Some Advanced Reliability Modelling Techniques

Miklós Telek

Department of Telecommunications
Technical University of Budapest
Budapest, Hungary

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Address: Building St,
Sztoczek u. 2.
H-1111, Budapest
Hungary

Phone/Fax: +36-1-1665824
E-mail: telek@plan.hit.bme.hu

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Chapter 1

Introduction

The solution of reliability evaluation problems consist of model construction and model analysis, where the model construction phase means the abstract description of the real systems by a given description tool, while in the model analysis phase the required measures of the model are derived. The reliability behaviour of real systems can be generally described by a continuous time random process whose state space are composed by the finite distinguishable states of the model.

Based on the well-known results available for the continuous time Markov chains the Markovian models of systems have been applied for more than 30 years, although in several cases these models yield only a rude representation of the reality[1]. The introduction of the class of Phase Type distributions and its applications for the conversion of non-Markovian systems into a continuous time Markov chain have provided an important opportunity towards the analysis of the non-Markovian models since the early 70th[2]. The other main direction in reliability modelling the consideration of more complex stochastic processes is the subject of this study. The application of semi-Markov processes for reliability analysis is widespread as well, but there are only few known results obtained by the use of Markov Regenerative Processes.

While the traditional analysis problems included only the steady state and/or the transient analysis of the reliability models, since the 70th the considered analysis problems have been also extended for the performance-like analysis of these models due to the introduction of reward variables.

Obviously, the evolution of the reliability modelling techniques is not separated from the evolution of the scientific research fields in which the same theory of the continuous time discrete state stochastic processes and their modelling tools are applied. Thus, the results achieved in queueing theory, and with a further application step, in the performance analysis of communication networks and dependable computer systems have always played significant role in the development of reliability modelling and analysis, as well.

The key modelling tools considered in this study are the timed Petri Nets. They have been widely applied in the stochastic modelling and simulation for a long time,

[1]Mentionable preliminaries in the application of Markovian reliability models are [56, 6] on the field of generation and analysis of Markovian models and [51, 52, 53, 54, 55] on the field of network reliability analysis.

[2]Preliminaries are summarized in [14, 17, 16].

however, their transient analysis have been restricted to the cases in which the marking processes are continuous time Markov or semi-Markov ones. The first results for the analysis of Petri nets with Markov regenerative marking processes appeared only in 1993.

This study is basically devoted to the investigation of modelling opportunities of system reliability, i.e. a class of stochastic processes with the so-called Generally Distributed Transition Stochastic Petri Nets (GDT_SPN). The numerical analysis problems of the introduced results are not studied, only simple example with small state space is evaluated. The study is organized as follows.

The following chapter gives an introduction to the considered stochastic processes (i.e. semi-Markov processes, Markov regenerative processes) with the techniques applied throughout the later chapters. Appendix B contains some related results which do not belong to the theoretical base of these processes, but which play important role in their applications. A short introduction and survey of stochastic reward models are discussed later on with an outlook in Appendix C.

New results about the measures of some special stochastic reward models are presented in Chapter 3. The considered stochastic processes are semi-Markov processes and Markov regenerative processes with subordinated semi-Markov processes and the results are given in transform domain in the former case and in integral equation form in the latter case. In the first part of the chapter some subsets of Markov regenerative processes with subordinated semi-Markov processes are taken into consideration in order to obtain simpler results for the measures when the states are of *prs* type. In the following sections (Section 3.2 - 3.4) the states of the models are only *prs*, *prd* and *pri* type, respectively, and an absorbing group of states are considered as well.

Chapter 4 contains the introduction of timed Petri nets and summarizes the available results on their transient analysis including the recent results for timed Petri nets with Markov regenerative marking processes.

Chapter 5 gives a more general approach to the analysis of Markov regenerative stochastic Petri nets. The regenerative time points of marking processes are introduced in the second section by the inspections of the age variables assigned to each timed transition. Then the life cycle of transitions and the features of memory policies are examined. The structural restrictions of Petri nets are classified which provide that the regeneration periods can be analyzed by single reward models. A method for the analysis of subordinated semi-Markov process is also described.

The potential application opportunities of the results are shown in Chapter 6 by an example described in the language of queueing theory. The study is concluded in Chapter 7.

Simple reliability examples are provided before the theoretical treatment. These examples are not discussed in detailes in the study, they only show the practical importance of the studied problems. The solution of the introduced simple examples requires the application of the subsequent results.

Chapter 2

Stochastic Reward Processes

Example 1:

Let us consider a reliability system of two operating machines (A and B) and one repair man. Machine B has higher priority than machine A. The repair man immediately starts the repair of the failed machine. If machine B fails during the repair of machine A, the repair man preempts the repair of machine A and starts to repair machine B. There are four states of this system:

- state 1: both of the machines are working,
- state 2: machine B is working and machine A is failed,
- state 3: machine A is working and machine B is failed,
- state 4: both of the machines are failed.

Case I: When both of the failure and repair times are exponentially distributed random variables, the stochastic process over this state space is a continuous time Markov chain.

When the failure times of the machines are exponentially distributed random variables, but their repair times are any other positive random variables the stochastic process describes the behaviour of the system, depending on the property of the preempted repair.

Case II: If the repair work on machine A done up to the preemption is lost, and the repair starts from the beginning after the completion of the repair of machine B, then the stochastic process is a semi-Markov process.

Case III: If the repair work on machine A done up to the preemption is resumed after the completion of the repair of machine B, then the stochastic process is a Markov regenerative process.

This chapter intends to provide a short introduction to the later studied subjects and a summary of the ideas as well as the used notations.

Before introducing the concept of reward processes we discuss the considered stochastic processes and their properties based on the pioneer work of Cinlar [29]. But according to the later studied analysis method we diverge a bit from the original work in the following. The differences are emphasized in each definition.

2.1 Continuous time discrete state stochastic processes with regeneration time points

In this work we pay special attention to the stochastic processes ($Z(t)$) defined over a discrete state space (Ω), whose features can be characterized by the existence of (random) time instants, at which the future of the stochastic process depends only on its current state. Theoretically the time instants of this kind cover the past history of the process, thus they are called regenerative time points[1].

Definition 2.1 *T_n is called a* **regenerative time point**[2] *(RTP) if*

$$E\{f(Z(T_n+t_1),\ldots,Z(T_n+t_m),)\,|\,Z(T_n),Z(u),0\le u<T_n\}$$

$$=E\{f(Z(T_n+t_1),\ldots,Z(T_n+t_m),)\,|\,Z(T_n)\}$$

for any $0\le t_1\le\ldots\le t_m$, and bounded function defined on Ω^n.

This property is referred to as strong Markov property of the process at T_n taking $m=1$ ([29]).

The sequence of the RTPs plays special role in the study of stochastic processes.

Definition 2.2 *The sequence of the random variables $\{X_n,T_n;n\ge 0\}$ is said to be a (time homogeneous)* **Markov renewal sequence**[3] *provided that*

$$Pr\{X_{n+1}=x,T_{n+1}-T_n\le t\,|\,X_0,\ldots,X_n,T_0,\ldots,T_n\}$$

$$=Pr\{X_{n+1}=x,T_{n+1}-T_n\le t\,|\,X_n\}$$

$$=Pr\{X_1=x,T_1-T_0\le t\,|\,X_0\}$$

for all $n\ge 0$, $x\in\Omega$ and $t\ge 0$.

It follows that the series of states $\{X_n;n\ge 0\}$ forms a Markov chain ([29]).
In the following we restrict the considered Markov renewal sequences to the ones whose RTPs compose a strictly monotone increasing series ($T_0<T_1<T_2<\ldots$). We generally suppose that the studied process starts at $T_0=0$.

Since we are interested in the probabilistic behaviour of the continuous time stochastic processes during a positive period of time, we will not take care of the states of the process visited for 0 duration of time.

Proposition 2.3 *We will interpret the studied stochastic processes to be right continuous by $Z(t)=Z'(t^+)$, $\quad t\ge 0$.*

---

[1]Considering Example 1 any time instant of state transition to State 1 or State 3 is a regenerative time point. The futher time instants of state transitions can also be regenerative time points depending on the given case of the example.

[2]It is referred as regeneration time in [29] p. 298 for renewal processes.

[3]This sequence of random variables is called Markov renewal process in [29], but it is referred to as Markov renewal sequence in some later works ([25, 24]).

2.1.1 Semi-Markov processes

The time continuous stochastic process defined as the continuous time extension of a Markov renewal sequence is called semi-Markov process.

Definition 2.4 *$Z(t)$ is a (homogeneous)* **semi-Markov process** *(SMP) if a $\{X_n, T_n; n \geq 0\}$ Markov renewal sequence exists and*

$$Z(t) = X_n \ , \qquad \text{if} \ \ T_n \leq t < T_{n+1} \ \ .$$

There are some obvious consequences of this definition:

- $T_n, \ n \geq 0$ are RTPs of the process,
- there is no state transition between two consecutive RTPs,
- there can be RTP without real state transition (this case is considered as a virtual state transition from state i to state i [47]).

From the definition of the time homogeneous Markov renewal sequence one can argue that the probability

$$Pr\{X_1 = j, T_1 - T_0 \leq t \,|\, X_0 = i\} \ \ , \qquad i, j \in \Omega$$

plays a central role in the description of the Markov renewal sequences and the corresponding probability

$$Q_{ij}(t) = Pr\{Z(T_1) = j, T_1 \leq t \,|\, Z(0) = i\} \qquad i, j \in \Omega$$

in the description of the SMPs as well. The matrix $\mathbf{Q}(t) = \{Q_{ij}(t)\}$ is called the **kernel** of the SMP and summarizes all the information on the process that is necessary for evaluating its probabilistic behaviour. However, it is not unique since there can be different kernels describing the same SMP[4].

The Markov chain $\{X_n; n \geq 0\}$ is called the **embedded Markov chain** (EMC) of the SMP. According to this approach the time points T_n are called embedded time points, since the embedded Markov chain is formed by sampling the SMP at these time instants. The $\{X_n, T_n; n \geq 0\}$ Markov renewal sequence is also called the embedded Markov renewal sequence.

The one step state transition matrix of the EMC ($\mathbf{\Pi} = \{p_{ij}\}$) is derived from the kernel as:

$$p_{ij} = Pr\{Z(T_1) = j \,|\, Z(0) = i\} = \lim_{t \to \infty} Q_{ij}(t) \ \ . \tag{2.1}$$

For the purpose of the later studied subjects, we discuss some further features of SMPs. During this discussion we diverge a bit from the usual way of analyzing SMPs.

There are two possible interpretations of the evolution of a SMP:

- being in a given RTP, first, the next state is chosen from a discrete distribution (independent on the waiting time) and then the waiting time is sampled considering the next state on a (generally) continuous distribution,

[4] See Appendix B.

- being in a given RTP, first, the waiting time is sampled on a (generally) continuous distribution (independent on the next state), then the next state is chosen on a discrete distribution considering the waiting time.

