbf{\Pi}}\{cv^2(\tau) \mid \mathbf{E}[\tau]\} \quad \text{and} \quad \min_{\mathbf{\Pi}}\{cv^2(\tau_i) \mid \mathbf{E}[\tau_i]\}$$

where τ, τ_i $(i \in \Omega)$ and $\mathbf{\Pi}$ are related through (9.2) and (9.3). Note also that the states are numbered according to (9.1) which plays an important role in the initial state dependent cases.

9.3 The minimal coefficient of variation of DPH distributions

Theorem 9.1. *The squared coefficient of variation of τ, $cv^2(\tau)$, satisfies the inequality:*

$$cv^2(\tau) \geq \begin{cases} \dfrac{\langle\mu\rangle(1 - \langle\mu\rangle)}{\mu^2} & \text{if } \mu < N\ , \\ \dfrac{1}{N} - \dfrac{1}{\mu} & \text{if } \mu \geq N\ . \end{cases} \tag{9.4}$$

- *a DPH distribution which satisfies the equality if $\mu \leq N$ is the following: the nonzero initial probabilities are $p_{N-\lfloor\mu\rfloor} = \langle\mu\rangle$, $p_{N-\lfloor\mu\rfloor+1} = 1 - \langle\mu\rangle$ and the transition probabilities are $\Pr(X_1 = i - 1 | X_0 = i) = 1, \forall i \in \Omega$ (Fig. 9.1),*
- *the only DPH distribution which satisfies the equality if $\mu > N$ is the following: the nonzero initial probability is $p_N = 1$ and the transition probabilities are $\Pr(X_1 = i - 1 | X_0 = i) = N/\mu, \Pr(X_1 = i | X_0 = i) = 1 - N/\mu, \forall i \in \Omega$ (discrete Erlang(N) distribution) (Fig. 9.2).*

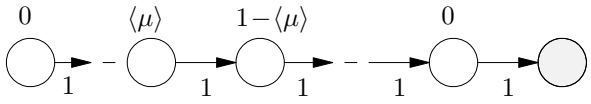


Figure 9.1: A DPH structure with minimal coefficient of variation, when $\mu \leq N$.

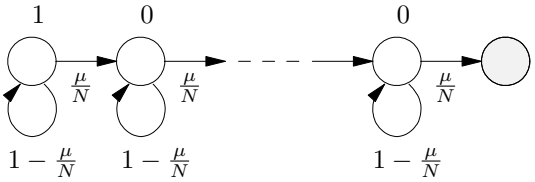


Figure 9.2: The only DPH structure with minimal coefficient of variation, when $\mu > N$.

To prove the theorem we need the following lemmas.

Lemma 9.2. *The minimal coefficient of variation of τ_i does not increase when n extra states, $i_1^*, i_2^*, \ldots, i_n^*$, with arbitrary mean time to absorption, $G(i_1^*), \ldots, G(i_n^*) \geq 1$, is added to the DPH structure.*

Proof of Lemma 9.2:

We prove the lemma by providing a DPH structure of $N+n$ phases that has the same coefficient of variation as the minimal coefficient of variation of τ_i without the extra states (Figure 9.3):

- the state transition probabilities between the original states are the same as in the DPH that provides the minimal coefficient of variation of τ_i without the extra states
- from each extra state the non-zero transition probabilities are $\pi_{i_j^*,0} = 1/G(i_j^*)$ and $\pi_{i_j^*,i_j^*} = 1 - 1/G(i_j^*)$

Lemma 9.3. *The minimal squared coefficient of variation of τ_i is as follows:*

$$cv^2(\tau_i) \geq \begin{cases} \dfrac{\langle G(i)\rangle(1-\langle G(i)\rangle)}{G^2(i)} & \text{if } G(i) < i \ , \\ \dfrac{1}{i} - \dfrac{1}{G(i)} & \text{if } G(i) \geq i \ . \end{cases} \tag{9.5}$$

Note that Lemma 9.3 is valid for all $i \in \Omega$ where the states are numbered according to (9.1).

Proof of Lemma 9.3:

The proof of Lemma 9.3 is composed by giving two lower bounds on the variance. According to our interpretation the first one is closely related to the degree

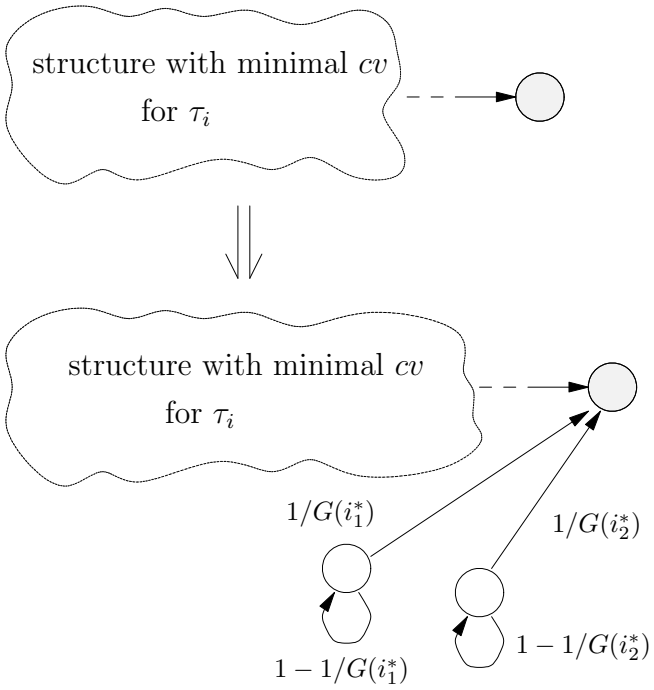


Figure 9.3: Adding extra states to the minimal structure

of the considered DPH distribution while the second one is related to the structural properties of the DPH class. We refer to the bounds based on this classification. (A short explanation of these properties is provided after the proof.) The lower bound of Lemma 9.3, Eq. (9.5), is obtained as the larger of the two bounds.

Bound of variance related to the degree of DPH distributions

A lower bound of the variance of the DPH distributions can be obtained by applying the elegant martingale approach proposed by Aldous and Shepp [3]. The discrete time stochastic process $\boldsymbol{Y} = \{Y_k, k = 0, 1, \ldots\}$ is defined as

$$Y_k = G(X_k) + \min(k, \tau_i) - G(X_0) \tag{9.6}$$

assuming $X_0 = i$.
$\boldsymbol{Y}$ is a martingale since,

- if $k \geq \tau_i$ then $Y_{k+1} = Y_k = \tau_i - G(i)$ is constant; and
- if $k < \tau_i$ then

$$\mathbf{E}[(Y_{k+1}|X_k)] = \mathbf{E}[(G(X_{k+1})|X_k)] + \min(k+1, \tau_i) - G(i) =$$

$$\sum_{j \in \Omega} \pi_{X_k,j} G(j) + \min(k+1, \tau_i) - G(i) =$$

$$G(X_k) - 1 + (k+1) - G(i) = G(X_k) + k - G(i) \ ,$$

where (9.2) has been applied in the second line.

By the definition of $\boldsymbol{Y}$, since $G(X_{\tau_i}) = 0$, we have

$$Y_{\tau_i} = \tau_i - G(i) \tag{9.7}$$

and

$$\mathbf{E}\big[Y_{\tau_i}^2\big] = \mathrm{var}(\tau_i) \ . \tag{9.8}$$

For $k \leq \tau_i$, using martingale properties, we have:

$$Y_k^2 = \sum_{i=1}^{k}(Y_i^2 - Y_{i-1}^2) = \sum_{i=1}^{k}(Y_i - Y_{i-1})^2 = \sum_{i=1}^{k}(G(X_i) - G(X_{i-1}) + 1)^2 =$$

$$\sum_{i=1}^{k}(G(X_i) - G(X_{i-1}))^2 + 2\ \sum_{i=1}^{k}(G(X_i) - G(X_{i-1})) + k$$

We define $S_k = \sum_{s=1}^{k}(G(X_s) - G(X_{s-1}))^2$, which gives

$$Y_k^2 = S_k + 2(G(X_k) - G(i)) + k \ .$$

Note that $G(X_s) - G(X_{s-1})$ takes non-zero values only at state transitions.
For $k = \tau_i$

$$Y_{\tau_i}^2 = S_{\tau_i} + 2(G(X_{\tau_i}) - G(i)) + \tau_i = S_{\tau_i} - 2G(i) + \tau_i \ ,$$

and

$$S_{\tau_i} = \sum_{s:X_s \neq X_{s-1}} (G(X_s) - G(X_{s-1}))^2 \geq \sum_{j=1}^{i}(G(j) - G(j-1))^2$$

$$\geq \frac{1}{i}\left(\sum_{j=1}^{i} G(j) - G(j-1)\right)^2 = \frac{1}{i}\ G^2(i) \ .$$

The first inequality says that the sequential path $(i, i-1, \ldots, 1, 0)$ results in the minimal squared differences and the second is Schwarz's inequality. Hence,

$$\mathbf{E}\big[Y_{\tau_i}^2\big] = \mathrm{var}(\tau_i) = \mathbf{E}[S_{\tau_i}] - 2G(i) + \mathbf{E}[\tau_i] = \mathbf{E}[S_{\tau_i}] - G(i) \geq \frac{1}{i}G^2(i) - G(i) \ , \tag{9.9}$$

results in a lower bound on the variance, that is:

$$cv^2(\tau_i) \geq \frac{1}{i} - \frac{1}{G(i)} \ . \tag{9.10}$$

Bound of variance provided by the structure of DPH distributions

An other lower bound on the variance of τ_i is obtained below by considering the structural properties of the DPH class and Lemma 9.2.

To simplify the notation we define

$$D(i) = \mathbf{E}[S_{\tau_i}] = \mathbf{E}\left[\sum_{s=1}^{\tau_i}(G(X_s) - G(X_{s-1}))^2\right].$$

From Eq. (9.9) it follows that $\mathrm{var}(\tau_i) = D(i) - G(i)$, (which implies, $D(i) \geq G(i), \forall i \in \Omega$). $D(i)$ satisfies

$$D(i) = \sum_{j \in \Omega} \pi_{ij} \left[D(j) + (G(i) - G(j))^2 \right] \ . \tag{9.11}$$

where

$$\begin{aligned} \sum_{j \in \Omega} \pi_{ij} (G(i) - G(j))^2 &\geq \left(\sum_{j \in \Omega} \pi_{ij} (G(i) - G(j)) \right)^2 \\ &= \left(G(i) - \sum_{j \in \Omega} \pi_{ij} G(j) \right)^2 = 1 \ . \end{aligned} \tag{9.12}$$

Eq. (9.12) comes from Jensen's inequality and from (9.2). The equality

$$\sum_{j \in \Omega} \pi_{ij} (G(i) - G(j))^2 = 1$$

holds when $\exists j^*$ such that $G(j^*) = G(i) - 1$ and $\pi_{ij^*} = 1$; i.e., equality can be attained only for $G(i) \geq 2$.

From (9.2) and (9.11) we have

$$\begin{aligned} var(\tau_i) &= D(i) - G(i) \\ &= \sum_{j \in \Omega} \pi_{ij} (G(i) - G(j))^2 - 1 + \sum_{j \in \Omega} \pi_{ij} (D(j) - G(j)) \\ &\geq \sum_{j \in \Omega} \pi_{ij} (G(i) - G(j))^2 - 1 \ = \ \sum_{j \in \Omega} \pi_{ij} G(j)^2 - (G(i) - 1)^2 \\ &\geq \sum_{j \in \Omega} \pi_{ij} G(j) - (G(i) - 1)^2 \ = \ (G(i) - 1) - (G(i) - 1)^2 \ , \end{aligned} \tag{9.13}$$

since $G(j)^2 \geq G(j); \forall j \in \Omega$. For $1 \leq G(i) < 2$ (i.e., $G(i) - 1 = \langle G(i) \rangle$) Eq. (9.13) means that

$$D(i) - G(i) \geq \langle G(i) \rangle - \langle G(i) \rangle^2 \ . \tag{9.14}$$

To show that (9.14) holds also for those states whose mean time to absorption is greater than 2 ($G(i) > 2$) we assume that there exists state i such that $n \leq G(i) < n + 1$ and $D(i) - G(i) < \langle G(i) \rangle - \langle G(i) \rangle^2$. Knowing that state insertions do not increase the minimal variance of τ_i according to Lemma 9.2 we insert new states $i_1^*, i_2^*, \ldots, i_{n-1}^*$ to the Markov chain in the following way:

$$\pi_{i,i_1^*} = 1, \quad \pi_{i_j^*, i_{j+1}^*} = 1, j = 1, 2, \ldots, n-2 \ ,$$

and the outgoing transition probabilities from i_{n-1}^* can be anything that fit with Eq. (9.2) (for the expanded Markov chain). Note that we maintain the numbering of the original states in the expanded Markov chain. This insertion of states results that $G(i_j^*) = G(i) - j$ (i.e., $\langle G(i) \rangle = \langle G(i_j^*) \rangle$) and $D(i_j^*) = D(i) - j$ for $\forall j \in \{1, 2, \ldots, n-1\}$, and hence

$$D(i_{n-1}^*) - G(i_{n-1}^*) < \langle G(i) \rangle - \langle G(i) \rangle^2 \ . \tag{9.15}$$

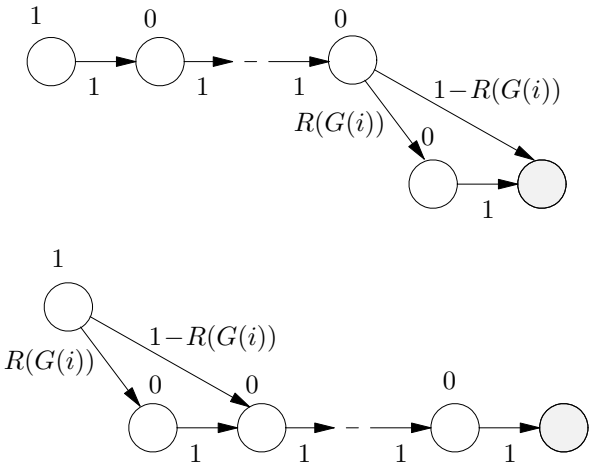
Eq. (9.15) is in conflict with Eq. (9.14) since $1 \leq G(i^*_{n-1}) < 2$, which means that for any i

$$\mathrm{var}(\tau_i) = D(i) - G(i) \geq \langle G(i) \rangle - \langle G(i) \rangle^2$$

and

$$cv^2(\tau_i) \geq \frac{\langle G(i) \rangle \ (1 - \langle G(i) \rangle)}{G^2(i)} \ . \tag{9.16}$$

The two DPH distributions below exhibit the lower bound provided by the DPH structure, and the first one demonstrates the structure of the state insertion described in the proof above.



Comparing the bounds in (9.10) and (9.16) it can be seen that (9.10) is meaningless (i.e., negative) when $G(i) < i$ and the structural bound (9.16) dominates. In contrast, for $G(i) > i$ the structural bound is less and (9.10) dominates. For $G(i) = i$ both bounds equal to 0. □

Lemma 9.3 has the following consequences:

- The minimal coefficient of variation of τ_i is obtained by a DPH with only downward transitions, i.e., $\pi_{ij} > 0$, iff $i \geq j$. Hence the minimal DPH is acyclic.
- As a result of the previous point the minimal coefficient of variation of τ_i is independent of n (the degree of τ), and it is equivalent with the minimal coefficient of variation that can be obtained by i phases. Hence (9.10) provides a relation of the degree (i), the mean ($G(i)$) and the minimal coefficient of variation of τ_i.
- The lower bound in (9.16) is independent of the degree of τ_i (which is i). This bound comes from the structural properties of the DPH distributions.

Proof of Theorem 9.1:
On the one hand, from Lemma 9.3, we have

$$cv^2(\tau) = \frac{\sum_{i \in \Omega} p_i \Big(cv^2(\tau_i) \cdot G^2(i) + G^2(i) \Big) - \mu^2}{\mu^2}$$

$$\geq \frac{\sum_{i\in\Omega} p_i \left(\frac{G^2(i)}{i} - G(i) + G^2(i)\right) - \mu^2}{\mu^2}$$

where the inequality comes from (9.10). We further have:

$$\begin{aligned} cv^2(\tau) &\geq \frac{\sum_{i\in\Omega} p_i \left(\left(\frac{1}{i}+1\right) G^2(i) - G(i)\right) - \mu^2}{\mu^2} \\ &\geq \frac{\sum_{i\in\Omega} \left(\frac{1}{N}+1\right) p_i G^2(i) - \sum_{i\in\Omega} p_i G(i) - \mu^2}{\mu^2} \\ &\geq \frac{\left(\frac{1}{N}+1\right)\mu^2 - \mu - \mu^2}{\mu^2} = \frac{1}{N} - \frac{1}{\mu} \end{aligned} \tag{9.17}$$

where $\sum_{i\in\Omega} p_i G^2(i) \geq \mu^2$ by Jensen's inequality.
On the other hand, using (9.16), we have

$$cv^2(\tau) = \frac{\sum_{i\in\Omega} p_i \Big(cv^2(\tau_i)\cdot G^2(i) + G^2(i)\Big) - \mu^2}{\mu^2}$$

$$\geq \frac{\sum_{i\in\Omega} p_i \Big(\langle G(i)\rangle \ (1 - \langle G(i)\rangle) + G^2(i)\Big) - \mu^2}{\mu^2}$$

Considering the sum in the numerator

$$\sum_{i\in\Omega} p_i \Big(\langle G(i)\rangle \Big(1 - \langle G(i)\rangle\Big) + G^2(i)\Big) =$$

$$\sum_{i\in\Omega} p_i \Big(\Big(1 - \langle G(i)\rangle\Big) \ \lfloor G(i)\rfloor^2 + \langle G(i)\rangle \ \Big(\lfloor G(i)\rfloor + 1\Big)^2\Big) =$$

$$\sum_{i\in\Omega} \Big(p_i \Big(1 - \langle G(i)\rangle\Big) \ \lfloor G(i)\rfloor^2 + p_i \langle G(i)\rangle \ \Big(\lfloor G(i)\rfloor + 1\Big)^2\Big)$$

The last expression is the second moment of a random variable with mean μ and support on $\mathbb{N}$. Among the random variables with mean μ and support on $\mathbb{N}$ the one with the minimal second moment is $\bar{X}$, defined as $\Pr(\bar{X} = \lfloor\mu\rfloor) = 1 - \langle\mu\rangle$ and $\Pr(\bar{X} = \lfloor\mu\rfloor + 1) = \langle\mu\rangle$ (i.e., the probability is concentrated around μ as much as possible), which means that:

$$\sum_{i\in\Omega} p_i\Big(1-\langle G(i)\rangle\Big)\ \lfloor G(i)\rfloor^2 + p_i\langle G(i)\rangle\ \Big(\lfloor G(i)\rfloor+1\Big)^2$$
$$\geq (1-\langle\mu\rangle)\ \lfloor\mu\rfloor^2 + \langle\mu\rangle\ (\lfloor\mu\rfloor+1)^2 = \mu^2 + \langle\mu\rangle(1-\langle\mu\rangle)$$

from which

$$cv^2(\tau) \geq \frac{\langle\mu\rangle(1-\langle\mu\rangle)}{\mu^2}$$

Since $\dfrac{\langle\mu\rangle(1-\langle\mu\rangle)}{\mu^2}$ is greater than $\dfrac{1}{N}-\dfrac{1}{\mu}$ when N is greater than μ and $\dfrac{\langle\mu\rangle(1-\langle\mu\rangle)}{\mu^2}$ is less than $\dfrac{1}{N}-\dfrac{1}{\mu}$ when N is less than μ, the theorem is given. □

Chapter 10

The Scale Factor: A New Degree of Freedom in Phase Type Approximation

A unified approach of discrete and continuous phase type approximation is presented in this chapter, in which the discrete and the continuous phase type models form a common model set. The models of this common set are assigned with a non-negative real parameter, the *scale factor*. The case when the scale factor is strictly positive results in discrete phase type distributions and the scale factor represents the time elapsed in one step. If the scale factor is 0, the resulting class is the class of continuous phase type distributions. Applying this unified view, it is shown that there is no qualitative difference between the discrete and the continuous phase type models and one can look for the best phase type approximation of a stochastic model by optimizing the scale factor.

The definition of discrete and continuous phase type distributions implies that DPH distributions have support on the set of the natural numbers while CPH distributions have support on the set of positive real numbers which is commonly associated with the positive half of the continuous time axis. When DPH distributions are used to model timed activities, the set of the natural numbers must be related to a time measure. Hence, a new parameter need to be introduced that represents the time span associated to each step. This new parameter is the scale factor of the DPH distribution, and can be viewed as a new degree of freedom, since its choice largely impacts the shape and properties of a DPH distribution over the continuous time axes. When DPH distributions are used to approximate a given continuous distribution, the scale factor affects the goodness of the fit.

10.1 Definition and Notation

A DPH distribution [64, 65] is the distribution of the time to absorption in a DTMC with n transient states, and one absorbing state numbered $(n+1)$. The one-step transition probability matrix of the corresponding DTMC can be partitioned as:

$$\widehat{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{b} \\ \mathbf{0} & 1 \end{bmatrix} \tag{10.1}$$

where $\boldsymbol{B} = [b_{ij}]$ is the $(n \times n)$ matrix collecting the transition probabilities among the transient states, $\boldsymbol{b} = [b_{i,n+1}]^T$ is the column vector of length n grouping the probabilities from any state to the absorbing one, and $\mathbf{0} = [0]$ is the zero vector. The initial probability vector $\widehat{\boldsymbol{\alpha}} = [\boldsymbol{\alpha}, \alpha_{n+1}]$ is of length $(n+1)$, with $\sum_{j=1}^{n} \alpha_j = 1 - \alpha_{n+1}$. Here, we consider only the class of DPH distributions for which $\alpha_{n+1} = 0$, but the extension to the case when $\alpha_{n+1} > 0$ is straightforward. The tuple $(\boldsymbol{\alpha}, \boldsymbol{B})$ is called the representation of the DPH distribution, and n the order.

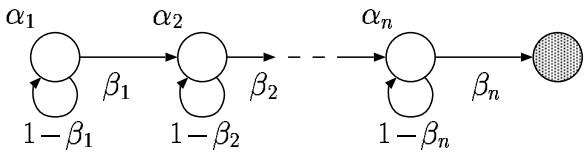
Similarly, a CPH distribution [65] is the distribution of the time to absorption in a CTMC with n transient states, and one absorbing state numbered $(n+1)$. The infinitesimal generator $\widehat{\mathbf{A}}$ of the CTMC can be partitioned in the following way:

$$\widehat{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{a} \\ \mathbf{0} & 1 \end{bmatrix} \tag{10.2}$$

where, $\boldsymbol{A}$ is a $(n \times n)$ matrix that describes the transient behavior of the CTMC and $\boldsymbol{a}$ is the column vector grouping the transition rates to the absorbing state. Let $\hat{\boldsymbol{\alpha}} = [\,\boldsymbol{\alpha},\ \alpha_{n+1}\,]$ be the $(n+1)$ initial probability (row) vector with $\sum_{i=1}^{n} \alpha_i = 1 - \alpha_{n+1}$. The tuple $(\boldsymbol{\alpha}, \boldsymbol{A})$ is called the representation of the CPH distribution, and n the order.

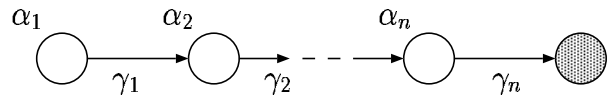
It has been shown in [10] for the discrete case and in [25] for the continuous case that the representations in (10.1) and (10.2), because of their too many free parameters, do not provide a convenient form for running a fitting algorithm. Instead, resorting to acyclic phase type distributions, the number of free parameters is reduced significantly since both in the discrete and the continuous case a canonical form can be used. The canonical form and its constraints for the discrete case [10] is depicted in Figure 10.1. Figure 10.2 gives the canonical form and associated constraints for the continuous case. In both cases the canonical form corresponds to a mixture of hypo-exponential distributions.

A fitting algorithm that provides acyclic CPH, acyclic DPH distributions has been provided in [8] and [10], respectively. Experiments suggests (an exhaustive comparison of fitting algorithms can be found in [53]) that, from the point of view of applications, the acyclic phase type class is as flexible as the whole phase type class.



$$\sum_{i=1}^{n} \alpha_i = 1,\ 0 < \beta_i \leq \beta_{i+1} \leq 1,\ 1 \leq i \leq n-1$$

Figure 10.1: Canonical representation of acyclic DPH distributions and its constraints



$$\sum_{i=1}^{n} \alpha_i = 1,\ 0 < \gamma_i \leq \gamma_{i+1},\ 1 \leq i \leq n-1$$

Figure 10.2: Canonical representation of acyclic CPH distributions and its constraints

10.2 Comparing properties of CPH and DPH distributions

CTMCs are defined as a function of a continuous variable, t, which is assumed to be the time in most cases. DTMCs are defined over the set of the natural numbers. In order to relate the number of jumps in a DTMC with a time measure, a time span must be assigned to each step. Let δ be (in some arbitrary units) the scale factor, i.e., the time span assigned to each step. The value of δ establishes an equivalence between the sentence "probability at the k-th step" and "probability at time $k\,\delta$", and hence, defines the time scale on which the properties of the DTMC are measured. The consideration of the scale factor δ introduces a new parameter, and consequently a new degree of freedom, in the DPH class with respect to the CPH class. In the following, we discuss how this new degree of freedom impacts the properties of the DPH class and how it can be exploited in practice.

Let u be an "unscaled" DPH distributed random variable (r.v.) of order n with representation ($\boldsymbol{\alpha}$, $\boldsymbol{B}$), defined over the set of the non-negative natural numbers. Let us consider a scale factor δ; the scaled r.v. $\tau = \delta\, u$ is defined over the discrete set of time points $(0,\, 1\,\delta,\, 2\,\delta,\, 3\,\delta,\, \ldots,\, k\,\delta,\, \ldots)$, being k a non-negative natural number. For the unscaled and the scaled DPH r.v. the following equations hold.

$$\begin{aligned} &F_u(k) = Pr\{u \leq k\} = 1 - \boldsymbol{\alpha}\boldsymbol{B}^k\,\boldsymbol{e} \\ &F_\tau(\delta k) = Pr\{\tau \leq \delta k\} = 1 - \boldsymbol{\alpha}\boldsymbol{B}^k\,\boldsymbol{e} \\ &m_u^i = E(u^i) \\ &m_\tau^i = E(\tau^i) = \delta^i\, E(u^i) \qquad i \geq 1, \end{aligned} \tag{10.3}$$

where $\boldsymbol{e}$ is the column vector of ones, and $E(u^i)$ is the i-th moment calculated from the factorial moments of u: $E(u(u-1)\ldots(u-i+1)) = i!\,\boldsymbol{\alpha}(\boldsymbol{I}-\boldsymbol{B})^{-i}\boldsymbol{B}^{i-1}\boldsymbol{e}$. It is evident from (10.3) that the mean m_τ of the scaled r.v. τ is δ times the mean m_u of the unscaled r.v. u. While m_u is an invariant of the representation ($\boldsymbol{\alpha}$, $\boldsymbol{B}$), δ is a free parameter; adjusting δ, the scaled r.v. can assume any mean value $m_\tau \geq 0$. On the other hand, one can easily infer from (10.3) that the coefficients of variation of τ and u are equal. A consequence of the above properties is that one can easily provide a scaled DPH of order ≥ 2 with arbitrary mean and arbitrary coefficient of variation with an appropriate scale factor. Or more formally: the unscaled DPH r.v. u of any order $n > 1$ can exhibit a coefficient of variation between $0 \leq cv_u^2 \leq \infty$. For $n = 1$ the coefficient of variation ranges between $0 \leq cv_u^2 \leq 1$.

As mentioned earlier, an important property of the DPH class with respect to

the CPH class is the possibility of exactly representing a deterministic delay. A deterministic distribution with value a can be realized by means of a scaled DPH distribution with n phases with scale factor δ if $n = a/\delta$ is integer. In this case, the structure of the DPH distribution is such that phase i is connected with probability 1 only to phase $i+1$ ($i = 1, \ldots, n$), and with an initial probability concentrated in state 1. If $n = a/\delta$ is not integer for the given δ, the deterministic behavior can only be approximated.

10.2.1 First order discrete approximation of CTMCs

Given a CTMC with infinitesimal generator $\widetilde{\boldsymbol{A}}$, the transition probability matrix over an interval of length δ can be written as:

$$e^{\widetilde{\boldsymbol{A}}\delta} = \sum_{i=0}^{\infty} (\widetilde{\boldsymbol{A}}\delta)^i / i! = \boldsymbol{I} + \widetilde{\boldsymbol{A}}\delta + \sigma(\delta),$$

hence the first order approximation of $e^{\widetilde{\boldsymbol{A}}\delta}$ is matrix $\boldsymbol{\Pi}(\delta) = \boldsymbol{I} + \widetilde{\boldsymbol{A}}\delta$. $\boldsymbol{\Pi}(\delta)$ is a proper stochastic matrix if $\delta < 1/q$, where $q = \max_{i,j} |\widetilde{\boldsymbol{A}}_{ij}|$. $\boldsymbol{\Pi}(\delta)$ is the exact transition probability matrix of the CTMC assumed that at most one transition occurs in the interval of length δ.

We can approximate the behavior of the CTMC at time $(0, \delta, 2\delta, 3\delta, \ldots, k\delta, \ldots)$ using the DTMC with transition probability matrix $\boldsymbol{\Pi}(\delta)$. The approximate transition probability matrix at time $t = k\delta$ is:

$$\boldsymbol{\Pi}(\delta)^k = (\boldsymbol{I} + \widetilde{\boldsymbol{A}}\delta)^{\frac{t}{\delta}}$$

The following theorem proves the property that the above first order approximation becomes exact as $\delta \to 0$.

Theorem 10.1. *As the length of the interval of the first order approximation, δ, tends to 0, such that $t = k\delta$ the approximate transition probability matrix tends to the exact one.*

Proof. The scalar version of the applied limiting behavior is well-known in the following form $\lim_{x\to 0}(1+ax)^{\frac{1}{x}} = e^a$. Since matrices $\boldsymbol{I}$ and $\widetilde{\boldsymbol{A}}$ commute we can obtain the matrix version of the same expression as follows

$$\lim_{\delta\to 0}(\boldsymbol{I} + \widetilde{\boldsymbol{A}}\delta)^{\frac{t}{\delta}} = \lim_{k\to\infty} \boldsymbol{\Pi}(t/k)^k = \lim_{k\to\infty} \left(\boldsymbol{I} + \frac{\widetilde{\boldsymbol{A}}t}{k}\right)^k =$$

$$\lim_{k\to\infty} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{\widetilde{\boldsymbol{A}}t}{k}\right)^j == \lim_{k\to\infty} \sum_{j=0}^{k} \frac{(\widetilde{\boldsymbol{A}}t)^j}{j!} \frac{k!}{k^j\ (k-j)!} = \sum_{j=0}^{\infty} \frac{(\widetilde{\boldsymbol{A}}t)^j}{j!} = e^{\widetilde{\boldsymbol{A}}t}.$$

□

An obvious consequence of Theorem 10.1 for PH distributions is given in the following corollary.

Corollary 10.2. *Given a scaled DPH distribution of order n, representation $(\boldsymbol{\alpha}, \boldsymbol{I} + \boldsymbol{A}\delta)$ and scale factor δ, the limiting behavior as $\delta \to 0$ is the CPH distribution of order n with representation $(\boldsymbol{\alpha}, \boldsymbol{A})$.*

10.2.2 The minimum coefficient of variation

It is known that one of the main limitation in approximating a given distribution by a PH one is the attainable minimal coefficient of variation, cv^2_{min}. In order to discuss this point, we recall the theorems that state the cv^2_{min} for the class of CPH and DPH distributions.

Theorem 10.3. (Aldous and Shepp [3]) *The cv^2_{min} of a* CPH *distributed r.v. of order n is $cv^2_{min} = 1/n$ and is attained by the Erlang(n) distribution independent of its mean m_c or of its parameter $\lambda = n/m_c$.*

The corresponding theorem for the unscaled DPH class has been proved in the previous chapter. The next section restate that theorem for scaled PDH distributions.

10.2.3 The minimum coefficient of variation of scaled DPH distributions

Theorem 10.4. *The cv^2_{min} of a scaled DPH r.v. of order n with scale factor δ and mean $m_\tau = \delta\, m_u$ is:*

$$\begin{cases} \dfrac{\left\langle \frac{m_\tau}{\delta} \right\rangle \left(1 - \left\langle \frac{m_\tau}{\delta} \right\rangle\right)}{\left(\frac{m_\tau}{\delta}\right)^2} & \text{if } m_\tau \leq n\,\delta\ , \\ \dfrac{1}{n} - \dfrac{\delta}{m_\tau} & \text{if } m_\tau > n\,\delta\ , \end{cases} \tag{10.4}$$

The scaled DPH r.v. which exhibits the cv^2_{min} has the same structure of Figures (9.1) and (9.2), as in the unscaled case (see Theorem 9.1).

Corollary 10.5. *For finite mean m_τ, $cv^2_{min} \to 1/n$ as $\delta \to 0$.*

Proof. As $\delta \to 0$ the second part of (10.4) ($m_\tau > n\,\delta$) becomes effective and $1/n - \delta/m_\tau \to 1/n$ as $\delta \to 0$. □

Corollary 10.5 proves that the cv^2_{min} of the DPH class converges to the cv^2_{min} of the CPH class of the same order as $\delta \leftarrow 0$. The following corollary presents a much stronger convergence result for the case of approximating distributions with low coefficient of variation. It is about the convergence in distribution.

Corollary 10.6. *The* best fitting *scaled DPH approximation of distributions with low coefficient of variation converges, in distribution, to the* best fitting *CPH approximation of the same distribution as δ tends to 0, where the* best fitting *PH approximation is defined as the one which exhibits the same mean and provides the closest approximation for the 2nd moment.*

Proof. Both the CPH and the DPH classes have limits in approximating distributions with low coefficient of variation. The best approximation of a distribution with coefficient of variation less than these limits is the Erlang(n) distribution in both the discrete and the continuous case (Theorem 10.3 and 9.1).

The representation ($\boldsymbol{\alpha}$, $\boldsymbol{A}$) of the continuous Erlang(n) with mean m_τ and the representation ($\boldsymbol{\alpha}$, $\boldsymbol{B}$) of the discrete Erlang(n) with mean m_τ, scale factor δ are:

$$\boldsymbol{\alpha} = \{1, 0, \ldots, 0\}, \quad \boldsymbol{A} = \begin{bmatrix} -\frac{n}{m_\tau} & \frac{n}{m_\tau} & 0 & \ldots & 0 \\ 0 & -\frac{n}{m_\tau} & \frac{n}{m_\tau} & \ldots & \\ & & \ddots & & \\ 0 & & & \ldots & -\frac{n}{m_\tau} \end{bmatrix}$$

$$\boldsymbol{\alpha} = \{1, 0, \ldots, 0\}, \quad \boldsymbol{B} = \begin{bmatrix} 1-\frac{n\delta}{m_\tau} & \frac{n\delta}{m_\tau} & 0 & \ldots & 0 \\ 0 & 1-\frac{n\delta}{m_\tau} & \frac{n\delta}{m_\tau} & \ldots & \\ & & \ddots & & \\ 0 & & & \ldots & 1-\frac{n\delta}{m_\tau} \end{bmatrix}$$

Note that $\boldsymbol{B} = \boldsymbol{I} - \boldsymbol{A}\delta$ and Corollary 10.6 follows from Corollary 10.2. □

In this particular case, when the structure of the best fitting scaled DPH and CPH distributions are known, we can show that the distribution of the best fitting scaled DPH distribution converges to the distribution of the best fitting CPH distribution when $\delta \to 0$. Unfortunately, the same convergence property cannot be proved in general, since the structural properties of the best fitting PH distributions are not known and they depend on the chosen (arbitrary) optimization criterion. Instead, in Section 10.3 we provide an extensive experimental study on the behavior of the best fitting scaled DPH and CPH distributions as a function of the scale factor δ .

10.2.4 DPH distributions with finite support

Another peculiar characteristic of the DPH class is to contain distributions with finite support. A DPH distribution has finite support if its structure does not contain cycles and self-loops (any cycle or self loop implies an infinite support).

Let $[a, b]$ be the finite support of a given distribution, with $a, b \geq 0$ and $a \leq b$ (when $a = b$ the finite support distribution reduces to a deterministic distribution with mass 1 at $a = b$). If a/δ and b/δ are both integers, it is possible to construct a scaled DPH of order b/δ for which the probability mass function has non-zero elements only for the values $a, a+\delta, a+2\delta, ..., b$. As an example, the discrete uniform distribution between $a = 2$ and $b = 6$ is reported in Figure 10.3, for scale factor $\delta = 1$.

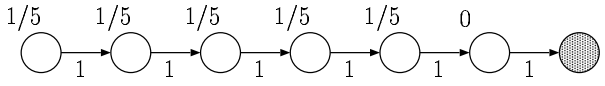


Figure 10.3: DPH representation of the discrete uniform distribution $[a = 2, b = 6]$

10.3 The optimal δ in PH fitting

The scale factor δ provides a new degree of freedom in fitting, and, furthermore, since the limit of a DPH distribution for $\delta \to 0$ is a CPH distribution, the optimization of

the scale factor in a fitting problem provides a quantitative way to decide whether a continuous or a discrete approximation performs better in the given problem. Hence, assuming δ as a decision variable, we can consider the CPH and the DPH class as a unique model set in which the choice among DPH or CPH classes is given by the optimal value of δ.

Let X be the continuous r.v. to be fit by a PH distribution, and let $F_X(x)$ be its cdf, $E(X^i)$ the i-th moment and $cv^2(X)$ the squared coefficient of variation. In order to define a fitting procedure, a distance measure between X and the approximating PH distribution needs to be defined. Then, the fitting algorithm provides the PH distribution which minimizes the chosen distance measure. In order to compare, in a unified framework, the goodness of the approximation reached by CPH and DPH distributions, we need to chose a distance measure that is meaningful and applicable both in the continuous as well as in the discrete setting. The selected distance measure is the squared area difference between the original cdf $F(\cdot)$ and the approximating cdf $\hat{F}(\cdot)$:

$$\mathcal{D} = \int_x (F(x) - \hat{F}(x))^2 dx \tag{10.5}$$

The distance measure $\mathcal{D}$ is easily applicable for any combination of discrete and continuous distributions. All the numerical experiments reported in the sequel are based on the minimization of the area difference given in (10.5).

10.3.1 Fitting distributions with low cv^2

The following considerations provide practical upper and lower bounds to guide in the choice of a suitable scale factor δ, and are mainly based on the dependence of the minimal coefficient of variation of a scaled DPH distribution on the order n and on the mean m_τ.

Since we only consider DPH distributions with no mass at zero, the mean of any unscaled DPH distribution is greater than 1. This means that δ should be less than $E(X)$. However, a more convenient upper bound that exploits the flexibility associated with the n phases, is given by:

$$\delta \leq \frac{E(X)}{n-1} \ . \tag{10.6}$$

If the squared coefficient of variation of the distribution to be approximated is less than $1/n$, δ should satisfy the following relation (see Theorem 10.4):

$$\delta > \left(\frac{1}{n} - cv^2(X)\right) E(X) \tag{10.7}$$

Let X be a Lognormal r.v. with parameters $(1, 0.2)$, whose mean is $E(X) = 1$ and $cv^2(X) = 0.0408$ (this distribution is the distribution L3 taken from the benchmark examined in [14, 10], hence we refer to it as L3). Table 10.1 reports the lower and upper bounds of δ, with $n = 2$, 4, 8, 12, computed from (10.7) and (10.6).

The cdf and pdf of the approximating CPH and DPH distributions of order $n = 10$, with different scale factors δ, are presented in Figure 10.4 and 10.5. When

considering the approximate DPH distribution, the $f(x)$ values are calculated at the discrete points $(\delta, 2\delta, 3\delta, \ldots, k\delta, \ldots)$ to which the following mass is assigned:

$$f(k\delta) = 1/\delta(F(k\delta) - F((k-1)\delta)) \tag{10.8}$$

For the ease of visual interpretation the points are connected with a line.

When δ is less than its lower bound the required cv^2 cannot be attained; when δ becomes too large the wide separation of the discrete steps increases the approximation error; when δ is in the proper range (e.g. $n = 10$; $\delta = 0.06$) a reasonably good fit is achieved. This example also suggests that an optimal value of δ exists that minimizes the chosen distance measure $\mathcal{D}$ in (10.5).

In order to display the goodness of fit for the L3 distribution, Figure 10.6 shows the distance measure $\mathcal{D}$ as a function of δ for various values of the order n. A minimum value of δ is attained in the range where the parameters fit the bounds of Table 10.1. Notice also that, as δ increases, the advantage of having more phases disappears, according to Theorem 10.4. The circles in the left part of Figure 10.6 (as well as in all the successive figures) indicate the corresponding distance measure $\mathcal{D}$ obtained from CPH fitting. The figure (and the subsequent ones as well) suggests that the distance measure obtained from DPH fitting converges to the distance measure obtained by the CPH approximation as δ tends to 0.

n	lower bound of δ equation (10.7)	upper bound of δ equation (10.6)
4	0.2092	0.333
8	0.0842	0.1428
12	0.0425	0.0909
16	0.0217	0.0666

Table 10.1: Upper and lower bound of δ for fitting distribution L3

10.3.2 Fitting distributions with high cv^2

We have seen in the previous subsections that it is beneficial to approximate distributions with a low coefficient of variation by means of a DPH distributions. In this subsection, we investigate the optimal value of δ when fitting distributions with a high coefficient of variation.

Let X be a Lognormal r.v. with parameters $(1, 1.8)$ (this is the distribution L1 taken from the benchmark in [14, 10]). For X we have $E(X) = 1$ and $cv^2(X) = 24.534$. Figure 10.7 shows the measure of the goodness of fit as a function of δ for various orders n (the cases when the number of phases are greater than 2 result in practically the same goodness of fit). The distance measures $\mathcal{D}$ decreases as $\delta \to 0$ indicating that the optimal fitting is achieved by applying CPH distribution. This example suggests that, for smooth distributions with infinite support and $cv^2(X) > 1/n$, the optimal value of δ tends to 0, implying that the best fit is obtained by a CPH. However, this conclusion might not be true for distributions with finite support, as it is explored in the next subsection.

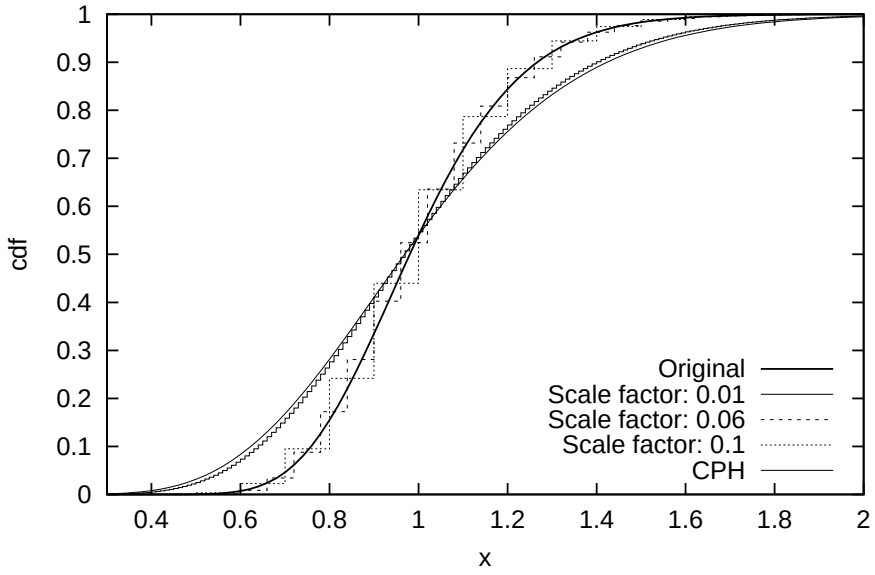


Figure 10.4: Approximating the L3 distribution with 10-phase PH approximations (CDF)

10.3.3 Fitting distributions with finite support

In this case, two features must be considered, namely the cv^2 and the maximum value of the finite support. It should be stressed that the chosen distance measure $\mathcal{D}$ in (10.5) can be considered as not completely appropriate in the case of finite support, since it does not force the approximating PH to have its mass confined in the finite support and 0 outside.

Let X be a uniform r.v. over the interval [1, 2], with $E(X) = 1.5$ and $cv^2(X) = 0.0370$ (this is the distribution U2 taken from the benchmark in [14, 10]). Figure 10.8 shows the distance measure as a function of δ for various orders n. It is evident that, for each n, a minimal value of δ is obtained, that provides the best approximation according to the chosen distance measure.

As a second example, let X be a uniform r.v. over the interval [0, 1], with $E(X) = 0.5$ and $cv^2(X) = 0.333$ (this is the distribution U1 taken from the benchmark in [14, 10]). Figure 10.9 shows the distance measure as a function of δ for various orders n. Since, in this example $cv^2(X) = 0.333$, an order $n = 3$ is large enough for a CPH to attain the coefficient of variation of the distribution. Nevertheless, the optimal δ in Figure (10.9), which minimizes the distance measure $\mathcal{D}$ for high order PH ($n > 2$), ranges between $\delta = 0.02$ and $\delta = 0.05$, thus leading to the conclusion that a DPH provides a better fit. This example evidences that the coefficient of variation is not the only factor which influences the optimal δ value. The shape of the distribution plays an essential role as well. Our experiments show that a discontinuity in the pdf (or in the cdf) is hard to approximate with CPH, hence in the majority of these cases DPH provides a better approximation.

Figure 10.10 shows the cdf and the pdf of the U1 distribution, compared with the best fit PH approximations of order $n = 10$, and various scale factors δ. In the case of DPH approximation, the $f(x)$ values are calculated as in (10.8). With respect

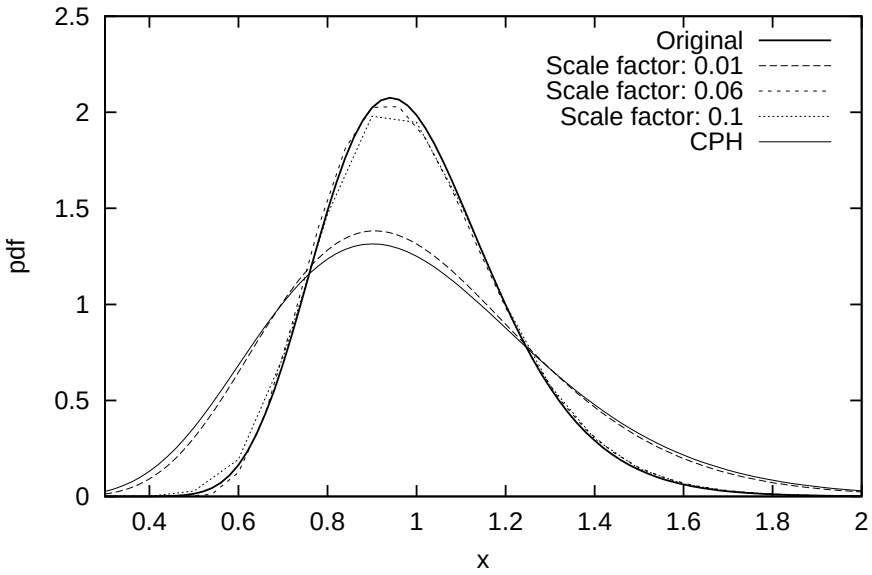


Figure 10.5: Approximating the L3 distribution with 10-phase PH approximations (PDF)

to the chosen distance measure, the best approximation is obtained for $\delta = 0.03$, which corresponds to a DPH distribution with infinite support. When $\delta = 0.1$ the approximate distribution has a finite support. Hence, the value $\delta = 0.1$ (for $n = 10$) provides a DPH able to represent the logical property that the random variable is less than 1. Another fitting criterion may, of course, stress this property.