Let us define these distributions respectively based on the kernel of the SMP. The (unconditional) distribution of the next state p_{ij}, which is sometimes referred as switching probability, has been already introduced (2.1). The probability distribution of the waiting time conditioned on the next state is written as

$$H_{ij}(t) = Pr\{T_1 \leq t \,|\, Z(T_1) = j, Z(0) = i\} = \frac{Q_{ij}(t)}{p_{ij}} \quad ,$$

the (unconditional) distribution of the waiting time is obtained as

$$Q_i(t) = Pr\{T_1 \leq t \,|\, Z(0) = i\} = \sum_{j \in \Omega} Q_{ij}(t) \quad ,$$

and finally the switching probability conditioned on the holding time is given by

$$p_{ij}(t) = Pr\{Z(T_1) = j \,|\, T_1 = t, Z(0) = i\} =$$

$$\lim_{\Delta \to 0} \frac{Pr\{Z(T_1) = j, t < T_1 \leq t + \Delta \,|\, Z(0) = i\}}{Pr\{t < T_1 \leq t + \Delta \,|\, Z(0) = i\}} =$$

$$\lim_{\Delta \to 0} \frac{Q_{ij}(t + \Delta) - Q_{ij}(t)}{Q_i(t + \Delta) - Q_i(t)} = \frac{dQ_{ij}(t)}{dQ_i(t)} \quad .$$

p_{ij} and $H_{ij}(t)$ are the functions for the description of the SMP according to the first interpretation, while $Q_i(t)$ and $p_{ij}(t)$ defines the distributions according to the second one.

There are available results for the steady state analysis of a SMP in time domain. We are mainly interested in the transient analysis and we note that the steady state result can be obtained as a particular case of the transient description by $t \to \infty$[5].

Let us denote the state transition matrix by $\mathbf{V}(t)$, whose elements are

$$V_{ij}(t) = Pr\{Z(t) = j \,|\, Z(0) = i\} \quad .$$

Theorem 2.5 *The state transition probability ($V_{ij}(t)$) satisfies the following equation [47]:*

$$V_{ij}(t) \;=\; \delta_{ij}\left[1 - Q_i(t)\right] \;+\; \sum_{k \in \Omega} \int_{h=0}^{t} V_{kj}(t-h)\, dQ_{ik}(h) \tag{2.2}$$

[5]The applied analysis method gives the transform domain description of the system behaviour. Hence the steady state results can be obtained without inverse transforming (by $s \to 0$), while the transient results can be obtained only by a numerical or a simbolical inverse transform method.

Proof:

Based on the above defined properties of the SMP in the RTPs and by conditioning on the time to the next RTP ($T_1 = h$) we have:

$$V_{ij}(t \,|\, T_1 = h) \;=\; \begin{cases} \delta_{ij} & \text{if} : h \;>\; t \\ \displaystyle\sum_{k \in \Omega} \frac{dQ_{ik}(h)}{dQ_i(h)} \cdot V_{kj}(t-h) & \text{if} : h \;\le\; t \end{cases} \tag{2.3}$$

where δ_{ij} is the Kronecker delta[6]. In (2.3) two mutually exclusive events are defined. If there is no RTP up to t the value of the state transition probability can be 1 (if $i = j$) or 0 (if $i \neq j$). If the first RTP occurs before t a state transition (real or virtual) happens and the state transition probability can be evaluated independently from that time.

Towards the analysis of a SMP based on (2.3), the first step is the evaluation of the unconditional state transition probabilities based on the distribution of T_1, which is $Q_i(t)$:

$$V_{ij}(t) \;=\; \int_{h=t}^{\infty} \delta_{ij}\, dQ_i(h) + \int_{h=0}^{t} \sum_{k \in \Omega} V_{kj}(t-h)\, dQ_{ik}(h) \tag{2.4}$$

Equation 2.2 is obtained from Equation 2.4 by evaluating the first integral.

□

By solving this integral equation set we have the transient behaviour of a SMP in time domain. The convolution in (2.2) suggests us to look for the solution also in transform domain.

Let us denote the Laplace transform *(LT)* and the Laplace-Stieltjes transform *(LST)* of $F(t), t \ge 0$ as $F^*(s)$ and $F^\sim(s)$ respectively[7], where:

$$F^*(s) \;=\; \int_0^{\infty} e^{-st}\, F(t)\, dt \qquad \text{and} \qquad F^\sim(s) \;=\; \int_0^{\infty} e^{-st}\, d\,F(t) \;\;.$$

The introduction of the second one is useful for the cases in which $F_X(t)$ is the distribution function of a positive random variable X, because

$$F_X^\sim(s) \;=\; E\left\{e^{-sX}\right\} \;\;.$$

By transforming (2.2) into LST domain we have:

$$V_{ij}^\sim(s) \;=\; \delta_{ij}\,[1 - Q_i^\sim(s)] + \sum_{k \in \Omega} Q_{ik}^\sim(s)\, V_{kj}^\sim(s) \tag{2.5}$$

---

[6] $\delta_{ij} \;=\; \begin{cases} 1 & \text{if} : i \;=\; j \\ 0 & \text{if} : i \;\neq\; j \end{cases}$

[7] Appendix A contains the main properties of these transforms.

By rearranging (2.5) into matrix form we obtain:

$$\mathbf{V}^{\sim}(s) \;=\; \mathbf{Q_D}^{\sim}(s) \;+\; \mathbf{Q}^{\sim}(s)\,\mathbf{V}^{\sim}(s) \tag{2.6}$$

where $\mathbf{Q_D}^{\sim}(s)$ is a diagonal matrix with elements $\{1 - Q_i^{\sim}(s)\}$. Finally the solution of (2.6) can be easily derived as:

$$\mathbf{V}^{\sim}(s) \;=\; [\mathbf{I} - \mathbf{Q}^{\sim}(s)]^{-1}\,\mathbf{Q_D}^{\sim}(s) \tag{2.7}$$

The $[\mathbf{I} - \mathbf{Q}^{\sim}(s)]^{-1}$ matrix is called the Markov renewal kernel, and its elements are called the Markov renewal functions in [29].

The transient analysis based on (2.7) requires the application of symbolical or numerical inversion methods. However, the steady state results can be directly obtained without the inversion as:

$$\mathbf{V}(\infty) = \lim_{t\to\infty} \mathbf{V}(t) \;=\; \lim_{s\to 0} \mathbf{V}^{\sim}(s) \tag{2.8}$$

where $\mathbf{V}(\infty)$ is the steady state probability matrix.

Continuous Time Markov Chain

Let us introduce as a special SMP the Continuous Time Homogeneous Markov Chain (CTMC) by its special property, that every time instant $t \geq 0$ is a RTP.

Definition 2.6 *$Z(t)$ is a* **Continuous Time Homogeneous Markov Chain** *if*

$$\begin{aligned} &Pr\,\{Z(t_{n+1}) = x_{n+1}\,|\,Z(t_n) = x_n, Z(t_{n-1}) = x_{n-1}, \ldots, Z(t_1) = x_1\} \\ &\quad = Pr\,\{Z(t_{n+1}) = x_{n+1}\,|\,Z(t_n) = x_n\} \\ &\quad = Pr\,\{Z(t_{n+1} - t_n) = x_{n+1}\,|\,Z(0) = x_n\} \end{aligned}$$

for all $0 \leq t_1 \leq \ldots \leq t_{n-1} \leq t_n \leq t_{n+1}$ *and* $x_1, \ldots, x_n, x_{n+1} \in \Omega$.

With reference to [47], we just mention that a CTMC can be described by its (time independent) infinitesimal generator matrix $\mathbf{A}$, whose $a_{ij};\ i \neq j$ elements are the transition rates from state i to state j ($a_{ij} \geq 0;\ i \neq j$) and whose diagonal elements are $a_{ii} = -\sum_{i\in\Omega, i\neq j} a_{ij}$ ($a_{ii} \leq 0$). The following (canonical) kernel defines a CTMC with infinitesimal generator matrix $\mathbf{A}$:

$$Q_{ij}(t) \;=\; \begin{cases} \dfrac{a_{ij}}{-a_{ii}}\,(1 \,-\, e^{a_{ii}\,t}) & \text{if} : i \,\neq\, j \\[2ex] 0 & \text{if} : i \,=\, j \end{cases} \tag{2.9}$$

2.1.2 Markov regenerative processes

In this subsection we introduce a more general stochastic process that for $t > 0$ contains RTPs as well, but state transitions between any two consecutive RTPs[8] are allowed.

Definition 2.7 *$Z(t)$ is a (homogeneous)* **Markov regenerative process** *(MRP) if there exists a Markov renewal sequence $\{X_n, T_n; n \geq 0\}$ that*

$$Pr\{Z(T_n + t_1) = x_1, \ldots, Z(T_n + t_m) = x_m \,|\, Z(T_n), Z(u), 0 \leq u < T_n\} =$$

$$Pr\{Z(T_n + t_1) = x_1, \ldots, Z(T_n + t_m) = x_m \,|\, Z(T_n)\}$$

for all $m \geq 1$, $0 < t_1 < \ldots < t_m$ and $x_1, \ldots, x_m \in \Omega$.

This definition can be expressed in words as, $Z(t)$ is a MRP if there exists a Markov renewal sequence $\{X_n, T_n;\ n \geq 0\}$ of random variables such that all the finite dimensional distributions of $\{Z(T_n + t);\ t \geq 0\}$ given $\{Z(u),\ 0 \leq u < T_n,\ X_n = i\}$ are the same as those of $\{Z(t);\ t \geq 0\}$ given $X_0 = i$.

On the other hand, from the homogeneity of the process, Definition 2.7 states that a MRP process viewed from two RTPs with the same states (for example $Z(t - T_n)$ and $Z(t - T_m)$ if $X_n = X_m$) forms the probabilistic replica of each other. The Markov renewal sequence $\{X_n, T_n;\ n \geq 0\}$ is sometimes referred to as the embedded Markov renewal sequence of the MRP.

Following the line of the former subsection we discuss the transient analysis of MRPs. At the beginning let us define the state transition probabilities of the process before the next RTP

$$G_{ij}(t) = Pr\{Z(t) = j \,|\, T_1 > t,\ Z(0) = i\} \quad ,$$

and the probabilities which describe the occurrence of the next RTP

$$K_{ij}(t) = Pr\{Z(T_1) = j, T_1 \leq t \,|\, Z(0) = i\} \quad .$$

The matrix $\mathbf{K}(t)$ is the kernel of the embedded Markov regenerative sequence ($\{X_n, T_n; n \geq 0\}$) and plays similar role as $\mathbf{Q}(t)$ for SMPs, but in order to emphasize the difference from the kernel of a SMP we use this different notation. Hence, the switching probability conditioned on the time to the next RTP is:

$$p_{ij}(t) = Pr\{Z(T_1) = j \,|\, T_1 = t, Z(0) = i\} = \frac{dK_{ij}(t)}{dK_i(t)} \quad .$$

In order to use the usual quantities ([29]) let us introduce the following notation:

$$E_{ij}(t) = G_{ij}(t)\,[1 - K_i(t)] = Pr\{Z(t) = j \,|\, T_1 > t,\ Z(0) = i\}\, Pr\{T_1 > t\}$$

$$= Pr\{Z(t) = j,\ T_1 > t,\ |\, Z(0) = i\} \quad ,$$

[8]It is the key property by which the class of MRPs is more general than the class of SMPs.