10.4 Approximating non-Markovian models

Section 10.3 has explored the problem of how to find the best fit among either a DPH or a CPH distribution by tuning the scale factor δ. When dealing with a stochastic model of a system that incorporates non exponential distributions, a well know solution technique consists in a markovianization of the underlying non-Markovian process by substituting the non exponential distribution with a best fit PH distribution, and then expanding the state space. A natural question arises also in this case, on how to decide among a discrete (using DPH) or a continuous (using CPH) approximation, in order to minimize the error in the performance measures we are interested in for the overall model.

One possible way to handle this problem could consist in finding the best PH fits for any single distribution and to plug them in the model. We only consider the case where the PH distributions are either all discrete (and with the same scale factor δ) or they are all continuous[1].

In order to quantitatively evaluate the influence of the scale factor on some performance measures defined at the system level, we have considered a preemptive

[1]Various embedding techniques have been explored in the literature for mixing DPH (with different scale factors) and CPH ([33, 46]).

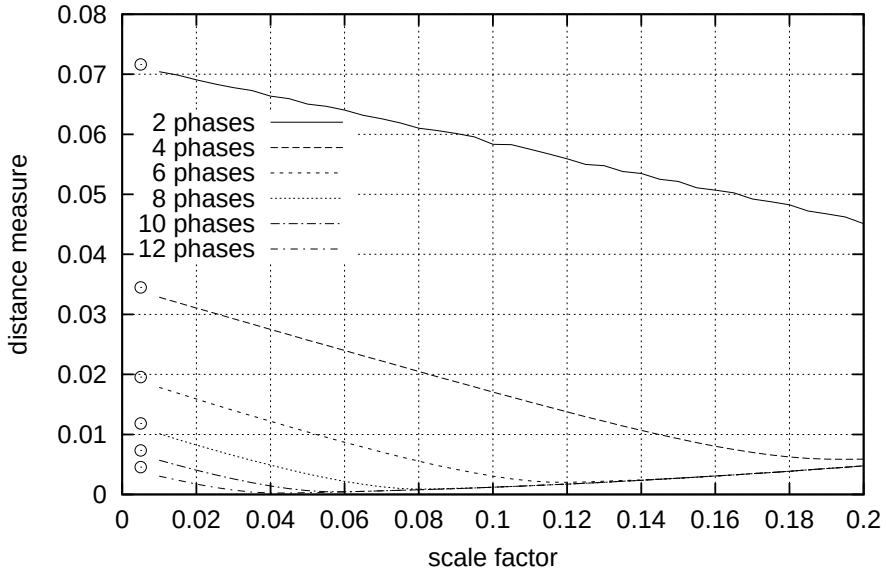


Figure 10.6: Distance measure as the function of the scale factor δ for low cv^2 (L3)

M/G/1/2/2 queue with two classes of customers. We have chosen this example because accurate analytical solutions are available both in transient condition and in steady-state using the methods presented in, e.g., [33]. The general distribution G is taken from the set of distributions (L1, L3, U1, U2) already considered in the previous section.

Customers arrive at the queue with rate $\lambda = 0.5$ in both classes. The service time of a higher priority job is exponentially distributed with parameter $\mu = 1$. The service time distribution of the lower priority job is either L1, L3, U1 or U2. Arrival of a higher priority job preempts the lower priority one. The policy associated to the preemption of the lower priority job is preemptive repeat different (prd), i.e. after the departure of the higher priority customer the service of the low priority customer starts from the beginning with a new service time sample.

The system has 4 states (Figure 10.12): in state *s1* the server is empty, in state *s2* a higher priority customer is under service with no lower priority customer in the system, in state *s3* a higher priority customer is under service with a lower priority customer waiting, in state *s4* a lower priority job is under service (in this case there cannot be a higher priority job).

Let p_i ($i = 1, \ldots, 4$) denote the steady state probability of the M/G/1/2/2 queue obtained from an exact analytical solution.

In order to evaluate the correctness of the PH approximation we have solved the model by substituting the original general distribution (either L1, L3, U1 or U2) with approximating DPH or CPH distributions. Let $\hat{p}_i$ ($i = 1, \ldots, 4$) denote the steady state probability of the M/PH/1/2/2 queue with the PH approximation.

The overall approximation error is measured in terms of the difference between the exact steady state probabilities p_i and the approximate steady state probabilities

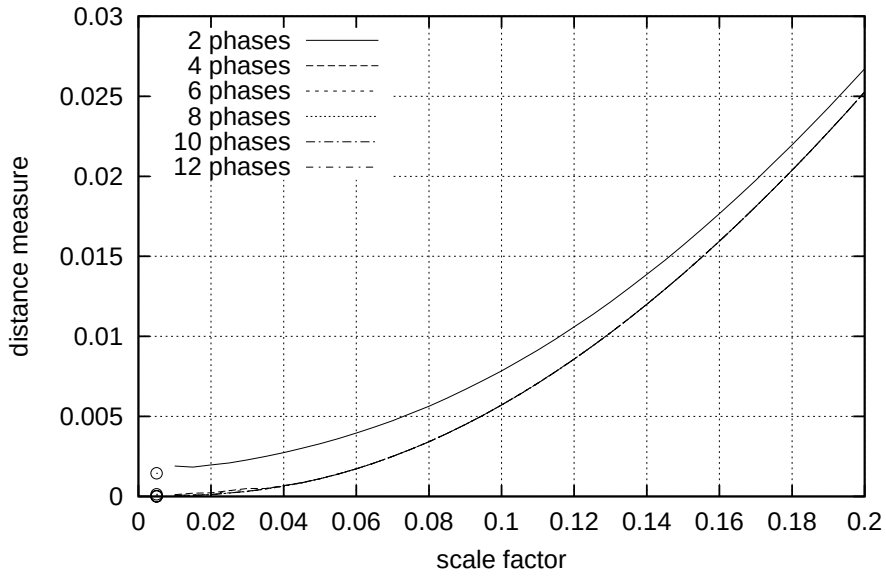


Figure 10.7: Distance measure as the function of the scale factor δ for high cv^2 (L1)

$\hat{p}_i$. Two error measures are defined:

$$\epsilon_{SUM} = \sum_i |p_i - \hat{p}_i| \quad \text{and} \quad \epsilon_{MAX} = \max_i |p_i - \hat{p}_i|.$$

The evaluated numerical values for ϵ_{SUM} and ϵ_{MAX} are reported in Figures 10.13 and 10.14 for the distribution L3. Since the behavior of ϵ_{MAX} is very similar to the behavior of ϵ_{SUM} in all the cases, for the other distributions we report ϵ_{SUM} only (Figures 10.15, 10.16, 10.17). The figures, which refer to the error measure in a performance index of a global stochastic model, show a behavior similar to the one obtained for a single distribution fitting. Depending on the coefficient of variation and on the shape of the considered non-exponential distributions an optimal value of δ is found which minimizes the approximation error. In these examples, the optimal value of δ is close to the one obtained for the single distribution fitting.

Based on our experiments, we guess that the observed property is rather general. If the stochastic model under study contains a single non-exponential distribution, then the approximation error in the evaluation of the performance indices of the global model can be minimized by resorting to a PH type approximation (and subsequent DTMC or CTMC expansion) with the optimal δ of the single distribution. The same should be true if the stochastic model under study contains more than one general distribution, whose best PH fit provides the same optimal δ.

In order to investigate the approximation error in the transient behavior, we have considered distribution U2 for the service time and we have computed the transient probability of state s_1 with two different initial conditions. Figure 10.18 depicts the transient probability of state s_1 with initial state s_1. Figure 10.19 depicts the transient probability of the same state, s_1, when the service of a lower priority job starts at time 0 (the initial state is s_4). All approximations are with DPH distributions of order $n = 10$. Only the DPH approximations are depicted because the CPH approximation is very similar to the DPH one with scale factor $\delta = 0.03$.

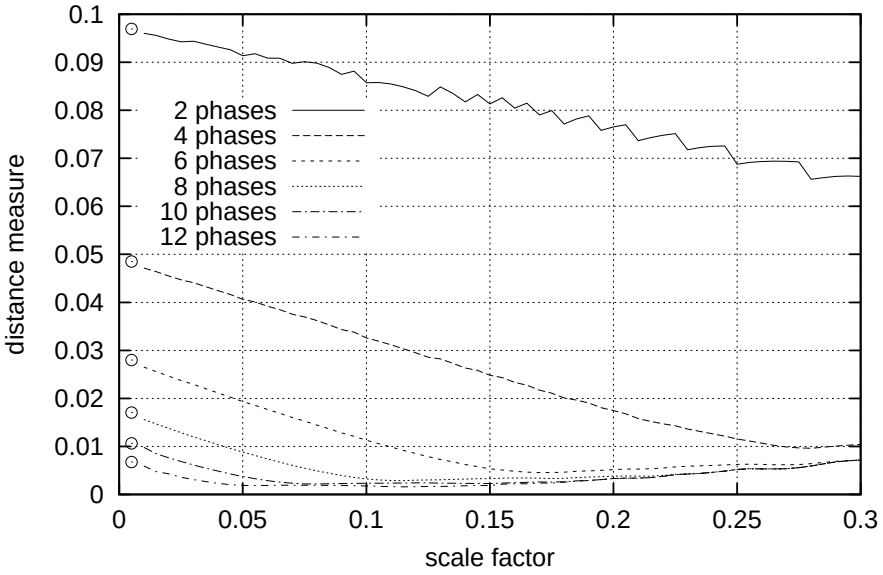


Figure 10.8: Distance measure as the function of the scale factor δ for Uniform(1,2) (U2)

In the first case, (Figure 10.18), the scale factor $\delta = 0.03$, which was the optimal one from the point of view of fitting the single distribution in isolation, provides the most accurate results for the transient analysis as well. Instead, in the second case, the approximation with a scale factor $\delta = 0.2$ captures better the sharp change in the transient probability. Moreover, this value of δ is the only one among the values reported in the figure that results in 0 probability for time points smaller than 1. In other words, the second example depicts the advantage given by DPH distributions to model durations with finite support. This example suggests also that DPH approximation can be of importance when preserving reachability properties is crucial (like in modeling time-critical systems) and, hence, DPH approximation can be seen as a bridge between the world of stochastic modeling and the world of functional analysis and model checking [9].

10.5 Remarks on the use of CPH versus DPH distributions

To conclude this chapter, we summarize the advantages and the disadvantages of applying approximate DPH models (even with optimal δ value) with respect to using CPH approximations.

Advantages of using DPH: An obvious advantage of the application of DPH distributions is that one can have a closer approximate of distributions with low coefficient of variation. An other important quantitative property of the DPH class is that it can capture distributions with finite support and deterministic values. This property allows to capture the periodic behavior of a complex stochastic model, while any CPH based approximation of the same model tends to a steady state.

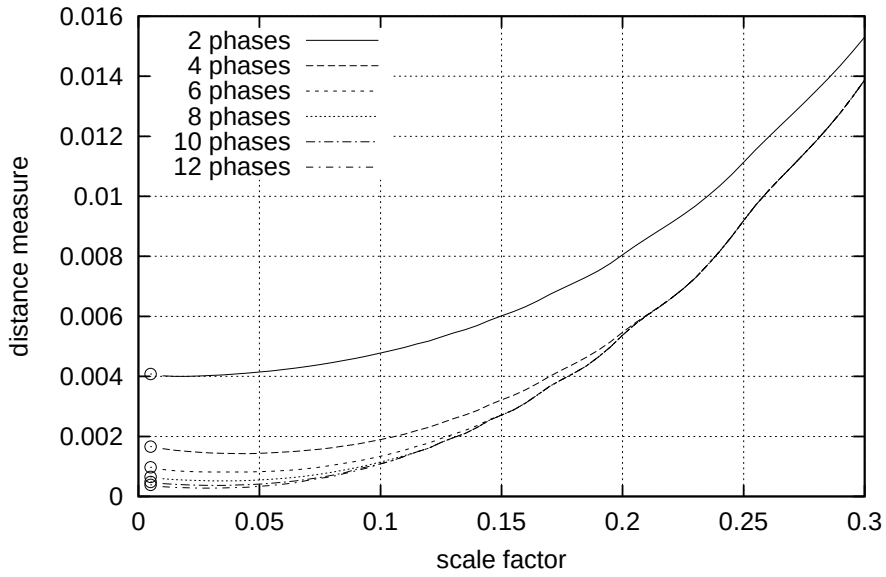


Figure 10.9: Distance measure as the function of the scale factor δ for Uniform(0,1) (U1)

Numerical experiments have also shown that DPH can better approximate distributions with some abrupt or sharp changes in the CDF or in the PDF.
Disadvantages of using DPH: There is a definite disadvantage of discrete time approximation of continuous time models. In the case of CPH approximation, coincident events do not have to be considered (they have zero probability of occurrence). Instead, when applying DPH approximation coincident events have to be handled, and their consideration may burden significantly the complexity of the analysis.

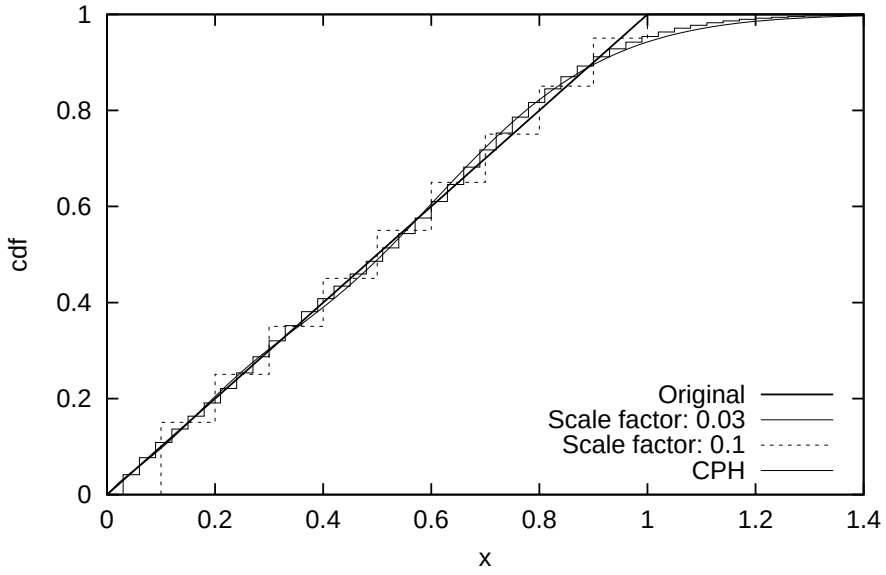


Figure 10.10: Approximating the Uniform $(0, 1)$ distribution (U1) (CDF)

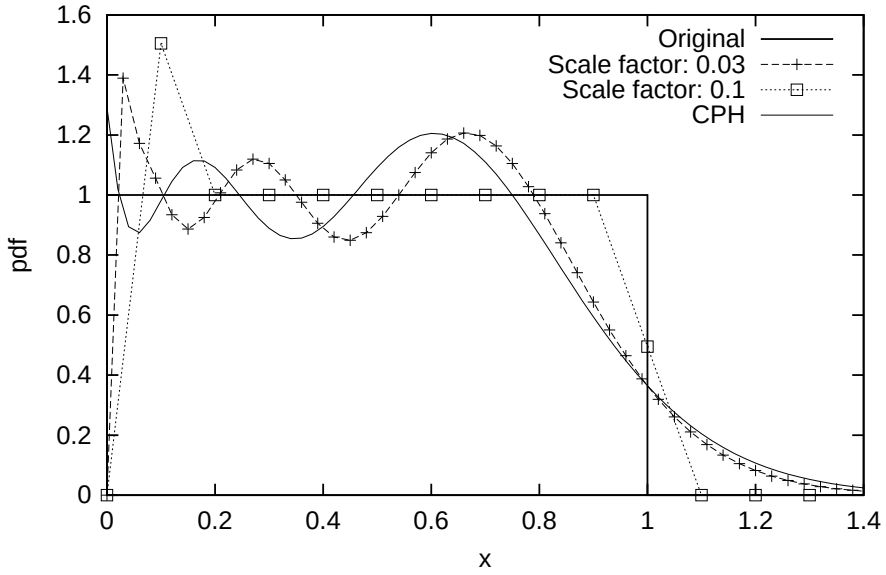


Figure 10.11: Approximating the Uniform $(0, 1)$ distribution (U1) (PDF)

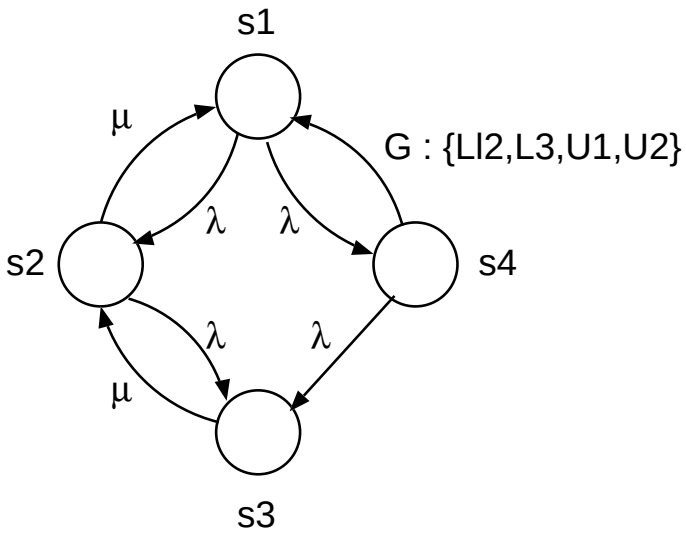


Figure 10.12: The state space of the considered M/G/1/2/2 queue

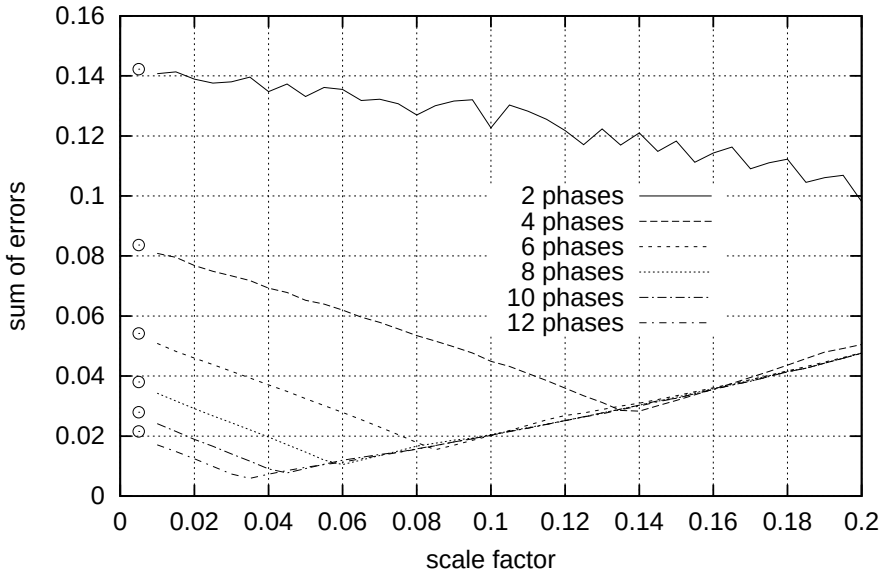


Figure 10.13: ϵ_{SUM} with scale factor δ and distribution L3

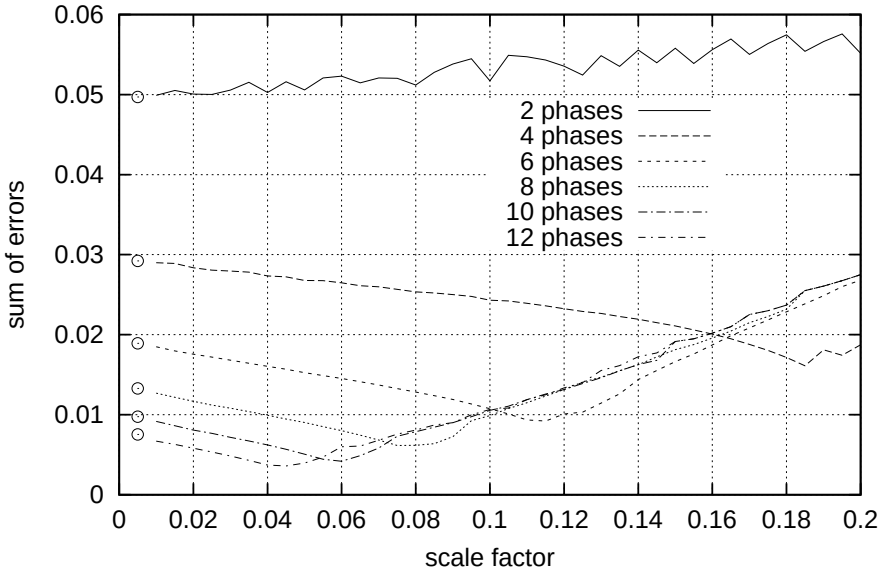


Figure 10.14: ϵ_{MAX} with scale factor δ and distribution L3

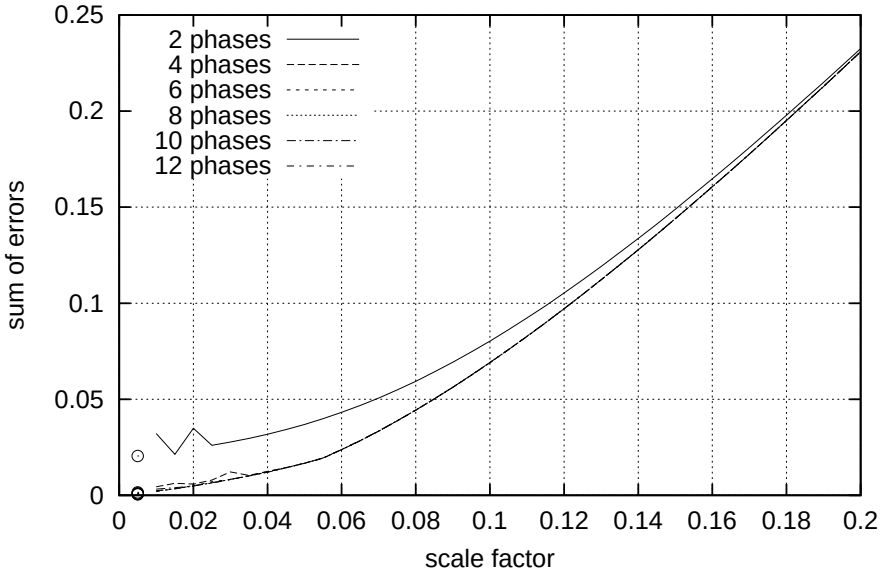


Figure 10.15: ϵ_{SUM} with scale factor δ and distribution L1

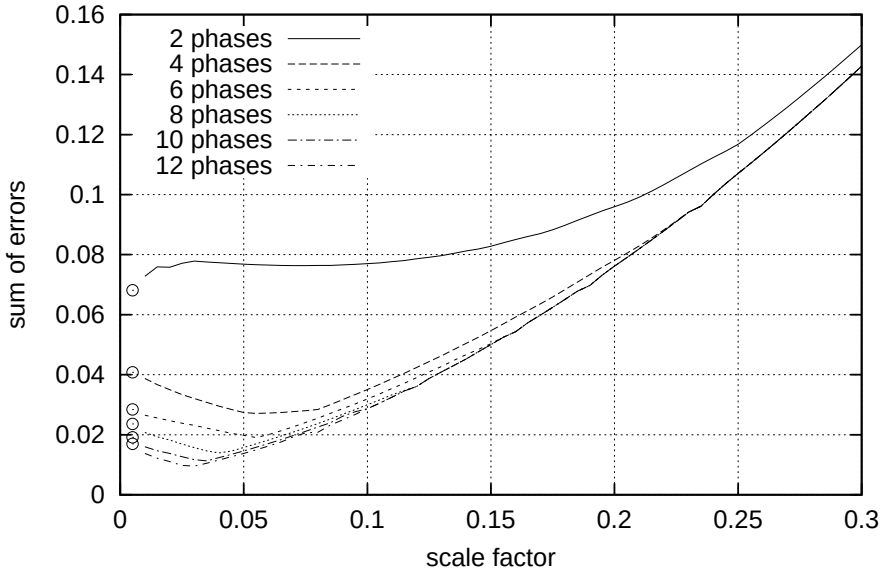


Figure 10.16: ϵ_{SUM} with scale factor δ and distribution U1

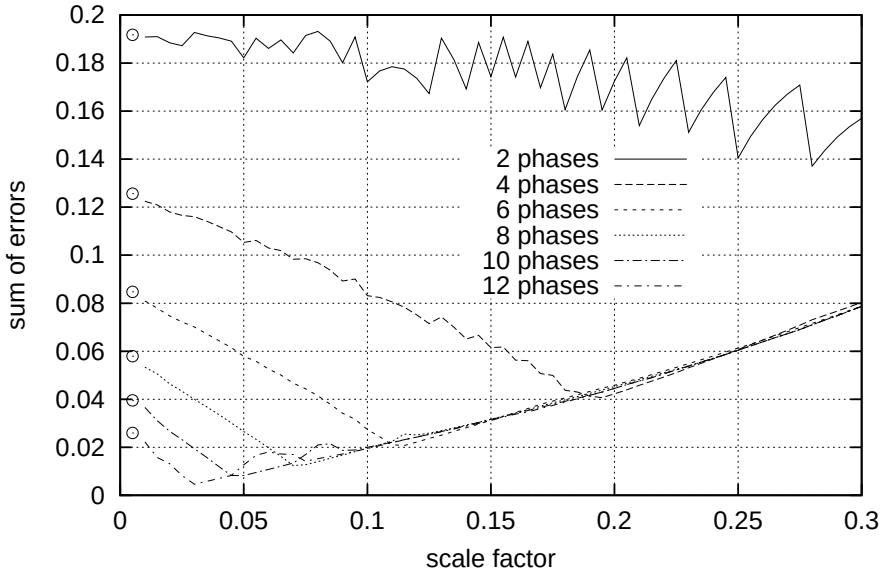


Figure 10.17: ϵ_{SUM} with scale factor δ and distribution U2

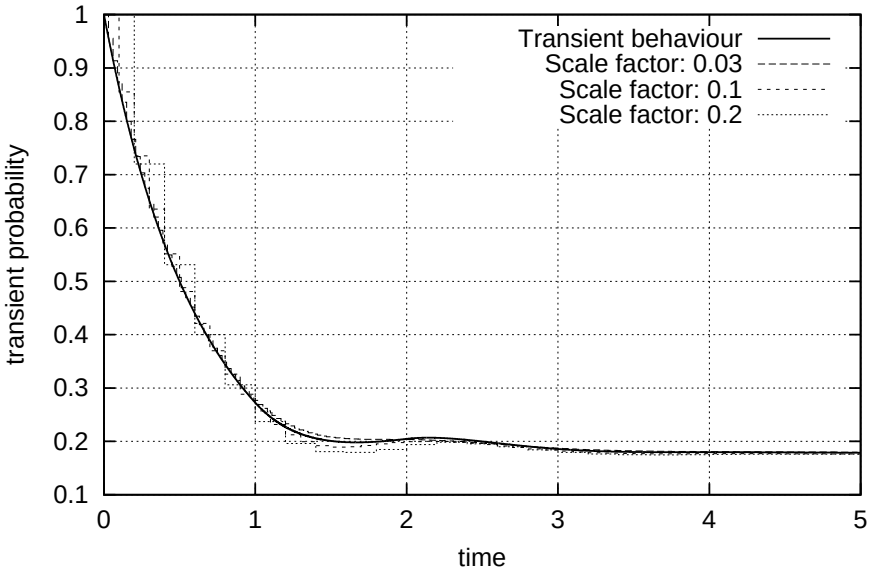


Figure 10.18: Approximating transient probabilities

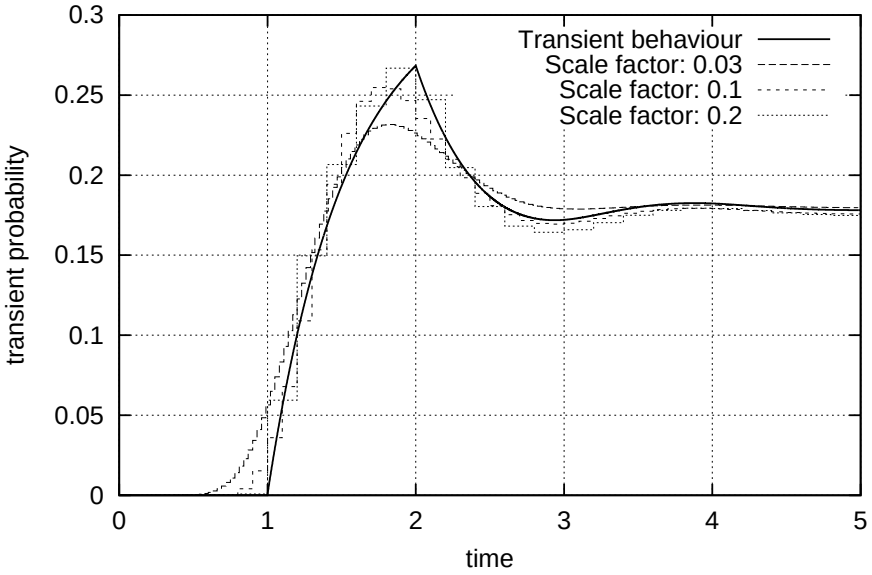


Figure 10.19: Approximating transient probabilities

Chapter 11

Moment Bounds for Acyclic Discrete and Continuous Phase Type Distributions of Second Order

The problem of matching moments to phase type (PH) distributions occurs in many applications. Often, low dimensions of the selected distributions are desired. It is obvious that the three parameters of acyclic PH distributions of second order – be they continuous (ACPH(2)) or discrete (ADPH(2)) – can be fitted to three given moments provided that these are feasible. For both types of PH distributions, this chapter provides the precise permissible ranges by giving the immanent lower and upper (if existing) bounds for the first three moments. For moments which obey these bounds an exact and minimal (with respect to the dimension of the representation) analytic mapping of three moments into ACPH(2) or ADPH(2) is presented.

The moments of ACPH(2) and ADPH(2) distributions are subject to a few restrictions. With respect to the second moments, the squared coefficients of variation (defined as the variance of the distribution divided by the squared mean ($= {f_1}^2$)) must be greater than or equal to 0.5 for ACPH(2) [3] and for ADPH(2) the squared coefficients of variation must be greater than or equal to $0.5 - \frac{1}{f_1}$ if $2 \leq f_1$ or to $2 \cdot (f_1 - 1)$ if $1 \leq f_1 < 2$ [77].

Here we present – for both the continuous and discrete case – the bounds of the third moment as a function of the first two, namely in the respective full range of the squared coefficient of variation (including the hypoexponential/hypogeometric region).

11.1 The canonical ACPH(2) distribution and moment bounds

Generally, the random variable X associated with an arbitrary continuous PH distribution function $F_X(t)$ represents the time to absorption in a finite continuous-time Markov chain (with n transient states), or more formally: $F_X(t) = 1 - \boldsymbol{\alpha} e^{\boldsymbol{T}t} \boldsymbol{e}$. The

nonsingular $(n \times n)$-matrix $\boldsymbol{T}$ denotes the generator of the transient Markov chain ($(\boldsymbol{T})_{ii} \leq 0$ for $1 \leq i \leq n$, $(\boldsymbol{T})_{ij} \geq 0$ for $i \neq j$ so that $(\boldsymbol{T}\boldsymbol{e})_i \leq 0$, but $\boldsymbol{T}\boldsymbol{e} \neq \boldsymbol{0}$). The n-dimensional vector $\boldsymbol{\alpha}$ is the initial distribution and $\boldsymbol{e}$ is the n-dimensional vector of ones. Note that the tuple $(\boldsymbol{\alpha}, \boldsymbol{T})$ completely characterizes the continuous PH distribution with power moments

$$m_i = E[X^i] = i!\, \boldsymbol{\alpha}\, (-\boldsymbol{T})^{-i} \boldsymbol{e} \ . \tag{11.1}$$

In this paper, we focus on the following specific class of continuous PH distributions: First, we consider the subclass of acyclic distributions, which admits minimal representations called canonical forms [25]. These distributions can be encoded by acyclic graphs so that $\boldsymbol{T}$ is an upper triangular matrix (with an appropriate ordering of the n states). Second, we study ACPH distributions of order 2, i.e., $s = 2$. The canonical representation $(\boldsymbol{\alpha}, \boldsymbol{T})$ is then given by

$$\boldsymbol{\alpha} = (p, 1-p) \quad \text{and} \quad \boldsymbol{T} = \begin{vmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{vmatrix} \ , \tag{11.2}$$

where $0 \leq p \leq 1$ and $0 < \lambda_1 \leq \lambda_2$. Figure 11.1 shows the related graph, where the filled circle depicts the absorbing state.

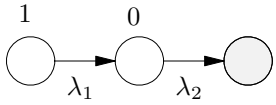


Figure 11.1: Canonical form of ACPH(2) distribution

Of course, the power moments can be computed directly from (11.1), but it might be more intuitive to have a look at the Laplace transform of the random variable X:

$$G_X(s) = E[e^{-sX}] = p\frac{\lambda_1}{s+\lambda_1}\frac{\lambda_2}{s+\lambda_2} + (1-p)\frac{\lambda_2}{s+\lambda_2} \ .$$

The first three power moments of X are:

$$m_1 = E[X] = -\frac{d}{ds}G_X(s)|_{s=0} = \frac{\lambda_1 + p\,\lambda_2}{\lambda_1\lambda_2} \ , \tag{11.3}$$

$$m_2 = E[X^2] = \frac{d^2}{ds^2}G_X(s)|_{s=0} = \frac{2\,({\lambda_1}^2 + p\,\lambda_1\lambda_2 + p\,{\lambda_2}^2)}{{\lambda_1}^2{\lambda_2}^2} \ , \tag{11.4}$$

$$m_3 = E[X^3] = -\frac{d^3}{ds^3}G_X(s)|_{s=0} = \frac{6\,({\lambda_1}^3 + p\,{\lambda_1}^2\lambda_2 + p\,\lambda_1{\lambda_2}^2 + p\,{\lambda_2}^3)}{{\lambda_1}^3{\lambda_2}^3} \ . \tag{11.5}$$

Having gone from the distribution parameters p, λ_1, λ_2 to the power moments m_1, m_2, m_3, we would now like to find the reverse way (and succeed therein in Section 11.3). First of all, we observe that not any arbitrary triple (m_1, m_2, m_3) can be transformed back to some valid parameter set $(p, \lambda_1, \lambda_2)$. For example, nonpositive values for m_1 will obviously render the triple infeasible (since ACPH(2) distributions describe nonnegative random variables). Analogously, the other moments are

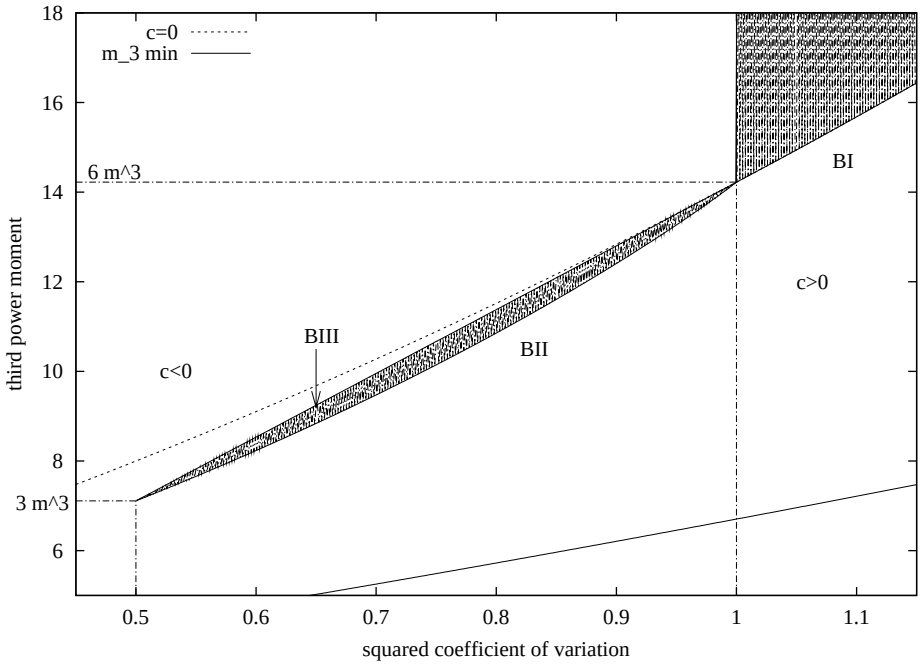


Figure 11.2: Third-moment bounds for ACPH(2) distribution with $m\,(=m_1)=\frac{4}{3}$

bounded - possibly from more than one side. For the second moment, Aldous and Shepp provided the (order-independent) result that "the least variable phase-type distribution is Erlang" [3]. In other words and for $s=2$, the squared coefficient of variation c_X^2 of an ACPH(2) distribution must satisfy:

$$c_X^2 = \frac{m_2}{{m_1}^2} - 1 \geq 0.5 \quad \Leftrightarrow \quad m_2 \geq 1.5\,{m_1}^2 \ .$$

Since the ACPH(2) class contains the Erlang-2 distribution ($p=1, \lambda_1=\lambda_2$), this bound is tight. It can be obtained from the formulae (11.3) and (11.4) by equating to 0 the derivative of m_2 with respect to m_1 (after having exploited the structural information $p=1, \lambda_1=\lambda_2$). Similarly, the bounds for the third moment m_3 can be found, where it turns out however that the bound behavior strongly depends on the precise value of c_X^2 or - expressed alternatively - on the relationship between the first two power moments. Figure 11.2 illustrates the typical features of the third-moment bounds for a fixed value $m_1=\frac{4}{3}$ ($=m$ in the figure). While for $c_X^2>1$ only a lower bound exists, both a lower and an upper bound limit m_3 to a rather small region for $0.5 \leq c_X^2 \leq 1$.

Table 11.1 gives the derived functions of the bounds along with the respectively employed structural information in the last column. This information documents which types of ACPH(2) distributions attain the specific bounds. At $c_X^2=1, m_3=6\,{m_1}^3$, we have a singular point. At this point the one-dimensional exponential distribution with parameter $\lambda_2=\frac{1}{m_1}$ ($p=0, \lambda_1=$ irrelevant) fulfills the conditions

mom.	condition	bounds	ACPH(2)
1.		$0 < m_1 < \infty$	-
2.(c_X^2)		$0.5 \leq c_X^2 < \infty$	-
3.	$0.5 \leq c_X^2 \leq 1$	$3\,{m_1}^3(3\,c_X^2 - 1 + \sqrt{2}\,(1-c_X^2)^{\frac{3}{2}}) \leq m_3 \leq 6\,{m_1}^3 c_X^2$	$\lambda_1 = \lambda_2$ (BII) $p = 1$ (BIII)
	$1 < c_X^2$	$\frac{3}{2}\,{m_1}^3(1+c_X^2)^2 \leq m_3 (< \infty)$	$\lambda_2 \to \infty$ (BI)

Table 11.1: Bounds for the first three moments of the ACPH(2) distributions

of the coordinates[1]. This point lies on the dotted line of Figure 11.2 defined by

$$c = 3\,{m_2}^2 - 2\,m_1 m_3 = 0 \quad \Leftrightarrow \quad m_3 = \frac{3}{2}\,{m_1}^3(c_X^2+1)^2 \ ,$$

which coincides with the lower bound in $c_X^2 \in (1,\infty)$. The importance of this dotted curve which separates the regions $c > 0$ and $c < 0$ will be discussed in Section 11.3. The lowest curve in Figure 11.2 marks the general lower bound for the third moment of any distribution on the nonnegative axis [82], where

$$m_1 \leq {m_2}^{\frac{1}{2}} \leq {m_3}^{\frac{1}{3}} \quad \Leftrightarrow \quad m_3 \geq {m_1}^3(1+c_X^2)^{\frac{3}{2}} \ .$$

Despite the obvious restrictions on the first three moments of ACPH(2) distributions, this subclass of continuous PH(2) distributions preserves an utmost flexibility in the sense that the presented bounds are identical with those of the more general class of matrix-exponential distributions [61].

11.2 The canonical ADPH(2) distribution and moment bounds

For the discrete case, we very much proceed along the same lines as for the continuous case - with the main difference that the factorial moments take the role of the power moments. As we will see, the bound behavior naturally bears similarities, but becomes a bit more involved. Again, we start by specializing the general notation (see [10]) of the discrete PH distributions to the canonical form of acyclic discrete PH distributions of order 2:

$$\boldsymbol{\alpha} = (p, 1-p) \quad \text{and} \quad \boldsymbol{B} = \begin{vmatrix} 1-\beta_1 & \beta_1 \\ 0 & 1-\beta_2 \end{vmatrix} \ , \tag{11.6}$$

where $0 \leq p \leq 1$ and $0 < \beta_1 \leq \beta_2 \leq 1$. Figure 11.3 displays the transient discrete-time Markov chain associated with this canonical representation. The discrete time to absorption (in unit time steps) will be denoted by the random variable N.

[1]Note that there are infinitely many ACPH(2) representation of the exponential distribution, but the minimal unique canonical representation of this distribution is the first order CPH distribution, the exponential.

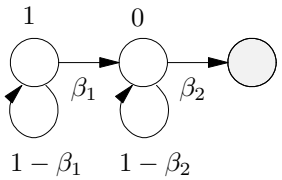


Figure 11.3: Canonical form of ADPH(2) distribution

Power moments might be derived directly from the probability mass function $f_N(k) = P\{N = k\} = \boldsymbol{\alpha}\boldsymbol{B}^{k-1}(\boldsymbol{I} - \boldsymbol{B})\boldsymbol{e}$ ($\boldsymbol{I}$ is the two-dimensional identity matrix) or indirectly via the factorial moments. These can be conveniently computed from the generator function of N

$$G_N(z) = E[z^N] = p\ \frac{\beta_1 z}{1-(1-\beta_1)z}\ \frac{\beta_2 z}{1-(1-\beta_2)z} + (1-p)\frac{\beta_2 z}{1-(1-\beta_2)z}$$

resulting in:

$$f_1 = E[N] = \frac{d}{dz}G_N(z)|_{z=1} = \frac{\beta_1 + \beta_2 p}{\beta_1\beta_2}\ , \tag{11.7}$$

$$\begin{aligned} f_2 &= E[N(N-1)] = \frac{d^2}{dz^2}G_N(z)|_{z=1} \\ &= \frac{2\left({\beta_1}^2(1-\beta_2) + p\,\beta_1\,\beta_2 + p\,{\beta_2}^2(1-\beta_1)\right)}{{\beta_1}^2\,{\beta_2}^2}\ , \end{aligned} \tag{11.8}$$

$$\begin{aligned} f_3 &= E[N(N-1)(N-2)] = \frac{d^3}{dz^3}G_N(z)|_{z=1} \\ &= \frac{6\left({\beta_1}^3(1-\beta_2)^2 + p\,\beta_1\,\beta_2\,(\beta_1 - 2\,\beta_1\,\beta_2 + \beta_2) + p\,{\beta_2}^3(1-\beta_1)^2\right)}{{\beta_1}^3\,{\beta_2}^3}\ . \end{aligned} \tag{11.9}$$

The first three factorial moments serve as the starting point on our way from such a partial description of a discrete random variable to the parameter specification (p, β_1, β_2) of the ADPH(2) canonical form. Also the moment bounds are given in the context of factorial moments. In [77], it was shown that the feasible range of the first factorial moment $f_1 \geq 1$ must be divided into two sections, in which the minimum squared coefficient of variation (scv: $c_N^2 = \frac{E[N^2]}{E[N]^2} - 1 = \frac{f_2 + f_1 - {f_1}^2}{{f_1}^2}$) follows different laws - both explicitly f_1-dependent though (see Table 11.2). These two ranges, $1 \leq f_1 < 2$ and $2 \leq f_1$, also have an effect on the third-moment behavior.

For $f_1 = \frac{4}{3}$ (< 2) - the same value as for the mean m_1 in Figure 11.2 - the third factorial moment f_1 is plotted over the squared coefficient of variation c_N^2. Although the shapes of the feasible regions of Figures 11.4 and 11.2 have much in common, several important differences are identified: First, the low-variability (here hypogeometric) range is not fixed (as to $(0.5, 1.0)$ for ACPH(2) distributions), but lies within boundaries which depend on f_1 ($1 \leq f_1 < 2$):

$$\frac{\langle f_1\rangle(1-\langle f_1\rangle)}{{f_1}^2} = \frac{(f_1-1)(2-f_1)}{{f_1}^2} = -(1 - \frac{3}{f_1} + \frac{2}{{f_1}^2}) \leq c_N^2 < 1 - \frac{1}{f_1}\ , \tag{11.10}$$

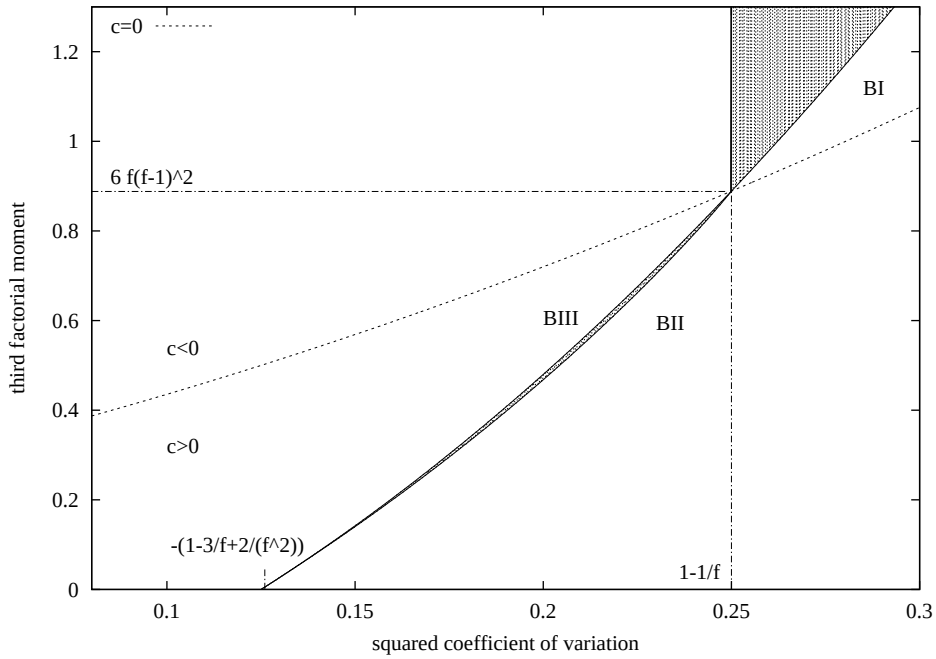


Figure 11.4: Third-moment bounds for ADPH(2) distribution with $f\,(=f_1)=\frac{4}{3}$ (<2)

where $\langle f_1\rangle$ denotes the fractional part of f_1, i.e., $\langle f_1\rangle = f_1 - \lfloor f_1\rfloor = f_1 - 1$ (since $1 < f_1 < 2$). Note that as f_1 approaches 1 or 2, the lower bound on the (nonnegative) squared coefficient of variation vanishes, i.e., $c_N^2 \geq 0$ in the limit. For $f_1 \to 1$, the ADPH(2) distributions converge towards the unit-step deterministic distribution ($p = 0, \beta_2 = 1$), while for $f_1 = 2$ (actually part of case $2 \leq f_1$), the respective minimum $c_N^2 = 0$ yields the deterministic distribution with $E[N] = 2$ ($p = 1, \beta_1 = \beta_2 = 1$). Our choice of $f_1 = \frac{4}{3}$ in Figure 11.4 imposes the strictest lower bound on c_N^2 in the range $1 \leq f_1 < 2$, i.e., the minimum squared coefficient of variation is maximal and equal to $c_N^2 = \frac{1}{8}$. The third factorial moment starts from zero at the minimum coefficient of variation and increases to $6\,f_1\,(f_1-1)^2$ for $c_N^2 \to 1 - \frac{1}{f_1}$, where only relatively little variation is tolerated in between.