Theorem 2.8 *The state transition probability ($V_{ij}(t)$) satisfies the following equation [29]:*

$$V_{ij}(t) \;=\; E_{ij}(t) \,+ \sum_{k\in\Omega} \int_{h=0}^{t} V_{kj}(t-h)\, dK_{ik}(h) \tag{2.10}$$

Proof: Let us define the state transition probabilities conditioning on $T_1 = h$:

$$V_{ij}(t \,|\, T_1 = h) \;=\; \begin{cases} G_{ij}(t) & \text{if} : h \,>\, t \\ \displaystyle\sum_{k\in\Omega} \frac{dK_{ik}(h)}{dK_i(h)} \cdot V_{kj}(t-h) & \text{if} : h \,\le\, t \end{cases} \tag{2.11}$$

In (2.11), similarly to (2.3) two mutually exclusive events are defined. If there is no RTP up to t, $G_{ij}(t)$ is the probability of the state transition by its definition. If there is at least one RTP before t the process jumps to the next regeneration state (which can be i as well in general) according to the switching probabilities and due to the property of the RTPs, the state transition probability is evaluated from that time.

By evaluating the unconditional state transition probability based on the distribution of T_1 ($K_i(t)$) (2.11) becomes:

$$\begin{aligned} V_{ij}(t) \;&=\; \int_{h=t}^{\infty} G_{ij}(t)\, dK_i(h) + \int_{h=0}^{t} \sum_{k\in\Omega} V_{kj}(t-h)\, dK_{ik}(h) \\ &=\; G_{ij}(t)\,[1-K_i(t)] \,+ \sum_{k\in\Omega} \int_{h=0}^{t} V_{kj}(t-h)\, dK_{ik}(h) \end{aligned} \tag{2.12}$$

Equation 2.12 yields Equation 2.10 by substituting $E_{ij}(t)$ for $G_{ij}(t)\,[1-K_i(t)]$.

□

The solution of (2.10) can be performed in the same manner as (2.2). The transformation of (2.10) into LST domain results in:

$$V_{ij}^{\sim}(s) \;=\; E_{ij}^{\sim}(s) \,+ \sum_{k\in\Omega} K_{ik}^{\sim}(s)\, V_{kj}^{\sim}(s) \tag{2.13}$$

whose matrix form is:

$$\mathbf{V}^{\sim}(s) \;=\; \mathbf{E}^{\sim}(s) \,+\, \mathbf{K}^{\sim}(s)\, \mathbf{V}^{\sim}(s) \tag{2.14}$$

and the matrix form solution can be written as:

$$\mathbf{V}^{\sim}(s) \;=\; [\mathbf{I} - \mathbf{K}^{\sim}(s)]^{-1}\, \mathbf{E}^{\sim}(s) \tag{2.15}$$

Equations (2.10) $-$ (2.15) are the usual closed form equations for the MRPs, and matrices $\mathbf{K}$(t) (called **external kernel**) and $\mathbf{E}$(t) (called **internal kernel**) are the usual description tools of a MRP. Similarly to the SMP case, the external kernel $\mathbf{K}$(t) is not unique, because of the possibility of having identical states in consecutive RTPs[9].

For the steady state solution we refer to (2.8).

The evolution of MRPs can be divided into independent parts by the RTPs.

[9]See Appendix B for canonical representation of MRPs.

Definition 2.9 *The stochastic process (denoted by $Z^i(t)$) subordinated to a MRP starting from state i in a RTP up to the next RTP is the restriction of the MRP $Z(t)$ for $t \leq T_1$ given $Z(T_0) = i; T_0 = 0$:*

$$Z^i(t) \; = \; [Z(t) \, : \, 0 \leq t \leq T_1, \, Z(0) = i]$$

referred to as the subordinated process of state i.

In this way, the evolution of a MRP is composed of an internal evolution inside the subordinated process (described by $\mathbf{E}(t)$) and an external evolution due to the occurrence of the RTPs (described by $\mathbf{K}(t)$).

2.1.3 Determined processes

In the rest of this study different properties of stochastic processes are studied, and for this purpose it is very important to see how much the sources of information on the considered stochastic process defines its properties.

Definition 2.10 *A stochastic process is said to be* **determined** *by a given description tool if all of its finite dimensional distributions*

$$Pr \left\{ Z(t_1) \leq x_1, \ldots, Z(t_m) \leq x_m \,|\, Z(0) = x_0 \right\}$$

(for all $m \geq 1$, $0 \leq t_1 \leq \ldots \leq t_m$ and $x_0, x_1, \ldots, x_m \in \Omega$), can be derived based on the description tool.

The CTMCs, and SMPs are determined by their infinitesimal generators ($\mathbf{A}$) and kernels ($\mathbf{Q}(t)$), respectively.

A MRP given by its external and internal kernels, $\mathbf{K}(t)$ and $\mathbf{E}(t)$ respectively is not determined, since the subordinated process is described only by the state probabilities

$$G_{ij}(t) = Pr \left\{ Z(t) = j \,|\, T_1 > t, \, Z(0) = i \right\} = \frac{E_{ij}(t)}{1 - K_i(t)}$$

which corresponds to case $m = 1$ of Definition 2.10.

The external evolution of a MRP (i.e. the occurrence of the RTPs and the states in RTPs) is indeed the evolution of the embedded Markov renewal sequence. This embedded Markov renewal sequence is determined by its kernel $\mathbf{K}(t)$, hence a MRP can become determined by its external kernel and by a description tool which determines the subordinated process. It is the case when the subordinated process is a CTMC determined by $\mathbf{A}$ and when the subordinated process is a SMP determined by $\mathbf{Q}(t)$; and moreover the subordinated process can be a determined MRP as well.

2.2 Introduction to Stochastic Reward Processes

Example 2:

Let us consider a reliability system of one machine and one repair man. The repair man immediately starts the repair of the machine, when it fails.

The company that sold the machine has to pay punishment for the time during the machine is down.

How much punishment should the company pay in a year?

When does the amount of punishment exceed 10.000 UC (unit of currency)?

The adopted modelling framework consists in describing the behaviour of the system configuration in time by means of a stochastic process, called the *structure-state process*, and by associating to each state of the structure-state process a non-negative real constant representing the effective working capacity or performance level or cost or stress of the system in that state. The real variable associated to each structure-state is called the *reward rate* [47]. The structure-state process together with the reward rates forms the Stochastic Reward Model (SRM) [80].

Let the *structure-state process* $Z(t)$ $(t \geq 0)$ be a (right continuous) stochastic process defined over a discrete and finite state space Ω of cardinality n. Let f be a non-negative real-valued function defined as:

$$f[Z(t)] = r_i \geq 0 \ , \qquad \text{if} \quad Z(t) = i \tag{2.16}$$

$f[Z(t)]$ represents the instantaneous reward rate associated to state i.

Definition 2.11 *The* **accumulated reward** $B(t)$ *is a random variable which represents the accumulation of reward in time.*

During the sojourn of $Z(t)$ in state i between t and $t + \delta$, $B(t)$ increases by $r_i \, \delta$. $B(t)$ is a stochastic process that depends on $Z(u)$ for $0 \leq u \leq t$ [29]. However, a transition in $Z(t)$ may induce a modification in the accumulation process depending whether the transition entails a *loss of accumulated reward* , or *no loss of accumulated reward*[10]. A transition which does not entail any loss of reward already accumulated by the system is called *preemptive resume* (first transition on Figure 2.1), and its effect on the model is that the functional $B(t)$ resumes the previous value in the new state. A transition which entails the loss of reward accumulated by the system is called *preemptive repeat* (second and third transitions on Figure 2.1), and its effect on the model is that the functional $B(t)$ is reset to 0 in the new state.

A state whose outgoing transitions are all of preemptive resume type is called a *preemptive resume (prs)* state, while a state whose outgoing transitions are all of preemptive repeat type is called a *preemptive repeat (prt)* state.

A possible realization of the accumulation process $B(t)$ is shown in Figure 2.1. State j is a preemptive resume state while state i and k are preemptive repeat states.

The complementary question concerning the reward accumulation of SRMs is the question of the time for completing a given (possibly random) work requirement (i.e. time to accumulate the required amount of reward).

[10]The class of the discussed SRMs is much wider [47, 85, 81], but we restrict our attention to the later studied cases.

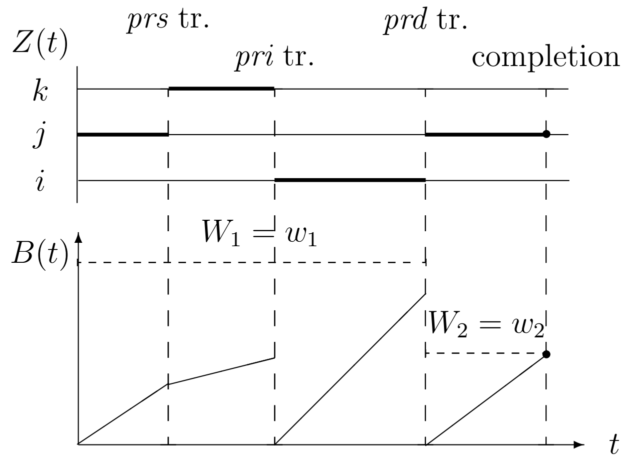

Figure 2.1: The behaviour of the functional $B(t)$ versus time.

Definition 2.12 *The* **completion time** C *is a random variable representing the time to accumulate a reward requirement equal to a random variable* W*:*

$$C \;=\; min\; [t \geq 0 \;:\; B(t) \;=\; W] \;\;.$$

C is the time at which the work accumulated by the system reaches the value W for the first time. Hence, W acts as an absorbing barrier for the functional $B(t)$. With reference to Figure 2.1, the completion time is the time at which $B(t)$ hits the barrier W for the first time and is absorbed.

We assume, in general, that W is a random variable with distribution $W(w)$ with support on $(0,\ \infty)$. The degenerate case, in which W is deterministic and the distribution $W(w)$ becomes the unit step function $U(w - w_d)$, can be considered as well. When W is a random variable and the preemption policy is *prt*, two cases arise depending whether the repeated task has the identical work requirement as the original preempted task (*preemptive repeat identical (pri)* - policy) (second transition on Figure 2.1), or a different work requirement sampled from the same distribution (*preemptive repeat different (prd)* - policy) (third transition on Figure 2.1). In the latter case, each when the functional $B(t)$ goes to zero, the barrier height W is resampled from the same distribution $W(w)$, while in the former case W maintains an identical value.
For a barrier height $W\;=\;w$, the completion time $C(w)$ is defined as:

$$C(w) \;=\; \min\; [t \geq 0 \;:\; B(t) \;=\; w] \;\;. \tag{2.17}$$

Let $C(t, w)$ be the *Cdf* of the completion time when the barrier height is w:

$$C(t, w) \;=\; Pr\,\{C(w) \;\leq\; t\} \tag{2.18}$$

The completion time C of a SRM with *prs* and *pri* transitions is characterized by the following distribution:

$$\hat{C}(t) \;=\; Pr\,\{\, C \leq t\} \;=\; \int_0^{\infty} C(t, w)\, dW(w) \tag{2.19}$$

The distribution of the completion time $C(t, w)$ incorporates the effect of a random variation of the execution speed consequent to a degradation and reconfiguration process, combined with the effect of the preemption and recovery policy on the execution of the task.