With f_1 entering the range $2 \leq f_1$, the expression $(1 - \frac{3}{f_1} + \frac{2}{{f_1}^2})$ (see formula (11.10)) turns nonnegative and - as indicated above - the lower bound of the squared coefficient of variation is replaced by $0.5 - \frac{1}{f_1}$. Nevertheless, the expression in brackets retains an important role also in the case $2 \leq f_1$, which is illustrated by Figure 11.5 for the specific $f_1 = \frac{8}{3}$ ($= f$ in the figure). Note that this doubled f_1 stipulates the same value for the minimum squared coefficient of variation as before. I.e., $0.5 - \frac{1}{f_1^{(2)}} = \frac{1}{8} = -(1 - \frac{3}{f_1^{(1)}} + \frac{2}{{f_1^{(1)}}^2})$, where $f_1^{(2)} = \frac{8}{3}$ and $f_1^{(1)} = \frac{4}{3}$.

In Figure 11.5, we observe that for $2 \leq f_1$ – as opposed to the case $1 \leq f_1 < 2$ – the third factorial moment does no longer reach down to zero at the minimum c_N^2. Furthermore, the upper bound in the low-variability range behaves differently for c_N^2 less or greater than $1 - \frac{3}{f_1} + \frac{2}{{f_1}^2} = \frac{(f_1-2)(f_1-1)}{{f_1}^2}$ (bounds BIV and BIII).

In both cases, $1 \leq f_1 < 2$ and $2 \leq f_1$ (i.e., $1 \leq f_1$) – in analogy to ACPH(2) – a

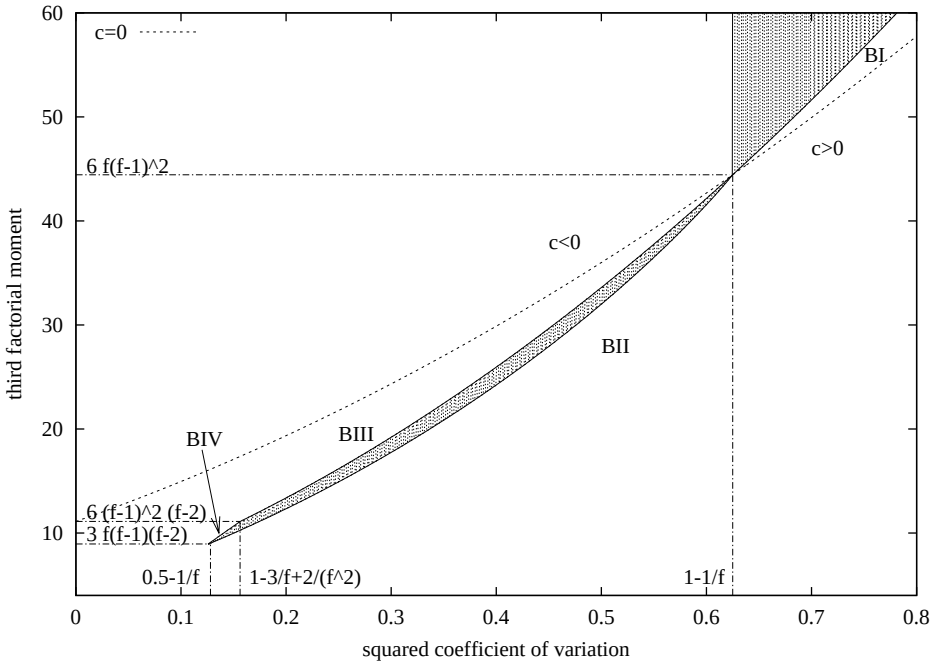


Figure 11.5: Third-moment bounds for ADPH(2) distribution with $f\,(=f_1)=\frac{8}{3}$ (>2)

singular point occurs, now at $(c_N^2 = 1-\frac{1}{f_1}, f_3 = 6\,f_1\,(f_1-1)^2)$ on the dotted lines

$$c = 3\,{f_2}^2 - 2\,f_1 f_3 = 0 \quad \Leftrightarrow \quad f_3 = \frac{3}{2}\,f_1\,\left(f_1\,(c_N^2+1)-1\right)^2 \quad .$$

The canonical representation of this point is the geometric distribution with parameter $\beta_2 = \frac{1}{f_1}$ $(p=0, \beta_1 = \text{irrelevant})$. In the hypergeometric range (i.e., $c_N^2 > 1-\frac{1}{f_1}$), only lower bounds exist for the third factorial moment for any feasible value of f_1 (see Figures 11.4 and 11.5). The exact formulae of the discussed bounds can be found in Table 11.2. They were derived in a similar manner as in the continuous case – again exploiting the structural information listed in the last column. To enhance the readability of Table 11.2, we left the variable f_2 in some expressions (instead of substituting it by $f_2 = {f_1}^2(c_N^2+1) - f_1$) and introduced the following auxiliary variable

$$g = \frac{6}{(2\,f_1+\sqrt{2\,d})^3}\left(f_1\,(2\,f_1+\sqrt{2\,d})(3\,f_2+2\,f_1)(f_2-2\,f_1+2) - 2\,{f_2}^2\,(f_2-\sqrt{2\,d})\right) \quad ,$$

where $d = 2\,{f_1}^2 - 2\,f_1 - f_2$. Variable d and the previously defined c will also appear in the moment fitting procedure to be outlined in the next section.

mom.	condition	bounds	ADPH(2)
1.		$1 \leq f_1 < \infty$	-
2.	$1 \leq f_1 < 2$	$\frac{(2-f_1)(f_1-1)}{{f_1}^2} \leq c_N^2 < \infty$	-
(c_N^2)	$2 \leq f_1$	$0.5 - \frac{1}{f_1} \leq c_N^2 < \infty$	-
3.	$1 \leq f_1 < 2$		
	$\frac{(2-f_1)(f_1-1)}{{f_1}^2} \leq c_N^2 < 1 - \frac{1}{f_1}$	$g \leq f_3$	$\beta_1 = \beta_2$ (BII)
		$\leq \frac{3\,f_2(f_2 - 2\,f_1 + 2)}{2\,(f_1 - 1)}$	$\beta_2 = 1$ (BIII)
	$2 \leq f_1$		
	$0.5 - \frac{1}{f_1} \leq c_N^2 < \frac{(f_1-2)(f_1-1)}{{f_1}^2}$	$g \leq f_3$	$\beta_1 = \beta_2$ (BII)
		$\leq 6\,{f_1}^2(f_1 - 1)c_N^2$	$p = 1$ (BIV)
	$\frac{(f_1-2)(f_1-1)}{{f_1}^2} \leq c_N^2 < 1 - \frac{1}{f_1}$	$g \leq f_3$	$\beta_1 = \beta_2$ (BII)
		$\leq \frac{3\,f_2(f_2 - 2\,f_1 + 2)}{2\,(f_1 - 1)}$	$\beta_2 = 1$ (BIII)
	$1 \leq f_1$		
	$1 - \frac{1}{f_1} \leq c_N^2$	$\frac{3\,f_2(f_2 - 2\,f_1 + 2)}{2\,(f_1 - 1)} \leq f_3$	$\beta_2 = 1$ (BI)

Table 11.2: Bounds for the first three moments of the ADPH(2) distributions

11.3 Method of moments for ACPH(2) and ADPH(2) distributions

The procedures of this section provide the best possible mapping of the first three moments of a generally distributed random variable into a PH representation of order 2 – in both the continuous and discrete setting. Though starting from power or factorial moments, respectively, the corresponding formulae for both cases resemble one another so strongly that the two methods of moments are treated in parallel. The moment bounds of the previous sections are crucial for these procedures in that they determine whether the given triple of moments is feasible or not.

Let us begin with the former situation (feasibility), in which all three moments fall into the related intervals within the derived boundaries. Solving each system of nonlinear algebraic equations – either (11.3), (11.4), (11.5) or (11.7), (11.8), (11.9) – for the parameters of the ACPH(2) or ADPH(2) distributions, respectively[2], one may finally arrive at the moment-fitting procedures of Table 11.3. In particular, the distinction of cases $c < 0, c = 0, c > 0$ can be graphically reproduced in Figures 11.2, 11.4 and 11.5. In this context, notice the congruent expressions for c in the discrete and continuous cases regardless of power or factorial moments.

[2]For example, we applied the Mathematica package with subsequent manipulations.

ACPH(2)	**ADPH(2)**
Power moments	Factorial moments
$m_1 = E[X],\ m_2 = E[X^2],$ $m_3 = E[X^3]$	$f_1 = E[N],\ f_2 = E[N(N–1)],$ $f_3 = E[N(N–1)(N–2)]$
Auxiliary variables	
$d = 2\,{m_1}^2 - m_2,\ c = 3\,{m_2}^2 - 2\,m_1\,m_3$ $b = 3\,m_1\,m_2 - m_3$ $a = b^2 - 6\,c\,d$	$d = 2\,{f_1}^2 - 2\,f_1 - f_2,\ c = 3\,{f_2}^2 - 2\,f_1\,f_3$ $b = 3\,f_1\,f_2 - 6\,(f_1 + f_2 - {f_1}^2) - f_3$ $a = b^2 - 6\,c\,d$
Moments fitting	
$m_1,\ m_2,\ m_3 \to p,\ \lambda_1,\ \lambda_2$	$f_1,\ f_2,\ f_3 \to p,\ \beta_1,\ \beta_2$
if $c > 0$	
$p = \dfrac{-b + 6\,m_1\,d + \sqrt{a}}{b + \sqrt{a}}$ $\lambda_1 = \dfrac{b - \sqrt{a}}{c}, \lambda_2 = \dfrac{b + \sqrt{a}}{c}$	$p = \dfrac{-b + 6\,f_1\,d + \sqrt{a}}{b + \sqrt{a}}$ $\beta_1 = \dfrac{b - \sqrt{a}}{c}, \beta_2 = \dfrac{b + \sqrt{a}}{c}$
if $c < 0$	
$p = \dfrac{b - 6\,m_1\,d + \sqrt{a}}{-b + \sqrt{a}}$ $\lambda_1 = \dfrac{b + \sqrt{a}}{c}, \lambda_2 = \dfrac{b - \sqrt{a}}{c}$	$p = \dfrac{b - 6\,f_1\,d + \sqrt{a}}{-b + \sqrt{a}}$ $\beta_1 = \dfrac{b + \sqrt{a}}{c}, \beta_2 = \dfrac{b - \sqrt{a}}{c}$
if $c = 0$	
$p = 0,\ \lambda_1 = 0,\ \lambda_2 = \dfrac{1}{m_1}$ (exp.)	$p = 0,\ \beta_1 = 0,\ \beta_2 = \dfrac{1}{f_1}$ (geom.)

Table 11.3: Moment fitting with ACPH(2) and ADPH(2) distributions

We now turn to the situation with initially infeasible moments. Generally – and according to [45] –, there are essentially three approaches to handle this problem:

option 1: matching the first two moments instead of three

option 2: adjusting the moments to be matched

option 3: using alternative three-moment matching techniques usually (in our case definitely) leading to higher-order (PH) representations

The presented moment bounds for ACPH(2) and ADPH(2) distributions make option 2 superior over option 1. They enable us to select the optimal moment-boundary values to enforce feasibility. In practice, one will merely set the third moment to the closest boundary value (computed for feasible first two moments), if the third power/factorial moment exceeds the limits. Moment fitting then follows Table 11.3. If the second power/factorial moment does not comply with the moment bounds significantly, avoiding higher-order representations hardly seems reasonable. For example, analytic option 3 alternatives are discussed in [45, 40] for the continuous and in [10] for the discrete case.

Chapter 12

Output Models of MAP/PH/1(/K) Queues for an Efficient Network Decomposition

For non-trivial (open) queueing networks, traffic-based decomposition often represents the only feasible solution method besides simulation. The network is partitioned into individual nodes which are analyzed in isolation with respect to approximate internal traffic representations. Since the correlations of network traffic may have a considerable impact on performance measures, they must be captured to some extent by the employed traffic descriptors. The decomposition methodology presented there is based on Markovian arrival processes (MAPs), whose correlation structure is determined from the busy-period behavior of the upstream queues. The resulting compact MAPs in connection with sophisticated moment matching techniques allow an efficient decomposition of large queueing networks.

Although most decomposition algorithms (e.g., [48, 83, 37, 72]) are based on renewal processes as traffic descriptors for ease of tractability, one should not neglect the correlation structures of the external and internal flows. These correlations have been demonstrated to significantly influence performance measures especially for bursty input traffic. For example, a simulation study [56] showed that the average waiting time in a queue with highly correlated arrivals can be 40 times larger than in the uncorrelated case. The following decomposition methods take into account the traffic correlations in different ways. In [4] truncation techniques for the infinite output MAP of a MAP/PH/1 queue are studied. For dual tandem queues, very good numerical results are reported. However, depending on the number of phases/states of the service distribution of the queue and its arrival process, the truncated MAPs still become quite large in general. More precisely, their orders depend multiplicatively on the orders of the PH distribution and the input MAP. Similar observations hold for the closely related and more flexible way [71] to obtain finite MAP representations of the departure processes of MAP/MAP/1 queues. While these truncated MAPs have been shown to match a size-dependent number of coefficients of correlations of lagged interdeparture times exactly [36], a different approach to output modeling is to fit a predefined set of traffic descriptors to selected performance indices of the true departure process.

The adopted approach is completely different from the previously applied ones in that it does not attempt to capture single elements of the correlation structure of the departure process directly (e.g., by matching the first coefficients of correlation). Instead the parameters of a MAP are chosen so that this traffic descriptor reflects the busy-period behavior of the considered queue.

12.1 MAP-Based Decomposition

Traffic-based decomposition assumes that dependences between queues are sufficiently conveyed by the traffic characterizations. In the first phase, the algorithm determines the parameters of these internal traffic representations. In the second phase, it derives performance indices for single nodes and network-wide results.

Our methodology progresses in the same way. The order in which the isolated queues are analyzed does not deviate from other (iterative) approaches. Without feedback loops, each node only needs to be treated once provided that the nodes have been reordered in advance with respect to external inputs and the network structure [38]; in the presence of feedback loops, the algorithm iterates over those nodes included therein until the rates and the squared coefficients of variation of the internal arrival flows, i.e., MAPs in our case, have converged. As for any other decomposition algorithm of this type, no general statements on the existence and uniqueness of a fixed point can currently be made for this iteration scheme.

In general, the following three operations are performed at each node: 1) MAP traffic descriptors directed to the node are merged into a single input MAP. 2) The departure process of the queue is approximated as a MAP. 3) The output MAP is split into MAP substreams according to the Markovian routing. For the output approximation, matrix-analytic techniques (for MAP/PH/1(/K) systems) deliver the relevant quantities via a busy-period analysis. Corresponding procedures yield the performance measures, like the first two moments of the waiting time and queue lengths as well as throughputs and loss probabilities. Global performance indices can be derived from these quantities as in [83]. Since the splitting and merging of MAPs in the context of traffic-based decomposition have been discussed in other publications (e.g., [71, 39]), there we concentrate on the output approximations of queues. It should, however, be mentioned that the commonly used merging procedure ignores possible cross-correlations among the involved traffic processes and therefore it is not exact in case of closed queueing networks. On this assumption, merging just like splitting of MAPs are simple matrix operations [65].

For the overall algorithm to work efficiently also for larger networks, the dimensions of the block matrices in the matrix-analytic methods ought to remain in a reasonable range. The major contribution of the presented approach in this respect consists in the fact that the orders of the output MAPs depend only linearly on the orders of the input MAP and the PH service distribution of the considered queue. Moreover, these traffic descriptors can be further compressed due to their structure: so more compact PH representations of the residual arrival time and/or of the service time may be sought for based on their moments[1]. Even more fundamentally,

[1]If the service time is specified by its moments, PH fitting will already be necessary during node

an output approximation may decide to ignore the second and third moment of the number of customers in a busy period (as in [40]) yielding reduced MAP skeletons, which are sufficiently accurate in many cases. All of the related moment matching techniques may be combined in comprehensive heuristics (which will also take into account merging situations, i.e., the sizes of the involved MAPs, see e.g., [39]) in order to enforce that the dimensions of the mentioned block matrices range below a given upper bound. This bound reflects the user's choice in the trade-off between accuracy and efficiency.

As described above, analytic moment fitting procedures occur in various situations of the proposed methodology – be it for continuous or discrete random variables – and impart a lot of flexibility to the MAP-based decomposition. Here we apply the methods, presented in the previous chapter, for matching an acyclic continuous/discrete phase-type distribution of order 2 to three given (power/factorial) moments, respectively.

If the second power/factorial moment falls outside the feasible range, we will resort to specific higher-order representations (see [40] for the continuous and [10] for the discrete case) during the moment matching to achieve an exact fit in the first two moments. If the third power/factorial moment does not fulfill the requirements, one option is to set it to the closest boundary value (computed for the given first two moments).

We once again point out the importance of compact representations of service/idle times or number of customers in a busy period for an efficient MAP-based decomposition. The applied procedures provide the best possible mapping of three moments into a continuous or discrete PH representation of order 2.

12.2 Markovian Arrival Processes (MAPs)

Markovian arrival processes are a rich subclass of Markov renewal processes with high popularity in the research community of traffic engineering. Let us consider a MAP with a finite state space of size m. This parameter is also called the order of the MAP and determines the dimensions of the matrices and vectors introduced below. Transitions of a MAP are distinguished whether they cause an arrival or not. Associated rates are correspondingly grouped into the two matrices $\boldsymbol{D_1}$ and $\boldsymbol{D_0}$:

- $\boldsymbol{D_1}$ is a nonnegative $(m \times m)$-rate matrix.
- $\boldsymbol{D_0}$ of the same dimension has negative diagonal elements and nonnegative off-diagonal elements.
- The irreducible infinitesimal generator $\boldsymbol{D}$ is defined by $\boldsymbol{D} = \boldsymbol{D_0} + \boldsymbol{D_1}$.

We require that $\boldsymbol{D_0}$ is invertible. Then $\boldsymbol{D} \neq \boldsymbol{D_0}$, i.e., the arrival process does not terminate. With probability $\frac{(\boldsymbol{D_0})_{ik}}{(-\boldsymbol{D_0})_{ii}}$ $(1 \leq i, k \leq m, k \neq i)$, there will be a transition from state i to state k without an arrival. With probability $\frac{(\boldsymbol{D_1})_{ik}}{(-\boldsymbol{D_0})_{ii}}$ $(1 \leq i, k \leq m)$, there will be a transition from state i to state k accompanied by an arrival.

analysis.

For the underlying Markov process with CTMC generator $\boldsymbol{D}$, we define the stationary probability vector $\boldsymbol{\pi}$ by

$$\boldsymbol{\pi D} = \mathbf{0}\ , \quad \boldsymbol{\pi e} = 1\ ,$$

where $\boldsymbol{e} = (1, \ldots, 1)^T$ is the column vector of ones.

The mean arrival rate and squared coefficient of variation of a MAP are [66]

$$\begin{aligned} \lambda_{MAP} &= \frac{1}{\mathrm{E}\left[\Gamma\right]} = \boldsymbol{\pi D_1 e} && \text{and} \\ c^2_{MAP} &= \frac{\mathrm{E}\left[\Gamma^2\right]}{(\mathrm{E}\left[\Gamma\right])^2} - 1 = 2\lambda_\Gamma \boldsymbol{\pi}(-\boldsymbol{D_0})^{-1}\boldsymbol{e} - 1, && \text{respectively,} \end{aligned} \tag{12.1}$$

where Γ denotes the marginal interevent (i.e., interarrival or interdeparture) time of the traffic process. In general, the interevent times of a MAP are correlated. The non-zero lag coefficients of correlation $\rho_\Gamma(j)$ $(j > 0)$ of an interval-stationary MAP can be derived [66]:

$$\rho_\Gamma(j) = \frac{\mathrm{E}\left[\Gamma_\odot \Gamma_{\odot+j}\right] - \mathrm{E}\left[\Gamma\right]^2}{\mathrm{E}\left[\Gamma^2\right] - \mathrm{E}\left[\Gamma\right]^2} = \frac{\lambda_\Gamma \boldsymbol{\pi}[(-\boldsymbol{D_0})^{-1}\boldsymbol{D_1}]^j(-\boldsymbol{D_0})^{-1}\boldsymbol{e} - 1}{2\lambda_\Gamma \boldsymbol{\pi}(-\boldsymbol{D_0})^{-1}\boldsymbol{e} - 1}\ .$$

$\Gamma_\odot$ and $\Gamma_{\odot+j}$ denote any two intervals j lags apart in the sequence of interevent times.

Many familiar arrival processes represent special cases of MAPs, among them Poisson processes, MMPPs, and – most important in view of MAP-based decomposition for general queueing networks – the superpositions of independent MAPs.

In steady state, the marginal distribution of the interevent time of a MAP is phase-type distributed with $\boldsymbol{\alpha} = \frac{\boldsymbol{\pi D_1}}{\boldsymbol{\pi D_1 e}}$ and $\boldsymbol{T} = \boldsymbol{D_0}$. On the other hand, the MAP notation of the $(\boldsymbol{\alpha}, \boldsymbol{T})$ PH renewal process is: $\boldsymbol{T} = \boldsymbol{D_0}$, $\boldsymbol{D_1} = (-\boldsymbol{Te})\boldsymbol{\alpha}$.

12.3 Busy-Period Analysis of MAP/PH/1(/K) Queues

The analytical tractability of MAPs manifests itself in efficient computational procedures of the matrix-analytic approach to queueing systems, which starts from a description of the level-defining queue length process as a quasi-birth-death process (QBD, [65]). We exploit corresponding methods for the proposed decomposition, where all nodes of the network are analyzed as MAP/PH/1 or MAP/PH/1/K systems. We adopt the following notation:

K the size of a finite buffer including the (single) server place

S the random variable for PH service time with representation $(\boldsymbol{\alpha}, \boldsymbol{T})$

N the number of customers served during a busy period with conditional factorial moments $\boldsymbol{\varphi_1}, \boldsymbol{\varphi_2}, \boldsymbol{\varphi_3}$ (defined as column vectors)

$\overline{\mathbf{y}} = (\boldsymbol{y_0}, \boldsymbol{y_1}, ..., \boldsymbol{y_K})$ the stationary queue length distribution (qld) at arbitrary time

$\boldsymbol{x_0}$ the stationary probabilities that a departure leaves behind an empty system

Throughout this chapter, subscripts A/S and superscripts $(A)/(S)$ indicate affiliation to the arrival process or service time, respectively. The scalars m_A and m_S are the orders of the input MAP $(\boldsymbol{D_0^{(A)}}, \boldsymbol{D_1^{(A)}})$ and of the PH service time distribution, which will also be denoted by $\boldsymbol{D_0^{(S)}} = \boldsymbol{T}$ and $\boldsymbol{D_1^{(S)}} = (-\boldsymbol{Te})\boldsymbol{\alpha}$ in the chosen QBD notation. Let $\rho = \lambda_A \cdot \mathrm{E}[S] = \boldsymbol{\pi D_1^{(A)} e} \cdot \boldsymbol{\alpha}(-\boldsymbol{T})^{-1}\boldsymbol{e}$ be the offered load of the queueing system with the following QBD generator matrix of block tridiagonal structure:

$$\widetilde{\boldsymbol{D}} = \begin{bmatrix} \widetilde{\boldsymbol{A}}_{\boldsymbol{1}}^{(\boldsymbol{0})} & \widetilde{\boldsymbol{A}}_{\boldsymbol{0}} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \widetilde{\boldsymbol{A}}_{\boldsymbol{2}} & \widetilde{\boldsymbol{A}}_{\boldsymbol{1}} & \widetilde{\boldsymbol{A}}_{\boldsymbol{0}} & \ddots & \vdots \\ \boldsymbol{0} & \ddots & \ddots & \ddots & \boldsymbol{0} \\ \vdots & \ddots & \widetilde{\boldsymbol{A}}_{\boldsymbol{2}} & \widetilde{\boldsymbol{A}}_{\boldsymbol{1}} & \widetilde{\boldsymbol{A}}_{\boldsymbol{0}} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \widetilde{\boldsymbol{A}}_{\boldsymbol{2}} & \widetilde{\boldsymbol{A}}_{\boldsymbol{1}}^{(\boldsymbol{K})} \end{bmatrix} \quad \text{with} \quad \begin{aligned} \widetilde{\boldsymbol{A}}_{\boldsymbol{1}}^{(\boldsymbol{0})} &= \boldsymbol{D_0^{(A)}} \otimes \boldsymbol{I} \\ \widetilde{\boldsymbol{A}}_{\boldsymbol{0}} &= \boldsymbol{D_1^{(A)}} \otimes \boldsymbol{I} \\ \widetilde{\boldsymbol{A}}_{\boldsymbol{1}} &= \boldsymbol{D_0^{(A)}} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{D_0^{(S)}} \\ \widetilde{\boldsymbol{A}}_{\boldsymbol{2}} &= \boldsymbol{I} \otimes \boldsymbol{D_1^{(S)}} \\ \widetilde{\boldsymbol{A}}_{\boldsymbol{1}}^{(\boldsymbol{K})} &= \widetilde{\boldsymbol{A}}_{\boldsymbol{0}} + \widetilde{\boldsymbol{A}}_{\boldsymbol{1}} \end{aligned}$$

The operator $\otimes$ denotes the Kronecker product [35]. For queues with unlimited capacity ($K = \infty$), the bottom line of matrix $\widetilde{\boldsymbol{D}}$ becomes irrelevant and its dimension as well as the bold-faced subscript in $\boldsymbol{y_i}$ run to infinity. Our definition of the QBD implies the same dimensions for the vectors $\boldsymbol{y_i}$ and $\boldsymbol{x_0}$, namely $m_A \cdot m_S$, which also is the dimension of each block row/level of matrix $\widetilde{\boldsymbol{D}}$. The matrix-analytic techniques [65, 55] efficiently compute various kinds of qlds (e.g., $\overline{\mathbf{y}}$), their moments and many other performance measures, like loss probabilities, etc. Formulae for the first two moments of the waiting time can be found in [39, 40]. In view of the output approximation in the next section, we discuss here how the moments of N – the number of customers served in a busy period – are determined for MAP/PH/1 and MAP/PH/1/K systems.

12.3.1 MAP/PH/1 queue: number of customers in a busy period

In order to obtain the generating function of the random variable N, we examine the discrete-time Markov chain (DTMC with transition probability matrix $\boldsymbol{\Pi}$) embedded in the QBD at the epochs of level switching:

$$\boldsymbol{\Pi} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A_0^{(0)}} & \boldsymbol{0} & \boldsymbol{0} & \cdots \\ \boldsymbol{A_2} & \boldsymbol{0} & \boldsymbol{A_0} & \boldsymbol{0} & \cdots \\ \boldsymbol{0} & \boldsymbol{A_2} & \boldsymbol{0} & \boldsymbol{A_0} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad \text{with} \quad \begin{aligned} \boldsymbol{A_0^{(0)}} &= (-\widetilde{\boldsymbol{A}}_{\boldsymbol{1}}^{(\boldsymbol{0})})^{-1}\,\widetilde{\boldsymbol{A}}_{\boldsymbol{0}} \\ \boldsymbol{A_0} &= (-\widetilde{\boldsymbol{A}}_{\boldsymbol{1}})^{-1}\,\widetilde{\boldsymbol{A}}_{\boldsymbol{0}} \\ \boldsymbol{A_2} &= (-\widetilde{\boldsymbol{A}}_{\boldsymbol{1}})^{-1}\,\widetilde{\boldsymbol{A}}_{\boldsymbol{2}} \end{aligned}$$

Furthermore, we define $f_{ij}(n) = P\{N = n, Z_{\gamma_0^{(1,i)}} = (0, j) \,|\, Z_0 = (1, i)\}$ and matrix $\widetilde{\boldsymbol{F}}(n) = \{f_{ij}(n)\}$ $(1 \le i, j \le m_A \cdot m_S)$, where Z_u stands for the state of the DTMC in terms of a level number and a block matrix index at the uth step. The stopping time $\gamma_0^{(1,i)}$ specifies the occurrence of the transition that ends the busy period having

started in $Z_0 = (1, i)$. The conditional generating function $\boldsymbol{F}(z)$ of the number of customers served in a busy period is given by

$$\boldsymbol{F}(z) = \sum_{n=1}^{\infty} \widetilde{\boldsymbol{F}}(n) \cdot z^n = z\,\boldsymbol{A_2} + \boldsymbol{A_0}\,\boldsymbol{F}(z)^2 \qquad \text{(see [58])} \ . \tag{12.2}$$

Note that $\boldsymbol{F}(1) = \boldsymbol{G}$, where $\boldsymbol{G}$ is the well-known fundamental-period matrix of both the DTMC and CTMC above – the key ingredient for the computational procedures of the matrix-analytic approach (e.g., see [54] for its computation). Since we assume $\rho < 1$ for the infinite-buffer queue (i.e., stability), $\boldsymbol{G}$ is a stochastic matrix (i.e., $\boldsymbol{Ge} = \boldsymbol{e}$).

Now, we derive the first three conditional factorial moments $\boldsymbol{\varphi_1}, \boldsymbol{\varphi_2}$ and $\boldsymbol{\varphi_3}$ of random variable N. For notational convenience, let $\boldsymbol{F^{(n)}} = \frac{d^n}{dz^n}\boldsymbol{F}(z)|_{z=1}$ $(n \geq 0$, where $\boldsymbol{F^{(0)}} = \boldsymbol{G})$. The derivatives of $\boldsymbol{F}(z)$ at $z = 1$ can be written in the general form (where $I_{\{\bullet\}}$ is the indicator of event $\bullet$):

$$\boldsymbol{F^{(\ell)}} = I_{\{\ell \in \{0,1\}\}} \cdot \boldsymbol{A_2} + \boldsymbol{A_0} \cdot \sum_{i=0}^{\ell} \begin{pmatrix} \ell \\ i \end{pmatrix} \boldsymbol{F^{(\ell-i)}}\,\boldsymbol{F^{(i)}} \qquad (\ell \geq 0) \ . \tag{12.3}$$

Algebraic manipulations yield the following simple iterative procedures for $\boldsymbol{F^{(1)}}$ (to be determined first) and $\boldsymbol{F^{(2)}}$ assuming $\boldsymbol{G}$ is known:

$$\begin{aligned} \boldsymbol{F^{(1)}_{m+1}} &= (\boldsymbol{I} - \boldsymbol{A_0 G})^{-1} \left(\boldsymbol{A_2} + \boldsymbol{A_0 F^{(1)}_m G}\right) \ , \\ \boldsymbol{F^{(2)}_{m+1}} &= (\boldsymbol{I} - \boldsymbol{A_0 G})^{-1} \boldsymbol{A_0} \left(\boldsymbol{F^{(2)}_m G} + 2\boldsymbol{F^{(1)}}^2\right) \ , \end{aligned}$$

with initial values $\boldsymbol{F^{(1)}_0} = \boldsymbol{F^{(2)}_0} = \boldsymbol{0}$.

Finally, vectors $\boldsymbol{\varphi_i} = \boldsymbol{F^{(i)}e}$ $(i = 1, 2, 3)$ for the conditional factorial moments are obtained from (12.2) as

$$\begin{aligned} \boldsymbol{\varphi_1} &= \{\ E[N|Z_0 = (1,i)]\ \} = (\boldsymbol{I} - \boldsymbol{A_0} - \boldsymbol{A_0 G})^{-1} \boldsymbol{A_2 e} \ , \\ \boldsymbol{\varphi_2} &= \{\ E[N(N-1)|Z_0 = (1,i)]\ \} = 2\,(\boldsymbol{I} - \boldsymbol{A_0} - \boldsymbol{A_0 G})^{-1} \boldsymbol{A_0 F^{(1)} \varphi_1} \ , \\ \boldsymbol{\varphi_3} &= \{\ E[N(N-1)(N-2)|Z_0 = (1,i)]\ \} \\ &= 3\,(\boldsymbol{I} - \boldsymbol{A_0} - \boldsymbol{A_0 G})^{-1} \boldsymbol{A_0} (\boldsymbol{F^{(2)} \varphi_1} + \boldsymbol{F^{(1)} \varphi_2}) \ . \end{aligned}$$

Note that (12.3) allows to compute the higher moments in a similar way, and to calculate the vectors of the first ℓ factorial moments we need to compute matrices $\boldsymbol{F^{(0)}} = \boldsymbol{G}, \ldots, \boldsymbol{F^{(\ell-1)}}$.

12.3.2 MAP/PH/1/K queue: number of customers in a busy period

Again, we start from the DTMC embedded in the QBD. The quadratic transition probability matrix $\boldsymbol{\Pi}$ ends with the $(K+1)$st block row (i.e., the one belonging to level K), in which the next to last block – the only nonzero block in the last row – has to be replaced by $\boldsymbol{A_2^{(K)}} = (-\widetilde{\boldsymbol{A}}_{\boldsymbol{1}}^{(\boldsymbol{K})})^{-1} \widetilde{\boldsymbol{A}}_{\boldsymbol{2}}$. Determining the conditional factorial

moments of N for the finite-buffer queue proceeds very much along the same lines as for the MAP/PH/1 system. But now – since the busy-period behavior is no longer level-independent – the corresponding definitions are expanded by a capacity information.

$$\boldsymbol{\Pi} = \begin{bmatrix} \mathbf{0} & \boldsymbol{A}_0^{(0)} & \mathbf{0} & \dots & \mathbf{0} \\ \boldsymbol{A_2} & \mathbf{0} & \boldsymbol{A_0} & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & \boldsymbol{A_2} & \mathbf{0} & \boldsymbol{A_0} \\ \mathbf{0} & \dots & \mathbf{0} & \boldsymbol{A}_2^{(K)} & \mathbf{0} \end{bmatrix} \quad \text{with} \quad \begin{array}{rcl} \boldsymbol{A}_0^{(0)} & = & (-\widetilde{\boldsymbol{A}}_1^{(0)})^{-1}\,\widetilde{\boldsymbol{A}}_0 \\ \boldsymbol{A_0} & = & (-\widetilde{\boldsymbol{A}}_1)^{-1}\,\widetilde{\boldsymbol{A}}_0 \\ \boldsymbol{A_2} & = & (-\widetilde{\boldsymbol{A}}_1)^{-1}\,\widetilde{\boldsymbol{A}}_2 \\ \boldsymbol{A}_2^{(K)} & = & (-\widetilde{\boldsymbol{A}}_1^{(K)})^{-1}\,\widetilde{\boldsymbol{A}}_2 \end{array}$$

Consequently, we have $f_{ij}(n,k) = P\{N = n, Z_{\gamma_0^{(1,i)}} = (0,j) \,|\, Z_0 = (1,i), \Delta = k\}$ and $\widetilde{\boldsymbol{F}}(n,k) = \{f_{ij}(n,k)\}$, where the variable Δ counts the number of levels starting from the current level to the greatest one. In analogy to (12.2), the conditional generating function $\boldsymbol{F}(z)$ of the number of customers served in a busy period of a MAP/PH/1/K system is given by:

$$\boldsymbol{F}(z,k) = \sum_{n=1}^{\infty} \widetilde{\boldsymbol{F}}(n,k) \cdot z^n = \begin{cases} z \cdot \boldsymbol{A}_2^{(K)} & \text{if } k = 1 \\ z\,\boldsymbol{A_2} + \boldsymbol{A_0}\,\boldsymbol{F}(z,k-1)\,\boldsymbol{F}(z,k) & \text{if } k > 1 \end{cases} \tag{12.4}$$

With $\boldsymbol{F}_{\boldsymbol{k}}^{(\boldsymbol{n})} = \frac{d^n}{dz^n}\boldsymbol{F}(z,k)|_{z=1}$ $(n \geq 0,$ where $\boldsymbol{F}_{\boldsymbol{k}}^{(\boldsymbol{0})} = \boldsymbol{F}(1,k))$, the derivatives are $(\ell \geq 0)$

$$\boldsymbol{F}_{\boldsymbol{k}}^{(\boldsymbol{\ell})} = \begin{cases} I_{\{\ell \in \{0,1\}\}}\,\boldsymbol{A}_2^{(K)} & \text{if } k = 1 \\ I_{\{\ell \in \{0,1\}\}}\,\boldsymbol{A_2} + \boldsymbol{A_0} \cdot \sum_{i=0}^{\ell} \binom{\ell}{i} \boldsymbol{F}_{\boldsymbol{k-1}}^{(\boldsymbol{\ell-i})} \boldsymbol{F}_{\boldsymbol{k}}^{(\boldsymbol{i})} & \text{if } k > 1 \end{cases} \tag{12.5}$$

We are interested in the conditional factorial-moment vectors $\boldsymbol{\varphi_i} = \boldsymbol{F}_{\boldsymbol{K}}^{(\boldsymbol{i})}\boldsymbol{e}$ $(i = 1,2,3)$ for the subscript $k = K$. Due to the more involved successive substitution scheme, we now have to compute all four matrices $\boldsymbol{F}_{\boldsymbol{K}}^{(\boldsymbol{0})}, \boldsymbol{F}_{\boldsymbol{K}}^{(\boldsymbol{1})}, \boldsymbol{F}_{\boldsymbol{K}}^{(\boldsymbol{2})}, \boldsymbol{F}_{\boldsymbol{K}}^{(\boldsymbol{3})}$ explicitly from:

$$\boldsymbol{F}_{\boldsymbol{k}}^{(\boldsymbol{\ell})} = (\boldsymbol{I} - \boldsymbol{A_0}\,\boldsymbol{F}_{\boldsymbol{k-1}}^{(\boldsymbol{0})})^{-1} \cdot \left(I_{\{\ell \in \{0,1\}\}}\,\boldsymbol{A_2} + \boldsymbol{A_0} \cdot \sum_{i=0}^{\ell-1} \binom{\ell}{i} \boldsymbol{F}_{\boldsymbol{k-1}}^{(\boldsymbol{\ell-i})} \boldsymbol{F}_{\boldsymbol{k}}^{(\boldsymbol{i})} \right) . \tag{12.6}$$

Starting with initial values $\boldsymbol{F}_{\boldsymbol{1}}^{(\boldsymbol{0})} = \boldsymbol{F}_{\boldsymbol{1}}^{(\boldsymbol{1})} = \boldsymbol{A}_{\boldsymbol{2}}^{(\boldsymbol{K})}, \boldsymbol{F}_{\boldsymbol{1}}^{(\boldsymbol{2})} = \boldsymbol{F}_{\boldsymbol{1}}^{(\boldsymbol{3})} = \boldsymbol{0}$, this substitution scheme suggests to calculate the terms $\boldsymbol{F}_{\boldsymbol{k}}^{(\boldsymbol{\ell})}$ consecutively in the order

$$\texttt{for}\ (\ \ell = 0 \text{ to } 3\ )\ \{\ \texttt{for}\ (\ k = 2 \text{ to } K\ )\ \{\ \boldsymbol{F}_{\boldsymbol{k}}^{(\boldsymbol{\ell})} = \dots \text{ Eq. (12.6)}\ \}\ \} .$$

Finally: $\boldsymbol{\varphi_1} = \boldsymbol{F}_{\boldsymbol{K}}^{(\boldsymbol{1})}\boldsymbol{e}$, $\boldsymbol{\varphi_2} = \boldsymbol{F}_{\boldsymbol{K}}^{(\boldsymbol{2})}\boldsymbol{e}$, $\boldsymbol{\varphi_3} = \boldsymbol{F}_{\boldsymbol{K}}^{(\boldsymbol{3})}\boldsymbol{e}$.

12.3.3 Quantities needed for the output approximation

As will be outlined in the next section, the proposed output approximation for MAP/PH/1(/K>1) queues attempts to match an ADPH(2) distribution to the first

three factorial moments f_1, f_2, f_3 of the random variable N^* – the number of customers served *after* the first customer of a busy period on the condition that more than one customers are served in this busy period. The relationship between N and N^* can be formulated by

$$P\{N^* = n\} = P\{N = n+1 | N > 1\} = \frac{P\{N = n+1\}}{1 - P\{N = 1\}} \qquad (n \geq 1) \ . \tag{12.7}$$

Before converting the (conditional) factorial moments $\boldsymbol{\varphi_i}$ $(i = 1, 2, 3)$ of N into the (unconditional) factorial moments f_i $(i = 1, 2, 3)$ of N^*, we state that for $K > 1$ (including $K = \infty$) $p_{00} \equiv P\{N = 1\}$ can be computed from:

$$p_{00} = P\{N = 1\} = \frac{\boldsymbol{x_0}}{\boldsymbol{x_0 e}}(-\boldsymbol{D_0^{(A)}} \otimes \boldsymbol{I})^{-1}(\boldsymbol{D_1^{(A)}} \otimes \boldsymbol{I}) \cdot \boldsymbol{A_2 e} = \widetilde{\boldsymbol{z}}_{\boldsymbol{e}} \boldsymbol{A_2 e} \ . \tag{12.8}$$

The vector $\widetilde{\boldsymbol{z}}_{\boldsymbol{e}} = \dfrac{\boldsymbol{x_0}}{\boldsymbol{x_0 e}}(-\boldsymbol{D_0^{(A)}} \otimes \boldsymbol{I})^{-1}(\boldsymbol{D_1^{(A)}} \otimes \boldsymbol{I})$ contains the distribution of the QBD, when the first customer of a busy period has just entered the system. The elements of matrix $\boldsymbol{A_2}$ can be interpreted as the conditional probabilities that no other customers arrive before the first customer's service is finished. For MAP/PH/1(/K) queues, $\boldsymbol{x_0}$ is obtained from

$$\boldsymbol{x_0} = \frac{1}{\lambda_A(1 - P_{\text{loss}})} \boldsymbol{y_0}(-\boldsymbol{D_0^{(A)}} \otimes \boldsymbol{I}) \qquad \text{(see [18])} \ ,$$

where P_{loss} denotes the loss probability (which naturally equals 0 for $K = \infty$). Vector $\widetilde{\boldsymbol{z}}_{\boldsymbol{e}}$ will also serve to uncondition the factorial moments of N. Exploiting expression (12.7) together with some algebraic manipulations, we can transform the factorial moments of N into those of N^*:

$$f_1 = \frac{\widetilde{\boldsymbol{z}}_{\boldsymbol{e}} \boldsymbol{\varphi_1} - 1}{1 - \widetilde{\boldsymbol{z}}_{\boldsymbol{e}} \boldsymbol{A_2 e}} \ , \quad f_2 = \frac{\widetilde{\boldsymbol{z}}_{\boldsymbol{e}} \boldsymbol{\varphi_2} - 2\,\widetilde{\boldsymbol{z}}_{\boldsymbol{e}} \boldsymbol{\varphi_1} + 2}{1 - \widetilde{\boldsymbol{z}}_{\boldsymbol{e}} \boldsymbol{A_2 e}} \ ,$$
$$f_3 = \frac{\widetilde{\boldsymbol{z}}_{\boldsymbol{e}} \boldsymbol{\varphi_3} - 3\,\widetilde{\boldsymbol{z}}_{\boldsymbol{e}} \boldsymbol{\varphi_2} + 6\,\widetilde{\boldsymbol{z}}_{\boldsymbol{e}} \boldsymbol{\varphi_1} - 6}{1 - \widetilde{\boldsymbol{z}}_{\boldsymbol{e}} \boldsymbol{A_2 e}} \ .$$

12.4 Output Models for MAP/PH/1(/K>1) Queues

In the output approximation of the systems above, we extend ideas from [40], where the departure processes are approximately modeled as MAPs with an SMP skeleton. The so-called busy-period approach leads to very compact and yet sufficiently accurate MAPs with intuitive physical interpretations. In analogy to [40], we distinguish between MAP/PH/1(/K>1) and MAP/PH/1/1 queues in principle. For the latter systems, the exact departure process might often be of a reasonable size (namely $m_A \cdot (m_S + 1)$) for efficient use in a MAP-based decomposition. The proposed output approximation has been designed for queueing systems, where more than a single customer may be served during a busy period (as opposed to MAP/PH/1/1 queues). Therefore, this section is dedicated to MAP/PH/1(/K>1) systems. First, we develop a DTMC model that approximates the behavior (i.e., more precisely the first

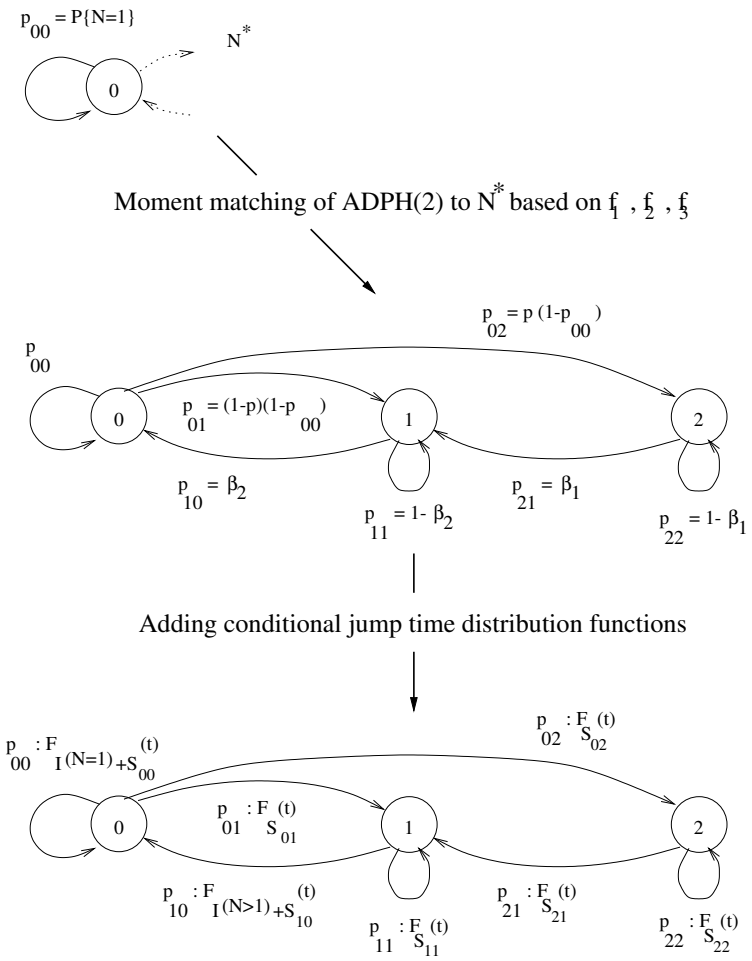


Figure 12.1: Via the DTMC to the SMP(3)

three moments, if it is possible with ADPH(2)) of the number of customers in a busy period. Enhancing this DTMC with conditional jump time distributions yields a semi-Markov process, from which the output MAP is easily derived by plugging in PH representations for service times and idle periods.