The following relationships between the different preemption policies can be easily established. If the work requirement W is an exponential random variable, the two policies prs and prd give rise to the same completion time (due to the memoryless property of the exponential distribution, the residual task requirement under the prs policy coincides with the resampled requirement under the prd policy). On the other hand, if W is deterministic, the two policies pri and prd are coincident (resampling a step function provides always the same constant value).

Moreover, assuming that the structure-states are all of prs type, so that no loss of reward occurs, the distribution of the completion time is closely related to the distribution of the accumulated reward by means of the following relation:

$$Pr\{B(t) \leq w\} = Pr\{C(w) \geq t\} \tag{2.20}$$

2.3 A survey of SRMs applied for analyzing dependable systems

Kulkarni et al. [60] derived the closed form Laplace transform equations of $C(t, w)$ when $Z(t)$ is a *CTMC* and all the states belong to the same preemption class. The extension to a semi-Markov $Z(t)$ process whose state space is partitioned into the three preemption classes has been considered in [61]. Bobbio and Trivedi [19] studied the case where $Z(t)$ is a *CTMC*, the work requirement W is a phase type (*PH*) random variable[11] [74] and the task execution policy is a probabilistic mixture of *prs* and *prd* policies. The combination of *prs* and *pri* policies has been investigated in [21, 22] having as an object the evaluation of the completion time of a program on a gracefully degradable computing system.

The properties of stochastic reward processes have been studied since a long time [66, 29, 57, 58, 47], however, only recently, SRMs have received attention as a modelling tool in performance/reliability evaluation. Indeed, the possibility of associating a reward variable to each structure state increases the descriptive power and the flexibility of the model.

Different interpretations of the structure-state process and of the associated reward structure give rise to different applications [69]. Common assignments of the reward rates are: execution rates of tasks in computing systems (the computational capacity) [5, 84], number of active processors (or processing power) [8, 42], throughput [68, 38, 45], average response time [48, 59, 63] or response time distribution [86, 79, 72].

To point out the reliability aspects, one of the most important interpretations is the accumulation of the stress of a real systems in the different down states. It is

[11] A random variable is said to be a phase type random variable, if it is the time to reach the absorbing state group of a finite state continuous time Markov chain. The distribution function of phase type random variables are rational functions in Laplace transform domain.

often the case in the practical reliability systems [76, 34, 40, 78, 82]. For example in a nuclear power plant the effect of the breakdown depends on the leaked radiation rather than on the state of the system. Moreover the most important measures of the classical reliability theory [4] can be viewed as a particular case of *SRM* obtained by constraining the reward rates to be binary variables.

Two main different points of view have been assumed in the literature when dealing with *SRM* for degradable systems [60]. In the *system oriented* point of view the most significant measure is the total amount of work done by the system in a finite interval. The accumulated reward is a random variable whose distribution function is sometimes called *performability* [67]. Various numerical techniques for the evaluation of the performability have been investigated in recent papers: [68, 49, 32, 41, 83, 35]. In the *user oriented* (or *task oriented*) point of view the system is regarded as a server, and the emphasis of the analysis is on the ability of the system to accomplish an assigned task in due time. Consequently, the most characterizing measure becomes the probability of accomplishing an assigned service in a given time. The task oriented point of view is a more direct representation of the quality of service.

Gaver [37] analyzed the distribution of the completion time for a two state server with different mechanisms of interruption and recovery policies. Extensions to the above model were provided in [75], while the completion time problem for fault tolerant computing systems was addressed in [20]. A unified formulation to the system oriented and the user oriented point of view was provided by Kulkarni et al. in [60, 61, 77]. An alternative interpretation of the completion time problem can be given in terms of the hitting time of an appropriate cumulative functional [29] against an absorbing barrier equal to the work requirement. The definition of a cumulative functional was first suggested by Kulkarni et al. [60] and then explicitly exploited in [13], where the completion time was modelled as a first hitting time against an absorbing barrier. This interpretation leads the above problem into the main stream of absorption problems in stochastic models and has proved to be useful in association with Stochastic Petri nets [9] and with the extension to multi-reward models [10, 13].

2.3.1 Classification of the SRM problems

To characterize the SRM problems we introduce a structure of the considered parameters.

Stochastic process The stochastic behaviour of the structure-state process gains a significant importance at the first sight. SRMs of the well behaved stochastic processes (CTMC, SMP) are detailed in several above referred papers [60, 61, 77, 85], but the analysis when the structure-state process is a MRP can be considered as a new issue of this work.

Preemption policy The effect of the state transitions, which can be *prs*, *prd* and *pri*, can depend on several parameters. The existence of the different policies in a single model increases its modelling power on the one hand, but it increases the complexity of the analysis on the other hand. This work does not discuss models in which the different policies are allowed simultaneously, however, there are models studied exclusively with *prs*, *prd* and *pri* states.

Evaluated measure The analysis of SRMs means indeed two analysis problems, i.e. evaluation of the distribution of the accumulated reward and of the completion time. For the below discussed Petri Net analysis purposes this work includes the study of both problems.

Absorbing subset of states There are practically important modelling problems in which the entrance of the structure state process in a special subset of states stops the accumulation of further reward independent of the later life of the model. For the purpose of the analysis a subset of this kind can be considered as an absorbing one. To have general results we suppose that our model can contain an absorbing subset of states. Hence the state space without the absorbing subset of states can be analyzed as a special case.

State dependent measures An other interesting approach of reward models is the "state dependent" analysis of the above introduced measures. This kind of measures are defined as:

- the probability of completion in a given state before t
- the probability of being in a given state at time t suppose that $C > t$

Some state dependent measures are introduced and analyzed in Section 5.6 for the first time (including the mentioned ones), but some other was introduced in [61].

Chapter 3

Analysis of reward Markov regenerative process

Example 3:

Let us consider Example 1. The company that sold the machines has to pay punishment for the time during any machine is down. The punishment is different in State 2, 3 and 4.

How much punishment should the company pay in a year?

When does the amount of punishment exceed 10.000 UC?

Let us consider these problems for Case III of Example 1.

This chapter summarizes the analysis of some of the considerable cases of the SRMs. The analyzed cases are chosen according to the purposes of the later studied problems, but the introduced methods and results can be extended to the evaluation of the similar measures of the processes not included in this study[1]. The subsections are organized from the general problems (MRPs) to the special ones (SMPs). At the end special cases are taken into consideration.

3.1 Completion time of a reward MRP with subordinated semi-Markov processes and with *prs* states

The aim of this section is the evaluation of the distribution function of the completion time of reward MRPs, but there are two main problems:

- The $\mathbf{K}(t)$ and $\mathbf{E}(t)$ matrices of a MRP do not contain enough information for the evaluation of the distribution of the completion time and the accumulated reward since only the state probabilities of the subordinated processes are known by them[2].

[1] An extended manuscript is available with the analysis of further measures.

[2] Only the mean of the accumulated reward can be evaluated based on $\mathbf{K}(t)$ and $\mathbf{E}(t)$. See Appendix C.

- In the applied analysis method the reward accumulation is evaluated state transition by state transition. The occurrence of the next state transition of a MRP depends on several reasons determined by the subordinated process, the last RTP, the last regeneration state and the present state.

The analysis of the distribution of the completion time and of the accumulated reward requires a complete description of the structure state process. Let us take into consideration the case when the subordinated processes are SMPs with given kernels $\mathbf{Q}^j(t) = \{Q^j_{ik}(t)\}$ (for subordinated process starting from state j). In this way, the distribution of the sojourn time of the states inside a regeneration period are known, and the analysis of the completion time becomes possible.

The internal kernel $\mathbf{E}(t)$ of a MRP of this kind is defined by its external kernel $\mathbf{K}(t)$ and by the kernels of the subordinated processes $\mathbf{Q}^j(t)$ as follows:

$$E_{ij}(t) \; = \; V^i_{ij}(t)[1 - K_i(t)] \tag{3.1}$$

where $V^i_{ij}(t)$ is the ij element of the state transition matrix of the subordinated process starting from state i, which can be evaluated based on $\mathbf{Q}^j(t)$ by Equation (2.7).

Due to the second problem of the evaluation of the completion time a complicated measure is defined which includes all of the information describing the occurrence of the next state transition of the process.

Let us introduce

$$C^j_i(t, w, b) = Pr(C(w) \leq t | Z(-b) = j, Z(0) = i, T_0 = -b, T_1 > 0)$$

where $C(w)$ is the time to complete a work requirement equal to w, b is the time passed from the last regeneration time point T_0, j is the last regeneration state (i.e. $Z(T_0) = j$).

In this way, $C^j_i(t, w, b)$ is the probability that the completion time $C(w)$ is not greater than t if the reward accumulation starts at time 0, inside the regeneration period started in state j at time $-b$.

Theorem 3.1 *For the completion time ($C^j_i(t, w, b)$) the following equation holds:*

$$\begin{aligned} C^j_i(t, w, b) \; = \; & U\left(t - \frac{w}{r_i}\right) \left[1 - K_j(\frac{w}{r_i} + b)\right] \left[1 - Q^j_i(\frac{w}{r_i})\right] \\ & + \int_{h=b}^{w/r_i + b} \sum_{k \in \Omega} C^k_k(t - (h - b), w - (h - b) r_i, 0) \; \left[1 - Q^j_i(h - b)\right] dK_{jk}(h) \\ & + \int_{g=0}^{w/r_i} \sum_{k \in \Omega} C^j_k(t - g, w - g\, r_i, b + g) \; [1 - K_j(g + b)] \; dQ^j_{ik}(g) \end{aligned} \tag{3.2}$$