In general, the proposed output approximations are very flexible with respect to the order of the corresponding MAPs, especially due to moment-matching techniques. To avoid ambiguities, many quantities related to the output process will be indexed with subscript D or superscript (D).

12.4.1 DTMC model for the number of customers in a busy period

An event in the departure process, i.e., a customer leaving the MAP/PH/1(/K>1) system, corresponds to a transition in the proposed DTMC model. Any move to state 0 exclusively signals the departure of the first customer in any busy period. Without any additional information – as depicted in Figure 12.1 (top part) – we can state that – if the DTMC follows the (solid) arc from 0 back to the same state

– a single-customer busy period must have occurred in the queueing system (with the corresponding interdeparture time being associated with the previous transition of the DTMC). Thus, the probability $p_{00} = P\{N = 1\}$ is attributed to transition $0 \to 0$. Any path originating in state 0 and leading to non-zero states comprises as many transitions as customers succeed the first customer in a busy period with more than a single customer, before this path returns back to state 0 for the first time. So, these paths describe the random variable N^*, which might have any distribution depending on the node specifications. If its moments are not entirely out of the feasible range (which would require a higher-order approximation), we will choose to match an ADPH(2) distribution (with parameters p, β_1, β_2) to the first three factorial moments of N^* (f_1, f_2, f_3). The moment-matching procedure of Chapter 11 results in the DTMC with three states of Figure 12.1 (middle part), which approximates the behavior of the random variable N^*.

12.4.2 From the DTMC to the SMP(3)

The above DTMC contains no information on the durations of the interdeparture times. However, an output model to be used in a traffic-based decomposition must reflect that interdeparture times consist of either a single service period or of the sum of a residual arrival time and a service period. To this end, we interpret the DTMC of the previous paragraph as a DTMC embedded in an SMP with three states (SMP(3)) and attach a jump time distribution function conditioned on both the source and target state to each transition (with transition probabilities p_{ij}, see Figure 12.1 (bottom part)). The interdeparture time preceding the departure of a customer associated with a move to state 1 or state 2 equals a service period S with distribution function $F_S(t)$ (where $S = S_{01} = S_{11} = S_{21} = S_{02} = S_{22}$). $I^{(N=1)}$ and $I^{(N>1)}$ stand for the random variables of the idle periods following a busy period with a single or more than one customer, respectively. The service period of the first customer in a busy period is taken into account in the conditional jump time distribution functions $F_{I^{(N=1)}+S_{00}}(t)$ and $F_{I^{(N>1)}+S_{10}}(t)$. This SMP(3) skeleton distinguishes only two idle periods (as a simplification). Generally, an idle period depends on the state of the input process right after the departure which finished the previous busy period of the MAP/G/1(/K) queue. The state of the input process at this instant, in turn, is influenced by the number of served customers in this busy period.

12.4.3 From the SMP(3) to the output MAP

By utilizing PH representations of service times and idle periods, we now derive compact output MAPs from the SMP(3) skeleton (Figure 12.2). The SMP(3) remains invariant, if we reverse the order of the idle periods $I^{(N=1)}$ and $I^{(N>1)}$ and their physically succeeding service times S_{00} and S_{10}, respectively, while keeping the event of departure at the end of each sum of random variables. In our MAP representation, we now contract the services contained within transitions originating from the same state into a single PH specification $(\boldsymbol{\alpha}, \boldsymbol{T})$ ($S_{00}, S_{01}, S_{02} \to$ 1st block row of $\boldsymbol{D_0^{(D)}}$, and analogously $S_{10}, S_{11} \to$ 3rd block row of $\boldsymbol{D_0^{(D)}}$ and $S_{21}, S_{22} \to$ 5th block row

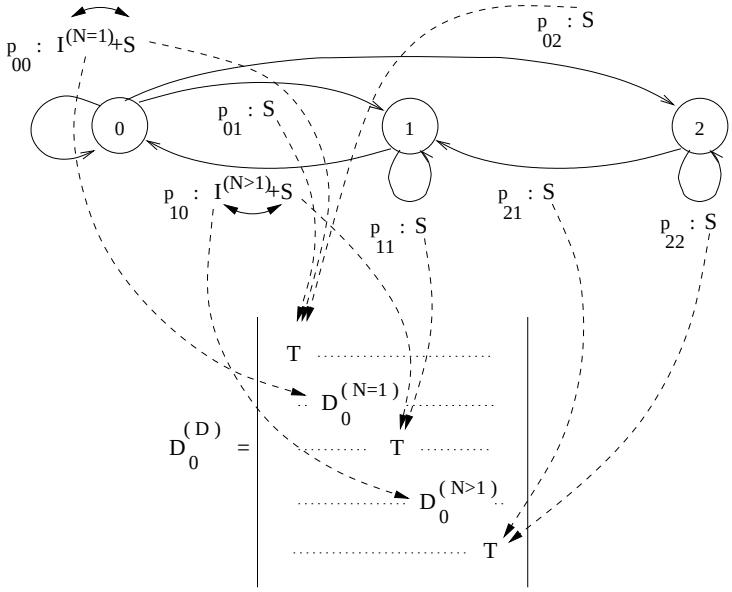


Figure 12.2: From the SMP(3) to the MAP

of $\boldsymbol{D_0^{(D)}}$). The interchange of random variables yields a more compact (and equally precise) MAP:

$$\boldsymbol{D_0^{(D)}} = \begin{vmatrix} \boldsymbol{T} & p_{00}(-\boldsymbol{Te}) \cdot \dfrac{\boldsymbol{x_0^{(N=1)}}(\boldsymbol{I}\otimes\boldsymbol{e})}{\boldsymbol{x_0^{(N=1)}e}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D_0^{(N=1)}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{T} & p_{10}(-\boldsymbol{Te}) \cdot \dfrac{\boldsymbol{x_0^{(N>1)}}(\boldsymbol{I}\otimes\boldsymbol{e})}{\boldsymbol{x_0^{(N>1)}e}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D_0^{(N>1)}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{T} \end{vmatrix} \tag{12.9}$$

$$\boldsymbol{D_1^{(D)}} = \begin{vmatrix} \boldsymbol{0} & \boldsymbol{0} & p_{01}(-\boldsymbol{Te})\boldsymbol{\alpha} & \boldsymbol{0} & p_{02}(-\boldsymbol{Te})\boldsymbol{\alpha} \\ \boldsymbol{D_1^{(N=1)}e\alpha} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & p_{11}(-\boldsymbol{Te})\boldsymbol{\alpha} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{D_1^{(N>1)}e\alpha} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & p_{21}(-\boldsymbol{Te})\boldsymbol{\alpha} & \boldsymbol{0} & p_{22}(-\boldsymbol{Te})\boldsymbol{\alpha} \end{vmatrix} \tag{12.10}$$

The MAPs $(\boldsymbol{D_0^{(N=1)}}, \boldsymbol{D_1^{(N=1)}})$ and $(\boldsymbol{D_0^{(N>1)}}, \boldsymbol{D_1^{(N>1)}})$ describe the idle periods after a busy period with a single customer or more than one customer, respectively. The probability vectors $\dfrac{\boldsymbol{x_0^{(N=1)}}(\boldsymbol{I}\otimes\boldsymbol{e})}{\boldsymbol{x_0^{(N=1)}e}}$ and $\dfrac{\boldsymbol{x_0^{(N>1)}}(\boldsymbol{I}\otimes\boldsymbol{e})}{\boldsymbol{x_0^{(N>1)}e}}$ are appropriate initial distributions (the term $(\boldsymbol{I}\otimes\boldsymbol{e})$ reduces the dimension from $m_A \cdot m_S$ to m_A). If we want to capture the full behavior of the input MAP $(\boldsymbol{D_0^{(A)}}, \boldsymbol{D_1^{(A)}})$ in the output model, we may set $\boldsymbol{D_0^{(N=1)}} = \boldsymbol{D_0^{(N>1)}} = \boldsymbol{D_0^{(A)}}$ and $\boldsymbol{D_1^{(N=1)}} = \boldsymbol{D_1^{(N>1)}} = \boldsymbol{D_1^{(A)}}$. Then the descriptions of the idle periods only differ in their initial distributions and the output

MAP can be compressed to

$$\boldsymbol{D}_0^{(D)} = \left| \begin{array}{cccc} \boldsymbol{T} & \boldsymbol{0} & \boldsymbol{0} & p_{00}(-\boldsymbol{T}\boldsymbol{e}) \cdot \dfrac{\boldsymbol{x}_0^{(N=1)}(\boldsymbol{I}\otimes\boldsymbol{e})}{\boldsymbol{x}_0^{(N=1)}\boldsymbol{e}} \\ \boldsymbol{0} & \boldsymbol{T} & \boldsymbol{0} & p_{10}(-\boldsymbol{T}\boldsymbol{e}) \cdot \dfrac{\boldsymbol{x}_0^{(N>1)}(\boldsymbol{I}\otimes\boldsymbol{e})}{\boldsymbol{x}_0^{(N>1)}\boldsymbol{e}} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{T} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{D}_0^{(A)} \end{array} \right| \tag{12.11}$$

$$\boldsymbol{D}_1^{(D)} = \left| \begin{array}{cccc} \boldsymbol{0} & p_{01}(-\boldsymbol{T}\boldsymbol{e})\boldsymbol{\alpha} & p_{02}(-\boldsymbol{T}\boldsymbol{e})\boldsymbol{\alpha} & \boldsymbol{0} \\ \boldsymbol{0} & p_{11}(-\boldsymbol{T}\boldsymbol{e})\boldsymbol{\alpha} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & p_{21}(-\boldsymbol{T}\boldsymbol{e})\boldsymbol{\alpha} & p_{22}(-\boldsymbol{T}\boldsymbol{e})\boldsymbol{\alpha} & \boldsymbol{0} \\ \boldsymbol{D}_1^{(A)}\boldsymbol{e}\boldsymbol{\alpha} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{array} \right| \tag{12.12}$$

In the following, we outline how the unknown quantities are determined from the MAP/PH/1(/K>1) queue.

Determining $\boldsymbol{x}_0^{(N=1)}$ and $\boldsymbol{x}_0^{(N>1)}$

As indicated by the notation, $\boldsymbol{x}_0^{(N=1)}$ is the vector of the stationary probabilities of ending a single-customer busy period in the QBD. Obviously (see also 12.3.3), $\boldsymbol{x}_0^{(N=1)}$ can be computed from

$$\boldsymbol{x}_0^{(N=1)} = \widetilde{\boldsymbol{z}}_e \boldsymbol{A}_2$$

Vector $\boldsymbol{x}_0^{(N>1)}$ is a compound analogue of $\boldsymbol{x}_0^{(N=1)}$ for the idle period after a busy period with more than one customer resulting from $\boldsymbol{x}_0^{(N=1)} + \boldsymbol{x}_0^{(N>1)} = \frac{1}{\boldsymbol{x}_0\boldsymbol{e}}\boldsymbol{x}_0$.

Moment fitting for the idle periods and service times

Unless the order of the output MAP becomes too large, $(\boldsymbol{D}_0^{(N=1)}, \boldsymbol{D}_1^{(N=1)})$ and $(\boldsymbol{D}_0^{(N>1)}, \boldsymbol{D}_1^{(N>1)})$ are chosen identical to the input MAP matrices $(\boldsymbol{D}_0^{(A)}, \boldsymbol{D}_1^{(A)})$. The corresponding output model (12.11)/(12.12) has the order $m_A + 3\,m_S$, which is linear in m_A and m_S. Considering the second and third moments of the number of customers served in a busy period only added m_S additional states (compared to [40]). If the distinction between $I^{(N=1)}$ and $I^{(N>1)}$ is completely ignored, we will substitute $\dfrac{\boldsymbol{x}_0(\boldsymbol{I}\otimes\boldsymbol{e})}{\boldsymbol{x}_0\boldsymbol{e}}$ for $\dfrac{\boldsymbol{x}_0^{(N=1)}(\boldsymbol{I}\otimes\boldsymbol{e})}{\boldsymbol{x}_0^{(N=1)}\boldsymbol{e}}$ and $\dfrac{\boldsymbol{x}_0^{(N>1)}(\boldsymbol{I}\otimes\boldsymbol{e})}{\boldsymbol{x}_0^{(N>1)}\boldsymbol{e}}$ in $\boldsymbol{D}_0^{(D)}$ of (12.11), which allows to find an even more concise output MAP. Then we might as well match a low-order PH distribution $(\boldsymbol{\beta}, \boldsymbol{U}^{(I)})$ to the first moments of the idle period (preferentially an ACPH(2) one to the first three power moments). The residual arrival time corresponds to the absorption time of a CTMC (with initial distribution $\dfrac{\boldsymbol{x}_0(\boldsymbol{I}\otimes\boldsymbol{e})}{\boldsymbol{x}_0\boldsymbol{e}}$). So, it is itself a PH distribution with representation $(\dfrac{\boldsymbol{x}_0(\boldsymbol{I}\otimes\boldsymbol{e})}{\boldsymbol{x}_0\boldsymbol{e}}, \boldsymbol{D}_0^{(A)})$, whose moments can easily be calculated. This results in the following replacements in (12.11)/(12.12):

$$\boldsymbol{D}_0^{(A)} \leftarrow \boldsymbol{U}^{(I)} \qquad \boldsymbol{D}_1^{(A)}\boldsymbol{e} \leftarrow -\boldsymbol{U}^{(I)}\boldsymbol{e} \qquad \frac{\boldsymbol{x}_0(\boldsymbol{I}\otimes\boldsymbol{e})}{\boldsymbol{x}_0\boldsymbol{e}} \leftarrow \boldsymbol{\beta}$$

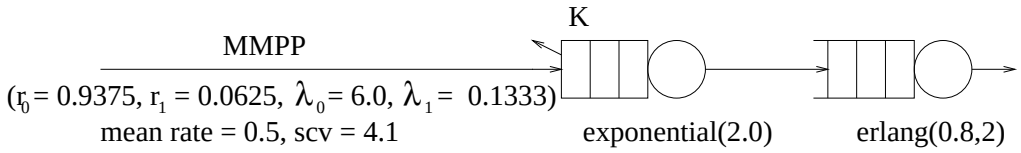


Figure 12.3: The dual tandem queue

Similar substitutions – typically of order 2 in form of an ACPH(2) distribution, unless the squared coefficient of variation is less than 0.5) – can be performed for the idle periods of the output model (12.9)/(12.10) (of order $2\,m_A + 3\,m_S$) and in general for possibly unnecessarily large PH service time distributions. Especially, when the two types of idle periods need to be distinguished (for reasons of accuracy), the application of moment matching to (12.9)/(12.10) often yields the most compact approximation of the departure process.

The busy queue

A special situation arises, if the system almost never becomes empty, i.e., $\boldsymbol{x_0 e} \approx 0$. Then, the output process can be modeled as a PH renewal process, where the PH interarrival time distribution corresponds to the service time $(\boldsymbol{\alpha}, \boldsymbol{T})$ (either exact or approximate).

12.5 Numerical experiments

In this section, we examine the output approximation (12.11)/(12.12) of the previous section. We concentrate on the mean queue length $\mathrm{E}[N_t]$ at arbitrary time (see [55, 63] for the computation for MAP/PH/1(/K) systems). In order to assess the accuracy of the decomposition results, we perform simulations by means of the SPNL component of TimeNET [84] with 99% confidence level and a maximum relative error of 1%. We first study the dual tandem queue in Figure 12.3 taken from [40]. External arrivals occur according to a bursty and nonrenewal MMPP with two states whose parameters are given in the figure and result in the MAP notation

$$\boldsymbol{D_0^{(A)}} = \begin{vmatrix} -(r_0+\lambda_0) & r_0 \\ r_1 & -(r_1+\lambda_1) \end{vmatrix} \quad \text{and} \quad \boldsymbol{D_1^{(A)}} = \begin{vmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{vmatrix} .$$

While the first queue processes requests in exponentially distributed service times (with rate 2.0), the second queue (with infinite capacity) has an Erlang-2 service time distribution of expectation 0.8. Since in the MAP-based decomposition the analysis of the first node in a tandem queueing network will always be exact (except for numerical errors), we focus on the mean queue length at the second node. In three sets of experiments, we vary specifications at the first queue (i.e., buffer size, service rate and mean arrival rate) in order to investigate their impact on the proposed output approximation as observed in the queueing behavior of the downstream queue.

Table 12.1 lists simulation data and decomposition results for different values of capacity K at the first queue. In [40], where the MAP-based decomposition ignores

	Simulation		Decomposition			Simulation		Decomposition	
K	mql	conf. int.	mql	rel. err.	K	mql	conf. int.	mql	rel. err.
∞	2.0401	$\pm$0.0128	2.0795	+1.9%	10	1.2779	$\pm$0.0090	1.2809	+0.2%
	(results from ref. [40]:		1.8789	−7.9%				1.2671	−0.8%)
30	1.9696	$\pm$0.0141	2.0157	+2.3%	6	0.9017	$\pm$0.0086	0.8847	−1.9%
25	1.9199	$\pm$0.0159	1.9540	+1.8%	4	0.6748	$\pm$0.0036	0.6451	−4.4%
20	1.8083	$\pm$0.0127	1.8380	+1.6%	3	0.5632	$\pm$0.0045	0.5025	−10.8%
15	1.6068	$\pm$0.0118	1.6287	+1.4%	2	0.4311	$\pm$0.0023	0.3307	−23.3%

Table 12.1: Mean queue lengths (mql) at second node for the dual tandem queue (varied K)

	Series for varied parameter at first queue							
	service rate				mean arrival rate			
	Simulation		Decomposition		Simulation		Decomposition	
ρ	mql	conf. int.	mql	rel. err.	mql	conf. int.	mql	rel. err.
0.1	2.8038	$\pm$0.0219	2.5636	−8.6%	0.2032	$\pm$0.0020	0.2035	+0.1%
0.2	2.3016	$\pm$0.0155	2.3267	+1.1%	0.4809	$\pm$0.0041	0.4880	+1.5%
0.3	1.7402	$\pm$0.0170	1.8152	+4.3%	0.8187	$\pm$0.0060	0.8422	+2.9%
0.4	1.2543	$\pm$0.0095	1.3200	+4.6%	1.2542	$\pm$0.0099	1.3120	+4.6%
0.5	0.9479	$\pm$0.0073	0.9762	+3.0%	1.8458	$\pm$0.0152	1.9696	+6.7%
0.6	0.7964	$\pm$0.0052	0.8064	+1.3%	2.7272	$\pm$0.0242	2.9567	+8.4%
0.7	0.7141	$\pm$0.0060	0.7163	+0.3%	4.1579	$\pm$0.0325	4.6005	+10.6%
0.8	0.6514	$\pm$0.0045	0.6619	+1.6%	6.9917	$\pm$0.0399	7.8729	+12.6%
0.9	0.6290	$\pm$0.0055	0.6258	−0.5%	15.402	$\pm$0.1299	17.573	+14.1%

Table 12.2: Mean queue lengths (mql) at second node for the dual tandem queue ($K = \infty$)

higher moments of the number of customers served in a busy period, the considered dual tandem queue is evaluated for $K = \infty$ and $K = 10$. Comparing rows 3 and 4 shows that an additional matching of the second and third moment of this random variable N significantly improves the numerical accuracy (from −7.9% to +1.9% and from −0.8% to +0.2%, respectively). At the same time, the order of the output MAP approximations only increases from 4 to 5. Note that the orders of the exact output MAPs are substantially larger (i.e., infinite for $K = \infty$ or $m_D^{\text{exact}} = m_A(1 + Km_S) = 22$ for $K = 10$). Medium-sized and large capacities lead to satisfactory relative errors, even though in cases $K = 20, 15, 10, 6$ the third (factorial) moment is set to the closest permissible boundary value as outlined in Chapter 11. The largest relative modification occurs for $K = 15$, where the true value $f_3 = 2098.0$ is replaced by 2222.9. Very small buffer sizes (see $K = 2, 3$) appear to be unfavorable to the proposed output approximation. This drawback, however, need not be overrated, since in these cases the exact output MAPs are usually so compact themselves that they can directly be employed in the context of MAP-based decomposition (as it is

q.-no.	MAP m_D	Simulation mql	Decomposition mql	rel. err.	q.-no.	MAP m_D	Simulation mql	Decomposition mql	rel. err.
1	8	0.2800	0.2804	+0.1%	6	38	0.2527	0.2680	+6.1%
2	14	0.2661	0.2716	+2.1%	7	44	0.2544	0.2671	+5.0%
3	20	0.2615	0.2706	+3.5%	8	50	0.2538	0.2662	+4.9%
4	26	0.2584	0.2698	+4.4%	9	56	0.2536	0.2654	+4.7%
5	32	0.2542	0.2689	+5.8%	10	(62)	0.2493	0.2646	+6.1%

Table 12.3: Mean queue lengths (mql) for 10-node tandem network

queue number	input MAP order m_A	output MAP order m_D	Simulation mql	conf. int.	Decomposition mql	rel. err.
1	2	8	0.4630	±0.0042	0.4635	+0.1%
2	32	41	0.7994	±0.0078	0.8240	+3.1%
3	41	(44)	0.2726	±0.0016	0.2799	+2.7%
4	41	(47)	0.2613	±0.0024	0.2683	+2.7%

Table 12.4: Mean queue lengths (mql) for four-node queueing network

done for the MAP/PH/1/1 system, see [40]).

In the next two series of experiments, we look into the dependence of decomposition results on the utility of the first queue, which is tuned in two ways: either by changing the service rate of the exponential distribution or by uniformly scaling all parameters of the arrival process so that its squared coefficient of variation (see (12.1)) remains constant, while the mean arrival rate varies. Capacity K is fixed to infinity. In the first series (left-hand side of Table 12.2), all other specifications of the network of Figure 12.3 are left untouched so that the utility at the second queue does not change. In the second series (right-hand side of Table 12.2), the expectation of the Erlang-2 distribution is additionally altered to 0.5 so that we have identical utilities at both queues. The last column suggests that the approximations of the mean queue lengths at the second node deteriorate with increasing utility of the first queue, which however cannot be confirmed in general with respect to the fifth column. While overall results might be regarded acceptable, the deviations of more than 10% for few values in the last column arouse the conjecture that in some cases the fourth and fifth moments of the random variable N ought to be taken into account, too.