Proof: Conditioning on the time duration of the regeneration period $H = T_1 - T_0 = h$ and on the time to the next transition in the subordinated process $G = g$, let us define:

$$
C_i^j(t,w,b|\,H=h,\,G=g) \;=\; \begin{cases} U\left(t-\dfrac{w}{r_i}\right) & \\ \qquad\qquad\qquad\qquad \text{if} : (h-b)\,r_i \geq w \text{ and } g\,r_i \geq w & \\ \displaystyle\sum_{k\in\Omega} \frac{dK_{jk}(h)}{dK_j(h)} \cdot C_k^k(t-(h-b), w-(h-b)r_i, 0) & \\ \qquad\qquad\qquad\qquad \text{if} : h-b \leq g \text{ and } (h-b)\,r_i < w & \\ \displaystyle\sum_{k\in\Omega} \frac{dQ_{ik}^j(g)}{dQ_i^j(g)} \cdot C_k^j(t-g, a-gr_i, b+g) & \\ \qquad\qquad\qquad\qquad \text{if} : g < h-b \text{ and } g\,r_i < w & \end{cases} \tag{3.3}
$$

where $U(t)$ is the unit step function. In (3.3), three mutually exclusive events are identified. If $r_i \neq 0$ and $(h-b)\,r_i \geq w$ and $g\,r_i \geq w$, the completion time equals w/r_i. If $(h-b)\,r_i < w$ and $(h-b) < g$ then the regeneration period completes and a transition occurs to state k with probability $dK_{jk}(h)/dK_j(h)$ and the residual service $(w-(h-b)r_i)$ should be accomplished in a new regeneration period starting from state k at time $(t-(h-b))$. If $g\,r_i < w$ and $g < (h-b)$ then a transition occurs inside the subordinated process to state k with probability $dQ_{ik}^j(g)/dQ_i^j(g)$ and the residual service $(w-g\,r_i)$ should be accomplished starting from state k at time $(t-g)$ still inside the same regeneration period.
The mean of the conditional expression in Equation (3.3) with respect to H is:

$$
C_i^j(t,w,b\,|\,G=g) \;=\; \begin{cases} \displaystyle\int_{h=b}^{w/r_i+b} \sum_{k\in\Omega} C_k^k(t-(h-b), w-(h-b)r_i, 0)\; dK_{jk}(h) & \\ \quad + U\left(t-\dfrac{w}{r_i}\right) \left[1-K_j(\dfrac{w}{r_i}+b)\right] & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} : g\,r_i \geq w & \\ \displaystyle\int_{h=b}^{g+b} \sum_{k\in\Omega} C_k^k(t-(h-b), w-(h-b)r_i, 0)\; dK_{jk}(h) & \\ \quad + \displaystyle\sum_{k\in\Omega} \frac{dQ_{ik}^j(g)}{dQ_i^j(g)} \cdot C_k^j(t-g, w-g\,r_i, b+g)\;\; [1-K_j(g+b)] & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} : g\,r_i < w & \end{cases} \tag{3.4}
$$

and the mean of $C_i^j(t,w,b\,|\,G=g)$ in (3.4) is:

$$
\begin{aligned}
C_i^j(t,w,b) \;=\; & U\left(t - \frac{w}{r_i}\right) \left[1 - K_j(\frac{w}{r_i} + b)\right] \left[1 - Q_i^j(\frac{w}{r_i})\right] \\
& + \int_{h=b}^{w/r_i+b} \sum_{k\in\Omega} C_k^k(t-(h-b), w-(h-b)r_i, 0)\; dK_{jk}(h) \left[1 - Q_i^j(\frac{w}{r_i})\right] \\
& + \int_{g=0}^{w/r_i} \int_{h=b}^{g+b} \sum_{k\in\Omega} C_k^k(t-(h-b), w-(h-b)r_i, 0)\; dK_{jk}(h)\; dQ_i^j(g) \\
& + \int_{g=0}^{w/r_i} \sum_{k\in\Omega} C_k^j(t-g, w-g\,r_i, b+g)\; \left[1 - K_j(g+b)\right]\; dQ_{ik}^j(g)
\end{aligned}
\tag{3.5}
$$

Equation (3.2) is obtained from (3.5) by changing the order of the integrals.

□

The integral equation set (3.2) gives the distribution of the completion time of a MRP whose subordinated processes are SMPs. The evaluation of (3.2) in general cases, is very complicated, however we can study two restricted classes of reward MRPs, which provide tractable results.

3.1.1 Memoryless regeneration periods

Example 4:

Let us consider a reliability system with one machine and one repair man. Both of the failure and repair times are exponentially distributed random variables. The company that sold the machine has to pay punishment for the time interval during the down time exceeds a given (random) limit.

When does the amount of punishment exceed 10.000 UC?

The main difficulty of the evaluation of $C_i^j(s,w,b)$ is the fact that it depends on the time passed from the beginning of the regeneration period. If the sojourn time of the regeneration period is memoryless (i.e. exponentially distributed) than the analysis of the completion time becomes simpler since the dependence of the future on the past history of the system in the embedded points of the subordinated processes and in the regeneration points is comprised by two discrete variables which are the identifiers of the present state and the initial state of the subordinated process.

Let us define

$$C_i^j(t,w) = C_i^j(t,w,0) = Pr(C(w) \le t | Z(T_0) = j, Z(0) = i, T_0 \le 0, T_1 > 0)$$

for the completion time of a MRP of this kind.

Corollary 3.2 *For $C_i^j(t,w)$ the following transform domain equation holds:*

$$
\begin{aligned}
C_i^{j\sim}(s,w) \;=&\\
&e^{-(s+\lambda_i)\,w\,/\,r_i}\;\left[1-Q_i^j(\frac{w}{r_i})\right]\\
&+\int_{h=0}^{w/r_i} e^{-s\,h}\sum_{k\in\Omega} C_k^{k\sim}(s,w-h\,r_i)\;\left[1-Q_i^j(h)\right]dK_{jk}(h)\\
&+\int_{g=0}^{w/r_i} e^{-(s+\lambda_i)\,g}\;\sum_{k\in\Omega} C_k^{j\sim}(s,w-g\,r_i)\;dQ_{ik}^j(g)
\end{aligned}
\tag{3.6}
$$

Proof:

Conditioning on the time to the next regeneration time point $H = T_1 = h$ and on the time to the next transition in the subordinated process $G = g$, let us define:

$$
C_i^j(t,w|\,H=h,\,G=g) \;=\; \begin{cases}
U\left(t\;-\;\dfrac{w}{r_i}\right) & \\
\qquad\qquad\qquad \text{if}: h\,r_i \geq w \text{ and } g\,r_i \geq w & \\
\displaystyle\sum_{k\in\Omega}\frac{dK_{jk}(h)}{dK_j(h)}\;\cdot\;C_k^k(t-h,w-hr_i) & \\
\qquad\qquad\qquad \text{if}: h \leq g \text{ and } h\,r_i < w & \\
\displaystyle\sum_{k\in\Omega}\frac{dQ_{ik}^j(g)}{dQ_i^j(g)}\;\cdot\;C_k^j(t-g,w-gr_i) & \\
\qquad\qquad\qquad \text{if}: g < h \text{ and } g\,r_i < w &
\end{cases}
\tag{3.7}
$$

The mean of the conditional completion time in Equation (3.7) with respect to H and G is:

$$
\begin{aligned}
C_i^j(t,w) \;=&\\
&U\left(t\;-\;\frac{w}{r_i}\right)\;\left[1-K_j(\frac{w}{r_i})\right]\;\left[1-Q_i^j(\frac{w}{r_i})\right]\\
&+\int_{h=0}^{w/r_i}\sum_{k\in\Omega} C_k^k(t-h,w-h\,r_i)\;\left[1-Q_i^j(h)\right]dK_{jk}(h)\\
&+\int_{g=0}^{w/r_i}\sum_{k\in\Omega} C_k^j(t-g,w-g\,r_i)\;\left[1-K_j(g)\right]\;dQ_{ik}^j(g)
\end{aligned}
\tag{3.8}
$$

For an exponentially distributed regeneration period starting from state j $K_j(t) = 1 - e^{-\lambda_j t}$. By using this fact and by taking the Laplace-Stieltjes transform with respect to t (3.6) is obtained.

□

3.1.2 Embedded Reward SMP

Example 5:

Let us consider Example 1 with Case III. The company that sold the machines has to pay punishment for the time during any machine is down. The punishment is the same in State 2, 3 and 4.

How much punishment should the company pay in a year?

When does the amount of punishment exceed 10.000 UC?

In the above MRPs the state transition inside the regeneration periods of the MRPs plays important role in the amount of the accumulated reward. There are special Markov regenerative reward processes in which the internal state transitions do not effect the process of reward accumulation.

An MRP whose regenerative periods have constant reward rates independent of the state available in the given regenerative period forms a simple SMP with kernel $\mathbf{K}(t)$ from the point of view of the reward accumulation.

This requirement can be formulated by the non-zero elements of the $\mathbf{E}(t)$ matrix and by the reward rates:

- A Markov regenerative reward process fits the requirements of this class if for all j for which $E_{ij}(t) > 0$ for any t (i.e. state j available in the regeneration period) $r_j = r_i$, being r_i the reward rate at the beginning of the regeneration period.

Starting the reward accumulation in a regeneration time point let us define

$$C_i(t,w) = Pr(C(w) \leq t | Z(T_0) = i)$$

for the completion time of a MRP of this kind.

Corollary 3.3 *The completion time of a MRP of this kind is [61, 15]:*

$$C_i^{\sim *}(s,v) = \frac{r_i \left[1 - K_i^{\sim}(s + v\, r_i)\right]}{s + v\, r_i} + \sum_{k \in \Omega} K_{ik}^{\sim}(s + v\, r_i)\, C_k^{\sim *}(s,v) \tag{3.9}$$

Proof: Conditioning on the time to the next regeneration time point $H = T_1 = h$

$$C_i(t, w | \, H = h) = \begin{cases} U\left(t - \dfrac{w}{r_i}\right) & \text{if}: h\, r_i \geq w \\ \displaystyle\sum_{k \in \Omega} \frac{dK_{ik}(h)}{dK_i(h)} \cdot C_k(t-h, w - hr_i) & \text{if}: h\, r_i < w \end{cases} \tag{3.10}$$

Evaluating the mean of $C_i(t, w | \, H = h)$ taking the Laplace-Stieltjes transform with respect to t and taking the Laplace transform with respect to w (3.10) becomes (3.9).

□

3.2 Analysis of SRMs with absorbing state group and with *prs* states

Example 6:

Let us consider a reliability system of three operating machines (A, B and C) and one repair man. Machine C has higher priority than machine A and B, while machine B has higher priority than machine A. The repair of a lower priority machine is always interrupted by the failures of a higher priority machine.

The failure times of the machines are exponentially distributed random variables, but their repair times are any other positive random variables. The repair work on a lower priority machine done up to the preemption is resumed after the completion of the repair of higher priority machines.

The company that sold the machines has to pay punishment for the time during any machine is down. The punishment is different for the different machines. The contract is given up if all of the three machines are failed at the same time.

How much punishment should the company pay in a year in the frame of the contract?

When does the amount of punishment exceed 10.000 UC?

A stochastic process with *prs* states accumulates continuously the reward without any loss of the accumulated reward; thus in every realization of the stochastic process the accumulated reward up to t ($B(t)$) is a monotonically increasing function.

3.2.1 MRP with subordinated SMPs

In this subsection we take into consideration MRP whose subordinated processes are SMPs with finite state space Ω partitioned in two exhaustive and mutually exclusive subsets R and R^c. The considered reward accumulation starts at time 0 in any state of R at a regeneration time point or in an embedded time point of one of the subordinated SMP and stops if it enters R^c or if it reaches the required reward limit i.e. it completes. There is no state transition after the stop of the process.

The reward accumulation is restricted to subset R, and R^c can be considered as an absorbing subset. There is no reward accumulation in R^c: i.e. $r_i = 0$ for $i \in R^c$. It follows from the above definition of the process that there is no transition inside the R^c subset.

Let us define the truncated distribution of the completion time for any state $i, j \in R$ as

$$C_i^j(t, w, b) = Pr(C(w) \leq t \,|\, Z(-b) = j, Z(0) = i, T_0 = -b, T_1 > 0)$$

A jump from R to R^c before the completion disables the further reward accumulation and the completion. Thus $C_i^j(t, w, b)$ is a defective distribution function ($\lim_{t \to \infty} C_i^j(t, w, b) \leq 1$).

Theorem 3.4 *The completion time ($C_i^j(t, w, b)$) satisfies the following equation:*

$$
\begin{aligned}
C_i^j(t,w,b) \;=\; & \\
& U\left(t - \frac{w}{r_i}\right) \left[1 - K_j(\frac{w}{r_i}+b)\right] \left[1 - Q_i^j(\frac{w}{r_i})\right] \\
& + \int_{h=b}^{w/r_i+b} \sum_{k\in R} C_k^k(t-(h-b), w-(h-b)r_i, 0) \left[1 - Q_i^j(h-b)\right] dK_{jk}(h) \\
& + \int_{g=0}^{w/r_i} \sum_{k\in R} C_k^j(t-g, w-g\,r_i, b+g) \; [1 - K_j(g+b)] \; dQ_{ik}^j(g)
\end{aligned} \tag{3.11}
$$

Proof: Conditioning on the time duration of the regeneration period $H = T_1 - T_0 = h$ and on the time to the next transition in the subordinated process $G = g$, let us define:

$$
C_i^j(t,w,b|\, H=h,\, G=g) \;=\; \begin{cases}
U\left(t - \dfrac{w}{r_i}\right) & \\
\qquad\qquad \text{if} : (h-b)\,r_i \geq w \text{ and } g\,r_i \geq w & \\
\displaystyle\sum_{k\in R} \frac{dK_{jk}(h)}{dK_j(h)} \cdot C_k^k(t-(h-b), w-(h-b)r_i, 0) & \\
\qquad\qquad \text{if} : h-b \leq g \text{ and } (h-b)\,r_i < w & \\
\displaystyle\sum_{k\in R} \frac{dQ_{ik}^j(g)}{dQ_i^j(g)} \cdot C_k^j(t-g, w-gr_i, b+g) & \\
\qquad\qquad \text{if} : g < h-b \text{ and } g\,r_i < w &
\end{cases} \tag{3.12}
$$

The evaluation of the mean of $C_i^j(t,w,b|\, H=h,\, G=g)$ results in 3.11.

□

The other obvious measure of SRMs of this kind is the probability of leaving the state group R before the completion. Due to the properties of the studied process we can define easily this measure starting from state $i \in R$:

$$
D_i^j(t,w,b) = Pr(Z(t) \in R^c, C(w) > t \,|\, Z(-b) = j, Z(0) = i, T_0 = -b, T_1 > 0)
$$

$$
= Pr(Z(t) \in R^c \,|\, Z(-b) = j, Z(0) = i, T_0 = -b, T_1 > 0, W = w)
$$

Corollary 3.5 *For $D_i^j(t,w,b)$ the following equation holds:*

$$
\begin{aligned}
D_i^j(t,w,b) \;=& \\
& \int_{h=b}^{w/r_i+b} \sum_{k\in R} D_k^k(t-(h-b), w-(h-b)r_i, 0) \; \Big[1-Q_i^j(h-b)\Big] \, dK_{jk}(h) \\
& + \int_{h=b}^{w/r_i+b} \sum_{k\in R^c} U(t-(h-b)) \; \Big[1-Q_i^j(h-b)\Big] \, dK_{jk}(h) \\
& + \int_{g=0}^{w/r_i} \sum_{k\in R} D_k^j(t-g, w-g\,r_i, b+g) \; [1-K_j(g+b)] \; dQ_{ik}^j(g) \\
& + \int_{g=0}^{w/r_i} \sum_{k\in R^c} U(t-g) \; [1-K_j(g+b)] \; dQ_{ik}^j(g)
\end{aligned}
\tag{3.13}
$$

Proof:
Conditioning on H and G as before, for state $i \in R$ let us define:

$$
D_i^j(t,w,b|\, H=h,\, G=g) \;=\; \begin{cases}
0 & \text{if} : (h-b)\,r_i \ge w \text{ and } g\,r_i \ge w \\[2ex]
\displaystyle\sum_{k\in R} \frac{dK_{jk}(h)}{dK_j(h)} \cdot D_k^k(t-(h-b), w-(h-b)r_i, 0) + \\
\qquad \displaystyle\sum_{k\in R^c} \frac{dK_{jk}(h)}{dK_j(h)} \cdot U(t-(h-b)) \\
 & \text{if} : h-b \le g \text{ and } (h-b)\,r_i < w \\[2ex]
\displaystyle\sum_{k\in R} \frac{dQ_{ik}^j(g)}{dQ_i^j(g)} \cdot D_k^j(t-g, w-gr_i, b+g) + \\
\qquad \displaystyle\sum_{k\in R^c} \frac{dQ_{ik}^j(g)}{dQ_i^j(g)} \cdot U(t-g) \\
 & \text{if} : g < h-b \text{ and } g\,r_i < w
\end{cases}
\tag{3.14}
$$

The mean of $D_i^j(t,w,b|\, H=h,\, G=g)$ with respect to H and G is (3.14).

□

We can study this MRP from a system oriented point of view, which means that we are interested in the amount of work (accumulated reward) done by the system up to time t, rather than a user who is interested in the time required to complete a task (accumulate a given amount of reward). To this end let us introduce $S_i^j(t,w,b)$, the distribution of the accumulated reward up to t, as:

$$
S_i^j(t,w,b) = Pr(B(t) \le w \,|\, Z(-b)=j, Z(0)=i, T_0=-b, T_1>0)
$$

$$
= Pr(C(w) > t \,|\, Z(-b)=j, Z(0)=i, T_0=-b, T_1>0)
$$

Corollary 3.6 *For $S_i^j(t,w,b)$ the following equation holds:*

$$
\begin{aligned}
S_i^j(t,w,b) \;=\; & \\
& \Big[U(t) - U\Big(t - \frac{w}{r_i}\Big)\Big]\ \Big[1 - K_j(\frac{w}{r_i} + b)\Big]\ \Big[1 - Q_i^j(\frac{w}{r_i})\Big] \\
& + \int_{h=b}^{w/r_i+b} \sum_{k\in R} S_k^k(t-(h-b), w-(h-b)r_i, 0)\ \Big[1 - Q_i^j(h-b)\Big]\, dK_{jk}(h) \\
& + \int_{h=b}^{w/r_i+b} \sum_{k\in R^c} U(t-(h-b))\ \Big[1 - Q_i^j(h-b)\Big]\, dK_{jk}(h) \\
& + \int_{h=b}^{w/r_i+b} U(t) - U(t-(h-b))\ \Big[1 - Q_i^j(h-b)\Big]\, dK_j(h) \\
& + \int_{g=0}^{w/r_i} \sum_{k\in R} S_k^j(t-g, w-g\,r_i, b+g)\ [1 - K_j(g+b)]\ dQ_{ik}^j(g) \\
& + \int_{g=0}^{w/r_i} \sum_{k\in R^c} U(t-g)\ [1 - K_j(g+b)]\ dQ_{ik}^j(g) \\
& + \int_{g=0}^{w/r_i} U(t) - U(t-g)\ [1 - K_j(g+b)]\ dQ_i^j(g)
\end{aligned}
\tag{3.15}
$$

Proof: By conditioning on H and G we have:

$$
S_i^j(t,w,b|\,H=h,\,G=g) \;=\; \begin{cases}
U(t) - U\Big(t - \dfrac{w}{r_i}\Big) \\
\qquad\qquad \text{if} : (h-b)\,r_i \ge w \text{ and } g\,r_i \ge w \\[2ex]
\displaystyle\sum_{k\in R} \frac{dK_{jk}(h)}{dK_j(h)} \cdot S_k^k(t-(h-b), w-(h-b)r_i, 0) + \\
\quad U(t) - U(t-(h-b)) + \\
\quad \displaystyle\sum_{k\in R^c} \frac{dK_{jk}(h)}{dK_j(h)} \cdot U(t-(h-b)) \\
\qquad\qquad \text{if} : h-b \le g \text{ and } (h-b)\,r_i < w \\[2ex]
\displaystyle\sum_{k\in R} \frac{dQ_{ik}^j(g)}{dQ_i^j(g)} \cdot S_k^j(t-g, w-gr_i, b+g) + \\
\quad U(t) - U(t-g) + \displaystyle\sum_{k\in R^c} \frac{dQ_{ik}^j(g)}{dQ_i^j(g)} \cdot U(t-g) \\
\qquad\qquad \text{if} : g < h-b \text{ and } g\,r_i < w
\end{cases}
\tag{3.16}
$$

which yields (3.16) by evaluating the mean of $S_i^j(t,w,b|\,H=h,\,G=g)$ with respect to H and G.

□

The former defined measure $S_i^j(t,w,b)$ has the disadvantage that it covers the fact that the system still has the ability to accumulate additional amount of reward or it is already absorbed by the R^c state group which excludes the further reward accumulation. Let us define the modified measure which is sensitive to this difference:

$$P_i^j(t,w,b) = Pr(B(t) \le w, Z(t) \in R \,|\, Z(-b) = j, Z(0) = i, T_0 = -b, T_1 > 0)$$

To explain in words, while $S_i^j(t,w,b)$ is the distribution of the accumulated reward up to t, $P_i^j(t,w,b)$ is the distribution of the accumulated reward up to t supposed that the process did not leave the subset R up to t.

Corollary 3.7 $P_i^j(t,w,b)$ *satisfies the following equation:*

$$\begin{aligned}
P_i^j(t,w,b) \; = \; & \\
& \left[U(t) - U\left(t - \frac{w}{r_i}\right)\right] \left[1 - K_j(\frac{w}{r_i} + b)\right] \left[1 - Q_i^j(\frac{w}{r_i})\right] \\
& + \int_{h=b}^{w/r_i+b} \sum_{k\in R} P_k^k(t-(h-b), w-(h-b)r_i, 0) \left[1 - Q_i^j(h-b)\right] dK_{jk}(h) \\
& + \int_{h=b}^{w/r_i+b} U(t) - U(t-(h-b)) \left[1 - Q_i^j(h-b)\right] dK_j(h) \\
& + \int_{g=0}^{w/r_i} \sum_{k\in R} P_k^j(t-g, w-g\,r_i, b+g) \; [1 - K_j(g+b)] \; dQ_{ik}^j(g) \\
& + \int_{g=0}^{w/r_i} U(t) - U(t-g) \; [1 - K_j(g+b)] \; dQ_i^j(g)
\end{aligned} \tag{3.17}$$

Proof: By the usual way of conditioning on H and G, let us define:

$$P_i^j(t,w,b|\, H=h,\, G=g) \; = \; \begin{cases} U(t) - U\left(t - \dfrac{w}{r_i}\right) \\ \qquad\qquad \text{if} : (h-b)\, r_i \ge w \text{ and } g\, r_i \ge w \\[2ex] U(t) - U(t-(h-b)) + \\ \qquad \displaystyle\sum_{k\in R} \frac{dK_{jk}(h)}{dK_j(h)} \cdot P_k^k(t-(h-b), w-(h-b)r_i, 0) \\ \qquad\qquad \text{if} : h-b \le g \text{ and } (h-b)\, r_i < w \\[2ex] U(t) - U(t-g) + \\ \qquad \displaystyle\sum_{k\in R} \frac{dQ_{ik}^j(g)}{dQ_i^j(g)} \cdot P_k^j(t-g, w-gr_i, b+g) \\ \qquad\qquad \text{if} : g < h-b \text{ and } g\, r_i < w \end{cases} \tag{3.18}$$

(3.18) is the mean of $P_i^j(t,w,b|\, H=h,\, G=g)$ with respect to H and G.