An important feature of the proposed MAP output models, which is indispensible for an efficient network decomposition, consists in their moderate orders. Table 12.3 demonstrates how these orders (see columns labeled m_D) grow only linearly in a tandem network of ten homogeneous infinite-buffer queues with Erlang-2 service distributions (mean rate 1.9). The two-state MAP depicted in Figure 12.4 as the arrival process to queue 1 also serves as the external input to the tandem network. However, it is scaled to a mean arrival rate of 0.38 (with the squared coefficient of variation kept at 8.1). The first two queues of this network are also analyzed by

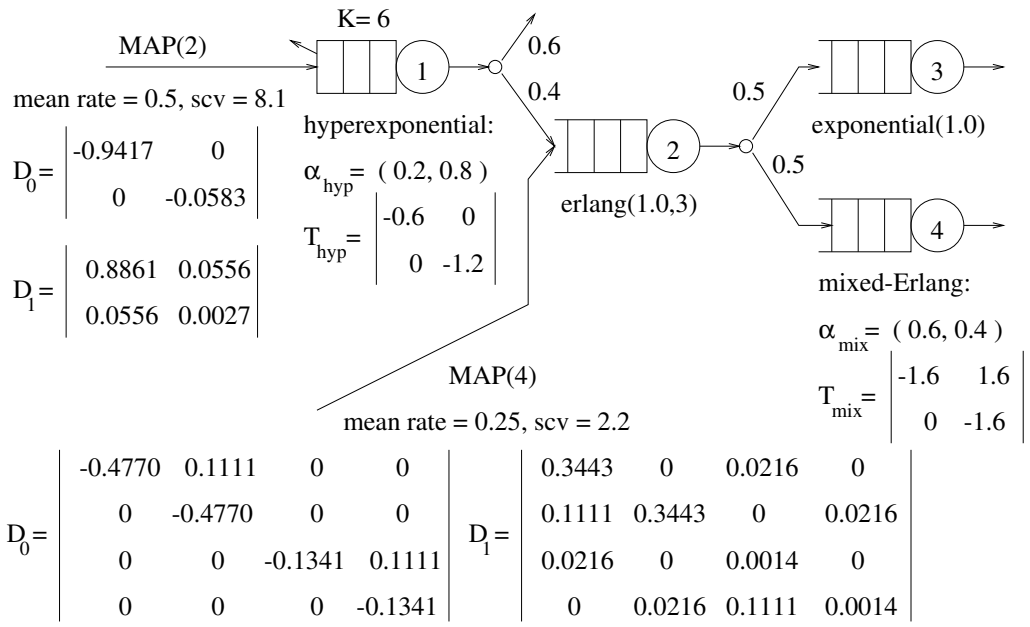


Figure 12.4: The four-node queueing network

MAP-based decomposition in [71]. Therein, MAP representations of order 134 for the departure process of the first queue deliver excellent results for the mean queue length at the second node. In order to proceed in the analysis of longer tandem networks, more compact representations are required. In the proposed methodology, the internal MAP sizes evolve according to the formula $m_D = m_A + 3 \cdot m_S = m_A + 6$ from queue to queue so that the output of the tenth queue in series is of order 62 only (brackets indicate that this MAP is actually not used in the computations). In a comparison between decomposition and simulation results (confidence intervals range from ± 0.0012 to ± 0.0026), the analytic values come off well both quantitatively and qualitatively. The mean queue lengths are slightly overestimated, but their falling off due to decreasing squared coefficients of variation of the internal traffic is correctly captured (unlike simulation, see queues 6/7).

Finally, we present a general four-node queueing network with splitting and merging (Figure 12.4) to emphasize the potential of an obvious decomposition approach to such networks based on the output approximation of Section 12.4. Again two bursty external inputs – MAPs of orders 2 and 4 with the given squared coefficients of variation (scv) – are taken from [71] with their mean rates being scaled to the stated values. Besides the known specifications for the exponential and Erlang distributions – here Erlang-3 at queue 2 –, a mixed Erlang and a hyperexponential service time distribution – as also used in [4] – are represented in PH notation in Figure 12.4 below the corresponding queues. They cover variabilities ranging from 1/3 to 11/9. Furthermore, routing probabilities and a finite buffer size are depicted. Table 12.4 collects the errors of the decomposition results (all below 3.1%) relative to the simulated values along with the orders of the involved traffic descriptors. Note that both splitting (invariant to MAP order) and merging (multiplies orders of involved MAPs) are performed as exact operations. The data illustrates that the provided

output approximation allows a reasonable trade-off between accuracy and efficiency.

Chapter 13

Concluding remarks

This dissertation provides a collection of selected research results obtained since 1995. The subjects are selected such that they belong to two main research fields: stochastic reward models, and state based techniques, but it was not intended to introduce the width of any of these fields completely.

The "distance" among the selected research results varies as well. Some chapters are closer to each others than to the other chapters of the same part. The notations of these chapters are more or less unified, but the notations of the chapters are independent in general.

Intensive research cooperation produced all of the introduced results. All results were already published in international conference proceedings or journals previously (as it is detailed in the associated thesis booklet), and parts of these publications are repeated in this dissertation. Only the set of results presented in the associated thesis booklet is meant to be the results of the author.

Last, but not least, I would like to thank the minutes of enjoy when we worked on the introduced research problems with the co-authors of the papers summarized in this dissertation: Andrea Bobbio, Armin Heindl, András Horváth, András Pfening and Sándor Rácz. I am really glad to meet and work with these exceptional people.

Bibliography

[1] M. Ajmone Marsan, G. Balbo, and G. Conte. A class of generalized stochastic Petri nets for the performance evaluation of multiprocessor systems. *ACM Transactions on Computer Systems*, 2:93–122, 1984.

[2] M. Ajmone Marsan and G. Chiola. On Petri nets with deterministic and exponentially distributed firing times. In *Lecture Notes in Computer Science*, volume 266, pages 132–145. Springer Verlag, 1987.

[3] D. Aldous and L. Shepp. The least variable phase type distribution is Erlang. *Stochastic Models*, 3:467–473, 1987.

[4] N. G. Bean, D. A. Green, and P. G. Taylor. Approximations to the output process of MAP/PH/1/ queues. In *Proc. 2nd Int. Workshop on Matrix-Analytic Methods*, pages 151–159, 1998.

[5] M.D. Beaudry. Performance-related reliability measures for computing systems. *IEEE Transactions on Computers*, C-27:540–547, 1978.

[6] K. Begain, G. Bolch, and H. Herold. *Practical Performance Modelling, Application of the MOSEL Language*. Kluwer Academic Publishers, 2001.

[7] A. Bobbio. A multi-reward stochastic model for the completion time of parallel tasks. In A. Jensen and V.B. Iversen, editors, *Teletraffic and Datatraffic (Proceedings 13-th International Teletraffic Congress, ITC-13)*, pages 577–582. Elesevier Science Publisher, 1991.

[8] A. Bobbio and A. Cumani. ML estimation of the parameters of a PH distribution in triangular canonical form. In G. Balbo and G. Serazzi, editors, *Computer Performance Evaluation*, pages 33–46. Elsevier Science Publishers, 1992.

[9] A. Bobbio and A. Horváth. Petri nets with discrete phase type timing: A bridge between stochastic and functional analysis. *Electronic Notes in Theoretical Computer Science*, 52(3), 2001.

[10] A. Bobbio, A. Horváth, M. Scarpa, and M. Telek. Acyclic discrete phase type distributions: Properties and a parameter estimation algorithm. *Performance Evaluation*, 2002. (to appear).

[11] A. Bobbio, V. G. Kulkarni, and M. Telek. Partial loss in reward models. In *2nd Int. Conf. on Mathematical Methods in Reliability*, pages 207–210, Bordeaux, France, July 2000.

[12] A. Bobbio, V.G. Kulkarni, A. Puliafito, M. Telek, and K. Trivedi. Preemptive repeat identical transitions in Markov Regenerative Stochastic Petri Nets. In *6-th International Conference on Petri Nets and Performance Models - PNPM95*, pages 113–122. IEEE Computer Society, 1995.

[13] A. Bobbio and M. Telek. Task completion time. In *Proceedings 2nd International Workshop on Performability Modelling of Computer and Communication Systems (PMCCS2)*, 1993.

[14] A. Bobbio and M. Telek. A benchmark for PH estimation algorithms: results for Acyclic-PH. *Stochastic Models*, 10:661–677, 1994.

[15] A. Bobbio and M. Telek. Markov regenerative SPN with non-overlapping activity cycles. In *International Computer Performance and Dependability Symposium - IPDS95*, pages 124–133. IEEE Computer Society Press, 1995.

[16] A. Bobbio and M. Telek. The task completion time in degradable sytems. In B. R. Haverkort, R. Marie, G. Rubino, and K. S. Trivedi, editors, *Performability Modelling: Techniques and Tools*, chapter 7, pages 139–161. Wiley, 2001.

[17] A. Bobbio and K.S. Trivedi. Computation of the distribution of the completion time when the work requirement is a PH random variable. *Stochastic Models*, 6:133–149, 1990.

[18] P. Bocharov. Analysis of the queue length and the output flow in single server with finite waiting room and phase type distributions. *Problems of Control and Information Theory*, 16(3):211–222, 1987.

[19] A. Brock. An analysis of checkpointing. *ICL Tech. Journal*, 1(3), 1979.

[20] K.M. Chandy, J.C. Browne, C.W. Dissly, and W.R. Uhrig. Analytic models for rollback and recovery strategies in database systems. *IEEE tr. on Software Engineering*, SE-1(1):100–110, 1975.

[21] Hoon Choi, V.G. Kulkarni, and K. Trivedi. Transient analysis of deterministic and stochastic Petri nets. In *Proceedings of the 14-th International Conference on Application and Theory of Petri Nets*, Chicago, June 1993.

[22] Hoon Choi, V.G. Kulkarni, and K. Trivedi. Markov regenerative stochastic Petri nets. *Performance Evaluation*, 20:337–357, 1994.

[23] G.L. Choudhury, D.M. Lucantoni, and W. Whitt. Multi dimensional transform inversion with applications to the transient M/G/1 queue. *Ann. Appl. Prob.*, 4:719–740, 1994.

[24] E. Cinlar. Markov renewal theory. *Advances in Applied Probability*, 1:123–187, 1969.

[25] A. Cumani. On the canonical representation of homogeneous Markov processes modelling failure-time distributions. *Microelectronics and Reliability*, 22:583–602, 1982.

[26] E. de Souza e Silva and H.R. Gail. Calculating availability and performability measures of repairable computer systems using randomization. *Journal of the ACM*, 36:171–193, 1989.

[27] E. de Souza e Silva, H.R. Gail, and R. Vallejos Campos. Calculating transient distributions of cumulative reward. In *Proceedings ACM/SIGMETRICS Conference*, Ottawa, 1995.

[28] L. Donatiello and V. Grassi. On evaluating the cumulative performance distribution of fault-tolerant computer systems. *IEEE Transactions on Computers*, 1991.

[29] R. Fricks, M. Telek, A. Puliafito, and K. Trivedi. Markov regenerative theory applied to performability evaluation. In K.K. Bagchi and G. Zobrist, editors, *Modeling and Simulation of Advanced Computer Systems: Applications and Systems*, State-of-the Art in Performance Modeling and Simulation, chapter 11, pages 193–236. Gordon and Breach Publishers, 1996.

[30] S. Garg, A. Puliafito, M. Telek, and K. Trivedi. Analysis of software rejuvenation using Markov regenerative stochastic Petri nets. In *Proceedings of the 6-th International Symposium on Software Reliability Engineering*, Toulouse, 1995.

[31] E. Gelenbe and D. Derochette. Performance of rollback recovery systems under intermittent failures. *Commun. ACM*, 21(6):493–499, 1978.

[32] R. German. New results for the analysis of deterministic and stochastic Petri nets. In *International Computer Performance and Dependability Symposium - IPDS95*, pages 114–123. IEEE CS Press, 1995.

[33] R. German. *Performance Analysis of Communication Systems: Modeling with Non-Markovian Stochastic Petri Nets*. John Wiley and Sons, 2000.

[34] A. Goyal and A.N. Tantawi. Evaluation of performability for degradable computer systems. *IEEE Transactions on Computers*, C-36:738–744, 1987.

[35] A. Graham. *Kronecker Products and Matrix Calculus with Applications*. Ellis Horwood, Chichester, UK, 1981.

[36] D. Green. Lag correlations of approximating departure processes of MAP/PH/1 queues. In *Proc. 3rd Int. Conf. on Matrix-Analytic Methods in Stochastic Models*, pages 135–151, 2000.

[37] B. R. Haverkort. Approximate analysis of networks of PH/PH/1/K queues with customer losses: Test results. *Annals of Operations Research*, 79:271–291, 1998.

[38] A. Heindl. Node reordering for improved performance of traffic-based decomposition. In *Proc. 5th Int. Workshop on Performability Modeling of Computer and Communication Systems*, pages 16–20, Erlangen, Germany, 2001.

[39] A. Heindl. *Traffic-Based Decomposition of General Queueing Networks with Correlated Input Processes*. Shaker Verlag, Aachen, Germany, 2001.

[40] A. Heindl and M. Telek. MAP-based decomposition of tandem networks of ·/PH/1(/K) queues with MAP input. In *Proc. 11th GI/ITG Conference on Measuring, Modelling and Evaluation of Computer and Communication Systems*, pages 179–194, Aachen, Germany, 2001.

[41] J. Hillston and M. Ribaudo. Stochastic process algebras: a new approach to performance modeling. In K. Bagchi, J. Walrand, and G. Zobrist, editors, *Modeling and Simulation of Advanced Computer Systems*, pages 235–256. Gordon Breach, 1998.

[42] R.A. Howard. *Dynamic Probabilistic Systems, Volume II: Semi-Markov and Decision Processes*. John Wiley and Sons, New York, 1971.

[43] B.R. Iyer, L. Donatiello, and P. Heidelberger. Analysis of performability for stochastic models of fault-tolerant systems. *IEEE Transactions on Computers*, C-35:902–907, 1986.

[44] D.L. Jagerman. An inversion technique for the Laplace transform. *The Bell System Technical Journal*, 61:1995–2002, October 1982.

[45] M.A. Johnson and M.R. Taaffe. Matching moments to Phase distributions: mixtures of Erlang distribution of common order. *Stochastic Models*, 5:711–743, 1989.

[46] R. Jones and G. Ciardo. On phased delay stochastic Petri nets: Definition and application. In *Proceedings 9th International Workshop on Petri Nets and Performance Models - PNPM01*. IEEE Computer Society, 2001.

[47] L. Kleinrock. *Queuing systems, Volume 1: Theory*. Wiley Interscience, New York, 1975.

[48] P. J. Kühn. Approximate analysis of general queueing networks by decomposition. *IEEE Trans. Communications*, COM-27:113–126, 1979.

[49] V.G. Kulkarni, V.F. Nicola, and K. Trivedi. On modeling the performance and reliability of multi-mode computer systems. *The Journal of Systems and Software*, 6:175–183, 1986.

[50] V.G. Kulkarni, V.F. Nicola, and K. Trivedi. The completion time of a job on a multi-mode system. *Advances in Applied Probability*, 19:932–954, 1987.

[51] V.G. Kulkarni, V.F. Nicola, and K. Trivedi. Effects of checkpointing and queueing on program performance. *Stochastic models*, 4(6):615–648, 1990.

[52] T. V. Laksman and U. Madhow. Performance analysis of window-based flow control using tcp/ip: the effect of high bandwidth-delay products and random loss. *IFIP Trans. High performance networking*, C-26:135–150, 1994.

[53] A. Lang and J. L. Arthur. Parameter approximation for phase-type distributions. In S. R. Chakravarty and A. S. Alfa, editors, *Matrix-analytic methods in stochastic models*, Lecture notes in pure and applied mathematics, pages 151–206. Marcel Dekker, Inc., 1996.

[54] G. Latouche and V. Ramaswami. A logarithmic reduction algorithm for quasi birth-and-death processes. *Journal of Applied Probability*, 30:650–674, 1993.

[55] G. Latouche and V. Ramaswami. *Introduction to Matrix-Analytic Methods in Stochastic Modeling*. Series on statistics and applied probability. ASA-SIAM, 1999.

[56] M. Livny, B. Melamed, and A. K. Tsiolis. The impact of autocorrelation on queueing systems. *Management Science*, 39:322–339, 1993.

[57] D. Logothetis, K. S. Trivedi, and A. Puliafito. Markov regenerative models. In *Int. Computer Performance and Dependability Symposium - IPDS95*, pages 134–142. IEEE CS Press, 1995.

[58] D. M. Lucantoni and M. Neuts. Some steady-state distributions for the MAP/SM/1 queue. *Commun. Statist.-Stochastic Models*, 10:575–598, 1994.

[59] J.F. Meyer. On evaluating the performability of degradable systems. *IEEE Transactions on Computers*, C-29:720–731, 1980.

[60] J.F. Meyer. Closed form solution of performability. *IEEE Transactions on Computers*, C-31:648–657, 1982.

[61] K. Mitchell and A. van de Liefvoort. Approximation models of feed-forward G/G/1/N queueing networks with correlated arrivals. In *Proc. 4th Int. Workshop on Queueing Networks with Finite Capacity*, pages 32/1–12, Ilkley, UK, 2000. Networks UK.

[62] H. Nabli and B. Sericola. Performability analysis: a new algorithm. *IEEE Transactions on Computers*, 45:491–494, 1996.

[63] V. A. Naoumov, U. Krieger, and D. Wagner. Analysis of a multi-server delay-loss system with a general Markovian arrival process. In Chakravarthy and Alfa, editors, *Matrix-Analytic Methods in Stochastic Models*, pages 43–66, New York, 1997. Marcel Dekker.

[64] M. Neuts. Probability distributions of phase type. In *Liber Amicorum Prof. Emeritus H. Florin*, pages 173–206. University of Louvain, 1975.

[65] M. Neuts. *Matrix-Geometric Solutions in Stochastic Models*. John Hopkins University Press, 1981.

[66] M. Neuts. *Algorithmic Probability: A Collection of Problems*. Chapman and Hall, 1995.

[67] V.F. Nicola, V.G. Kulkarni, and K. Trivedi. Queueing analysis of fault-tolerant computer systems. *IEEE Transactions on Software Engineering*, SE-13:363–375, 1987.

[68] S. Rácz and M. Telek. Performability analysis of Markov reward models with rate and impulse reward. In M. Silva B. Plateau, W. Stewart, editor, *Int. Conf. on Numerical solution of Markov chains*, pages 169–187, Zaragoza, Spain, 1999.

[69] S. Rácz and M. Telek. Mission time analysis of large dependable systems. In *IPDS 2000*, pages 13–22, Chicago, IL, USA, March 2000. IEEE CS Press.

[70] A. Reibman, R. Smith, and K.S. Trivedi. Markov and Markov reward model transient analysis: an overview of numerical approaches. *European Journal of Operational Research*, 40:257–267, 1989.

[71] R. Sadre and B. Haverkort. Characterizing traffic streams in networks of MAP/MAP/1 queues. In *Proc. 11th GI/ITG Conference on Measuring, Modelling and Evaluation of Computer and Communication Systems*, Aachen, Germany, 2001.

[72] R. Sadre, B. Haverkort, and A. Ost. An efficient and accurate decomposition method for open finite and infinite buffer queueing networks. In *Proc. 3rd Int. Workshop on Numerical Solution of Markov Chains*, pages 1–20, Zaragoza, Spain, 1999.

[73] R. Sahner, K.S. Trivedi, and A. Puliafito. *Performance and Reliability Analysis of Computer Systems: An Example-based Approach Using the SHARPE Software Package*. Kluwer Academic Publisher, 1996.

[74] W. H. Sanders, W. D. Obal, M. A. Qureshi, and F. K. Widjanarko. The ultrasan modeling environment. *Performance Evaluation*, 24(1-2):89–115, 1995.

[75] R. Smith, K. Trivedi, and A.V. Ramesh. Performability analysis: Measures, an algorithm and a case study. *IEEE Transactions on Computers*, C-37:406–417, 1988.

[76] W. J. Stewart. *Introduction to the Numerical Solution of Markov Chains*. Princeton University Press, Princeton, New Jersey, 1994.

[77] M. Telek. Minimal coefficient of variation of discrete phase type distributions. In *3rd International Conference on Matrix-Analitic Methods in Stochastic models, MAM3*, pages 391–400, Leuven, Belgium, 2000. Notable Publications Inc.

[78] M. Telek and A. Pfening. Performance analysis of Markov Regenerative Reward Models. *Performance Evaluation*, 27&28:1–18, 1996.

[79] M. Telek, A. Pfening, and G. Fodor. An effective numerical method to compute the moments of the completion time of Markov reward models. *Computers and mathematics with applications*, 36:8:59–65, 1998.

[80] M. Telek and S. Rácz. Numerical analysis of large Markovian reward models. *Performance Evaluation*, 36&37:95–114, Aug 1999.

[81] Miklós Telek. *Some advanced reliability modelling techniques*. Phd Thesis, Hungarian Academy of Science, 1994.

[82] W. Whitt. Approximating a point process by a renewal process, I. Two basic methods. *Operations Research*, 30:125–147, 1982.

[83] W. Whitt. The queueing network analyzer. *The Bell System Technical Journal*, 62:2779–2815, 1983.

[84] A. Zimmermann, J. Freiheit, R. German, and G. Hommel. Petri net modelling and performability evaluation with TimeNET 3.0. In *Proc. 11th Int. Conf. on Modelling Techniques and Tools for Computer Performance Evaluation*, pages 188–202, Chicago, USA, 2000.

Appendix A

List of notations

Ω	Finite state space of the structure state process
$Z(t)$	Continuous time finite state stochastic process
r_i	Reward rate in state i
W	Work requirement, r.v.
C	Completion time of the (random) work requirement W, r.v.
$\hat{C}(t)$	Distribution of the completion time of the work requirement W $\left(\hat{C}(t) = Pr\left\{C \leq t\right\}\right)$
$C(w)$	Completion time of the deterministic work requirement w, r.v.
$C(t,w)$	Distribution of $C(w)$ $\quad C(t,w) = Pr\left\{C(w) \leq t\right\}$
$W(w)$	Distribution of the work requirement W $W(w) = Pr\left\{W \leq w\right\}$ $\quad \left(\hat{C}(t) = \int_{a=0}^{\infty} C(t,w) d\, W(w)\right)$
$B(t)$	Accumulated reward at time t, r.v.
$B(t,w)$	Distribution of $B(t)$ $B(t,w) = Pr\left\{B(t) \leq w\right\}$
$\underline{P}(0)$	Row vector of initial state probabilities
$\underline{h}$	Column vector of ones
$\mathbf{A} = \{a_{ij}\}$	Infinitesimal operator of continuous time Markov chains $a_i = -a_{ii} = \sum_{j \in \Omega, j \neq i} a_{ij}$
H	Holding time of a regeneration period, r.v.
$\mathbf{Q}(t) = \{Q_{ij}(t)\}$	Kernel of semi-Markov process $Q_{ij}(t) = Pr\left\{H \leq t, Z(H) = j \middle\| Z(0) = i\right\}$

$\mathbf{K}(t) = \{K_{ij}(t)\}$	Global kernel of Markov regenerative process $K_{ij}(t) = Pr\{H \leq t, Z(H) = j \mid Z(0) = i\}$
$\mathbf{E}(t) = \{E_{ij}(t)\}$	Local kernel of Markov regenerative process $E_{ij}(t) = Pr\{H > t, Z(t) = j \mid Z(0) = i\}$
$\mathbf{\Pi} = \{p_{ij}\}$	One step state transition probability matrix of the embedded Markov chain $p_{ij} = Pr\{Z(H) = j \mid Z(0) = i\}$
$F(t) \rightarrow F^*(s)$	Laplace transform pair
$F(t) \rightarrow F^{\sim}(s)$	Laplace-Stieltjes transform pair

Abbreviations

CTMC	continuous time Markov chain
DTMC	discrete time Markov chain
RTP	regeneration time point
EMC	embedded Markov chain
SMP	semi-Markov process
MRP	Markov regenerative process
LT	Laplace transform
LST	Laplace-Stieltjes transform
SRM	stochastic reward model
MRM	Markov reward model
prs	preemptive resume
prd	preemptive repeat different
pri	preemptive repeat identical
PH	phase type distribution
MAP	Markovian arrival process
DPH	discrete time PH
CPH	continuous time PH