□

3.2.2 Results for SMP in double transform domain

Example 7:

Let us consider Example 6 with the only difference, that the repair work on a lower priority machine done up to the preemption is lost, and the repair restarts after the completion of the repair of higher priority machine(s).

Following the structure of the previous section we discuss the analysis of the special case when the structure state process is a SMP [61, 15, 85]. A linear equation set can be defined in this case which describes the solution in double transform domain.

Let us define the distribution of the completion time for any state of R as

$$C_i(t, w) = Pr(C(w) \le t | Z(0) = i).$$

A jump from R to R^c before the completion disables the further reward accumulation and the completion. Thus $C_i(t, w)$ is a defective distribution function.

Corollary 3.8 *$C_i(t, w)$ satisfies the following equation in double transform domain:*

$$C_i^{\sim *}(s, v) \;=\; \frac{r_i\,[1 \,-\, Q_i^{\sim}(s \,+\, v\,r_i)\,]}{s \,+\, v\,r_i} \;+\; \sum_{k \in R} Q_{ik}^{\sim}(s \,+\, v\,r_i)\, C_k^{\sim *}(s, v) \qquad (3.19)$$

Proof: Conditioning on the time to the next embedded time point H, let us define:

$$C_i(t, w | \, H = h) \;=\; \begin{cases} U\left(t \,-\, \dfrac{w}{r_i}\right) & \text{if} : h\, r_i \ge w \\[2ex] \displaystyle\sum_{k \in R} \frac{dQ_{ik}(h)}{dQ_i(h)} \cdot C_k(t-h, w-hr_i) & \text{if} : h\, r_i < w \end{cases} \qquad (3.20)$$

Evaluating the mean with respect to H, taking the Laplace-Stieltjes transform with respect to t and taking the Laplace transform with respect to w (3.20) becomes (3.19).

□

Let us define the probability of leaving the state group R before the completion as before:

$$D_i(t, w) = Pr(Z(t) \in R^c, C(w) > t | Z(0) = i) = Pr(Z(t) \in R^c | Z(0) = i, W = w)$$

Corollary 3.9 *For $D_i(t, w)$ the following equation holds in double transform domain:*

$$D_i^{\sim *}(s, v) \;=\; \frac{1}{v} \sum_{k \in R^c} Q_{ik}^{\sim}(s \,+\, v\,r_i) \;+\; \sum_{k \in R} Q_{ik}^{\sim}(s \,+\, v\,r_i)\, D_k^{\sim *}(s, v) \qquad (3.21)$$

Proof: Conditioning on the time to the next embedded time point H, let us define:

$$D_i(t,w|\,H=h) \;=\; \begin{cases} 0 & \text{if}: h\,r_i \ge w \\[2ex] \displaystyle\sum_{k\in R} \frac{dQ_{ik}(h)}{dQ_i(h)} \cdot D_k(t-h, w-hr_i) & \\ \qquad + \displaystyle\sum_{k\in R^c} \frac{dQ_{ik}(h)}{dQ_i(h)} \cdot U(t-h) & \text{if}: h\,r_i < w \end{cases} \tag{3.22}$$

Evaluating the mean with respect to H, taking the Laplace-Stieltjes transform with respect to t and taking the Laplace transform with respect to w results (3.21).

□

Let us introduce the appropriate distribution of the accumulated reward as:

$$S_i(t,w) = Pr(B(t) \le w|Z(0)=i) = Pr(C(w) > t|Z(0)=i)$$

Corollary 3.10 *For $S_i(t,w)$ the following equation holds in double transform domain:*

$$\begin{aligned} S_i^{\sim *}(s,v) \;=\; & \frac{1}{v}\left[1 + \sum_{k\in R^c} Q_{ik}^{\sim}(s\,+\,v\,r_i)\right] + \frac{r_i\,[Q_i^{\sim}(s\,+\,v\,r_i)\,-\,1\,]}{s\,+\,v\,r_i} \\ & + \sum_{k\in R} Q_{ik}^{\sim}(s\,+\,v\,r_i)\,S_k^{\sim *}(s,v) \end{aligned} \tag{3.23}$$

Proof: Conditioning on H, we can write:

$$S_i(t,w|\,H=h) \;=\; \begin{cases} U(t) - U\left(t\,-\,\dfrac{w}{r_i}\right) & \text{if}: h\,r_i \ge w \\[2ex] U(t)\;-\;U(t-h) & \\ \qquad + \displaystyle\sum_{k\in R^c} \frac{dQ_{ik}(h)}{dQ_i(h)} \cdot U(t-h) & \\ \qquad + \displaystyle\sum_{k\in R} \frac{dQ_{ik}(h)}{dQ_i(h)} \cdot S_k(t-h, w-hr_i) & \text{if}: h\,r_i < w \end{cases} \tag{3.24}$$

(3.24) results (3.23) by evaluating the mean of $S_i(t,w|\,H=h)$ with respect to H, by taking the Laplace-Stieltjes transform with respect to t and by taking the Laplace transform with respect to w.

□

And finally, the second system oriented measure which takes into consideration the ability of the further reward accumulation is:

$$P_i(t,w) = Pr(B(t) \le w, Z(t) \in R|Z(0)=i) = Pr(C(w) > t, Z(t) \in R|Z(0)=i)$$

Corollary 3.11 *For $P_i(t,w)$ the following equation holds in double transform domain:*

$$P_i^{\sim *}(s,v) \;=\; \frac{1}{v} + \frac{r_i\,[-1 + Q_i^{\sim}(s + v\,r_i)\,]}{s + v\,r_i} + \sum_{k\in R} Q_{ik}^{\sim}(s + v\,r_i)\,P_k^{\sim *}(s,v) \quad (3.25)$$

Proof: Conditioning on H, we have:

$$P_i(t,w|\,H=h) \;=\; \begin{cases} U(t) - U\left(t - \dfrac{w}{r_i}\right) & \text{if} : h\,r_i \geq w \\[2ex] U(t) \;-\; U(t-h) & \\ \qquad + \displaystyle\sum_{k\in R} \frac{dQ_{ik}(h)}{dQ_i(h)} \cdot P_k(t-h, w - hr_i) & \text{if} : h\,r_i < w \end{cases} \quad (3.26)$$

Evaluating the mean with respect to H, taking the Laplace-Stieltjes transform with respect to t and taking the Laplace transform with respect to w (3.26) becomes (3.25).

$\square$

3.3 Analysis of SRM with absorbing state group and with *prd* states

In a SMP with positive diagonal elements in $\mathbf{Q}(t)$ or in a MRP there can be two basically different events in an embedded or regenerative time points; there can be real state transition from state i to state j ($j \neq i$) and there can be virtual state transition from state i to state i. In the *prs* case (Section 3.2) there is no difference between the two cases concerning the reward accumulation process (i.e. the $B(t)$ function is continuous for any realization in both cases); but in case of *prd* or *pri* type states the reward accumulation restarts after a transition from state i to state j ($j \neq i$) and it is resumed continuously after a virtual transition in an embedded time point.

For the analysis of the *prd* and *pri* cases we have to use an accurate way to handle this difference. In case of SMP a practical technique is to introduce $\mathbf{Q}^u(t)$ the kernel of the SMP without positive diagonal element (i.e. virtual transition). In the Appendix B the generation of $\mathbf{Q}^u(t)$ from any $\mathbf{Q}(t)$ is defined. We use the notation $\mathbf{Q}^{uj}(t)$ for the kernel of the process subordinated to the regeneration period starting from state j without positive diagonal element. (The generation of $\mathbf{Q}^{uj}(t)$ from any $\mathbf{Q}^j(t)$ is the same as the generation of $\mathbf{Q}^u(t)$ by any $\mathbf{Q}(t)$.)

On the other hand, the problem of virtual and real state transitions in the RTPs can not be avoided without special restrictions of the external kernel and the kernels of the subordinated SMPs. For the analysis of the general cases we have to introduce an additional variable to indicate the amount of the accumulated reward at the RTPs. (The analysis of SMPs based on kernels with positive diagonal element requires the introduction of the same variable, but in that case the extra variable can be avoided by the canonical representation of the kernel $\mathbf{Q}^u(t)$.)

In the following we restrict the studied cases (to the measure $C_i(t)$[3]) noting that the results for the other cases can be obtained by applying a similar reasoning.

3.3.1 MRP with subordinated SMPs

Let us denote the random work requirement by W whose distribution is $W(w)$, and the completion time of the work requirement W by C whose distribution is $C(t)$. The relation of the former defined $C(w)$ (with distribution $C(t,w)$) and the unconditional C is obvious by

$$\hat{C}(t) = \int_{w=0}^{\infty} C(t,w)\, dW(w)$$

Let us define the distribution of the completion time starting from any state $i \in R$, inside a regeneration period starting from j at $-b$ when the actual value of the accumulated reward is d as:

$$C_i^j(t,b,d) = Pr(C \le t \,|\, Z(-b) = j,\, Z(0) = i,\, T_0 = -b,\, T_1 > 0,\, B(0) = d,\, W_1 > d)$$

where W_1 is the residual work requirement whose distribution under the given condition is

$$W(w,d) = \begin{cases} 0 & \text{if} : w \le d \\ \dfrac{W(w) - W(d)}{1 - W(d)} & \text{if} : w > d \end{cases}$$

The introduction of the positive amount of reward at the start of the examined period of the process is necessary because of the possibility of virtual state transition at the start which does not restart the reward accumulation.

Theorem 3.12 *The following equation holds for* $C_i^j(t,b,d)$*:*

$$\begin{aligned}
&C_i^j(t,b,d) = \\
&\int_{w=d}^{\infty} U\left(t - \frac{w-d}{r_i}\right) \left[1 - K_j\left(\frac{w-d}{r_i} + b\right)\right] \left[1 - Q_i^{uj}\left(\frac{w-d}{r_i}\right)\right] dW(w,d) \\
&+ \int_{h=b}^{\infty} \sum_{k \in R, k \neq i} C_k^k(t-(h-b),0,0) \left[1 - Q_i^{uj}(h-b)\right] \left[1 - W((h-b)r_i + d, d)\right] dK_{jk}(h) \\
&+ \int_{h=b}^{\infty} C_i^i(t-(h-b),0,(h-b)r_i + d) \left[1 - Q_i^{uj}(h-b)\right] \left[1 - W((h-b)r_i + d, d)\right] dK_{ji}(h) \\
&+ \int_{g=0}^{\infty} \sum_{k \in R} C_k^j(t-g, b+g) \left[1 - K_j^u(g+b)\right] \left[1 - W(gr_i)\right] dQ_{ik}^{uj}(g)
\end{aligned} \qquad (3.27)$$

[3] An extended version of this manuscript containing measures $D_i(t)$, $S_i(t)$, $P_i(t)$ for *prd* and *pri* preemption policies is available.

Proof: Conditioning on the time duration of the regeneration period $H = T_1 - T_0 = h$, on the time to the next transition in the subordinated process $G = g$, and on the work requirement $W_1 = w$ let us define:

$$
C_i^j(t,\, b,\, d \,|\, H = h,\, G = g,\, W_1 = w > d) \; =
$$

$$
\begin{cases}
U\left(t - \dfrac{w-d}{r_i}\right) & \\
\qquad\qquad \text{if} : (h-b)\, r_i \geq w - d \text{ and } g\, r_i \geq w - d & \\[2ex]
\displaystyle\sum_{k \in R, k \neq i} \frac{dK_{jk}(h)}{dK_j(h)} \cdot C_k^k(t - (h-b), 0, 0) & \\
\qquad + \dfrac{dK_{ji}(h)}{dK_j(h)} \cdot C_i^i(t - (h-b), 0, d + (h-b) r_i) & \\
\qquad\qquad \text{if} : h - b \leq g \text{ and } (h-b)\, r_i < w - d & \\[2ex]
\displaystyle\sum_{k \in R} \frac{dQ_{ik}^{uj}(g)}{dQ_i^{uj}(g)} \cdot C_k^j(t - g, b + g, 0) & \\
\qquad\qquad \text{if} : g < h - b \text{ and } g\, r_i < w - d &
\end{cases}
\tag{3.28}
$$

In (3.28), four mutually exclusive events are identified. If $r_i \neq 0$ and $(h-b)\, r_i \geq w$ and $g\, r_i \geq w$, the completion time equals w/r_i. If $g\, r_i < w$ and $g < (h-b)$ then a transition occurs inside the subordinated process to state k with probability $dQ_{ik}^j(g)/dQ_i^j(g)$ and the residual service $(w - g\, r_i)$ should be accomplished starting from state k at time $(t-g)$ still inside the same regeneration period. If $(h-b)\, r_i < w$ and $(h-b) < g$ then the regeneration period completes. Two mutually exclusive events can occur in this case. If there is a transition to state $k \in R$, $k \neq i$ (with probability $dK_{jk}(h)/dK_j(h)$) at the RTP then the accumulated work is lost (d is reset to 0). If there is no transition at the RTP (i.e. a former regeneration period is completed being in the next regeneration state) then the accumulation is resumed without any interruption.

The mean of the conditional expression in Equation (3.28) with respect to H, G and W_1 is (3.27).

□

3.3.2 Results for SMP in transform domain

Following the structure of the previous section we discuss the analysis of the SMP, to have the solution in transform domain.

Let us define the completion time for any state of R as

$$
C_i(t) = Pr(C \leq t \,|\, Z(0) = i).
$$

Corollary 3.13 *For $C_i(t)$ the following equation holds in Laplace-Stieltjes transform domain:*

$$
\begin{aligned}
C_i^{\sim}(s) \; = \; & \int_{w=0}^{\infty} e^{-s\,w/r_i} \left[1 - Q_i^u(\frac{w}{r_i})\right] \, dW(w) \\
& + \int_{h=0}^{\infty} \sum_{k\in R} C_k^{\sim}(s) \; [1 - W(hr_i)] \; dQ_{ik}^u(h)
\end{aligned}
\tag{3.29}
$$

Proof: Conditioning on the time to the next embedded time point H, let us define:

$$
C_i(t \,|\, H = h,\, W = w) \; = \; \begin{cases} U\left(t \; - \; \dfrac{w}{r_i}\right) & \text{if} : h\,r_i \geq w \\[2ex] \displaystyle\sum_{k\in R} \frac{dQ_{ik}^u(h)}{dQ_i^u(h)} \cdot C_k(t-h) & \text{if} : h\,r_i < w \end{cases}
\tag{3.30}
$$

Evaluating the mean with respect to H and W, taking the Laplace-Stieltjes transform with respect to t (3.30) becomes (3.29).

□

3.4 Analysis of SRM with absorbing state group and with *pri* states

In this section we follow the structure of the former one and we discuss only a restricted group of cases as well.

Similarly to the cases with *prs* states the derivations are explained for a given work requirement w, because the work requirement is the same throughout the lifetime of the process (up to completion or enter R^c).

3.4.1 MRP with subordinated SMPs

According to the above discussed problems of the virtual and real state transitions we also have to introduce the same additional variable d to indicate the amount of accumulated reward at the RTPs, because both kinds of transitions can occur.

Let us define the distribution of the completion time starting from any state $i \in R$, inside a regeneration period starting from j at $-b$ when the actual value of the accumulated reward is d as:

$$C_i^j(t, w, b, d) = Pr(C(w) \leq t \,|\, Z(-b) = j, Z(0) = i, T_0 = -b, T_1 > 0, B(0) = d,\, w > d)$$

Theorem 3.14 $C_i^j(t,w,b,d)$ *satisfies the following equation:*

$$
\begin{aligned}
C_i^j(t,w,b,d) \; = & \\
& U\left(t - \frac{w-d}{r_i}\right) \left[1 - K_j\left(\frac{w-d}{r_i} + b\right)\right] \left[1 - Q_i^{uj}\left(\frac{w-d}{r_i}\right)\right] \\
& + \int_{h=b}^{\frac{w-d}{r_i+b}} \sum_{k \in R, k \neq i} C_k^k(t-(h-b), w, 0, 0) \left[1 - Q_i^{uj}(h-b)\right] dK_{jk}(h) \\
& + \int_{h=b}^{\frac{w-d}{r_i+b}} C_i^i(t-(h-b), w, 0, (h-b)r_i + d) \left[1 - Q_i^{uj}(h-b)\right] dK_{jk}(h) \\
& + \int_{g=0}^{\frac{w-d}{r_i}} \sum_{k \in R} C_k^j(t-g, w, b+g, 0) \; [1 - K_j(g+b)] \; dQ_{ik}^{uj}(g)
\end{aligned}
\tag{3.31}
$$

Proof: Conditioning on the time duration of the regeneration period $H = T_1 - T_0 = h$ and on the time to the next transition in the subordinated process $G = g$, let us define:

$$
C_i^j(t,w,b,d|\, H=h,\, G=g,\, w>d) \; =
$$

$$
\begin{cases}
U\left(t - \dfrac{w-d}{r_i}\right) & \\
\qquad\qquad \text{if} : (h-b)\, r_i \geq w-d \text{ and } g\, r_i \geq w-d & \\
\displaystyle\sum_{k \in R, k \neq i} \frac{dK_{jk}(h)}{dK_j(h)} \cdot C_k^k(t-(h-b), w, 0, 0) & \\
\qquad + \dfrac{dK_{ji}(h)}{dK_j(h)} \cdot C_i^i(t-(h-b), w, 0, (h-b)r_i + d) & \\
\qquad\qquad \text{if} : h-b \leq g \text{ and } (h-b)\, r_i < w-d & \\
\displaystyle\sum_{k \in R} \frac{dQ_{ik}^{uj}(g)}{dQ_i^{uj}(g)} \cdot C_k^j(t-g, w, b+g, 0) & \\
\qquad\qquad \text{if} : g < h-b \text{ and } g\, r_i < w-d &
\end{cases}
\tag{3.32}
$$

(3.31) is resulted by evaluation the mean of (3.32) with respect to H and G.

□

3.4.2 Results for SMP in transform domain

Considering w work requirement the distribution of the completion time for any state of R is defined by

$$C_i(t,w) = Pr(C(w) \leq t | Z(0) = i).$$

Corollary 3.15 *The following transform domain equation holds for $C_i(t,w)$:*

$$C_i^{\sim}(s,w) \;=\; e^{-s\,w/r_i}\left[1 \;-\; Q_i^u(\frac{w}{r_i})\right] \;+\; \sum_{k\in R} C_k^{\sim}(s,w)\, \int_{h=0}^{w/r_i} e^{-sh}\, Q_{ik}^u(h) \qquad (3.33)$$

Proof: Conditioning on the time to the next embedded time point H, let us define:

$$C_i(t,w|\,H=h) \;=\; \begin{cases} U\left(t \;-\; \dfrac{w}{r_i}\right) & \text{if}: h\,r_i \geq w \\ \\ \displaystyle\sum_{k\in R} \frac{dQ_{ik}^u(h)}{dQ_i^u(h)} \;\cdot\; C_k(t-h,w) & \text{if}: h\,r_i < w \end{cases} \qquad (3.34)$$

Evaluating the mean with respect to H, taking the Laplace-Stieltjes transform with respect to t results (3.33).

□

3.5 Application of the general results for special cases

The above studied general cases of SRMs cover a wide range of special SRMs which are more often used for modelling real systems. The way of the derivation of special results from the general ones is summarized in this section.

First of all the consideration of the absence of the absorbing state group is very simple by the substitution of R by Ω and R^c by an empty set.

The first reduction of the generality of the underlying stochastic process is the use of the results for MRP with subordinated SMP to the analysis of the case when the underlying stochastic process is a SMP. For this goal we have to exclude the case of internal state transition inside the regeneration periods of the MRP, i.e. $E_{ij} = 0$ for all $i \neq j$. Considering a SMP with kernel $\mathbf{Q}(t)$ the following substitution can be used:

$$K_{ij}(t) = Q_{ij}(t) \quad Q_{ik}^j(t) = 0 \quad \text{for } \forall\, i,j,k \in \Omega \qquad (3.35)$$

The measures of this case can be described by a linear equation set in double transform domain.

3.5.1 Underlying CTMC

A further restriction of the underlying SMP is the consideration of exponentially distributed delays of the consecutive embedded time points, i.e. the analysis of the underlying CTMC (with infinitesimal generator **A**). The results of this case is obtained by the following substitution of the kernel:

$$Q_{ij}(t) = \begin{cases} \dfrac{a_{ij}}{-a_{ii}}\,(1 - e^{a_{ii}\,t}) & \text{if} : i \neq j \\ 0 & \text{if} : i = j \end{cases} \tag{3.36}$$

and since the results for SMPs are given in transform domain the appropriate substitution of the kernel is as follows:

$$Q_{ij}^{\sim}(s) = \begin{cases} \dfrac{a_{ij}}{s - a_{ii}} & \text{if} : i \neq j \\ 0 & \text{if} : i = j \end{cases} \tag{3.37}$$

As it already appeared in the pioneer work ([60]) the results for underlying CTMC often can be organized into a matrix form by the utilization of the fact that

$$a_{ii} = \sum_{j\in\Omega, j\neq i} a_{ij} \ .$$

To have a view on this fact let us evaluate the probability of leaving the state group R before the completion considering *prs* states, w work requirement, and CTMC underlying stochastic process (with infinitesimal generator **A**).

Without any loss of generality we can suppose that the states numbered $1, 2, \ldots, m$ belong to R ($1, 2, \ldots, m \in R$) and the states numbered $m+1, m+2, \ldots, n$ belong to R^c ($m+1, m+2, \ldots, n \in R^c$). By this ordering of states **A** can be partitioned into the following submatrices $\mathbf{A} = \begin{array}{|c|c|}\hline \mathbf{B} & \mathbf{C} \\ \hline \mathbf{D} & \mathbf{E} \\ \hline \end{array}$ where B contains the transition rates inside R, and C contains the transition rates from R to R^c.

The equation (3.21) describes the examined quantity for underlying SMP. By the substitution (3.37) for underlying CTMC we have:

$$D_i^{\sim *}(s,v) = \frac{1}{v} \sum_{k\in R^c} \frac{a_{ik}}{s + v\,r_i - a_{ii}} + \sum_{k\in R,\, k\neq i} \frac{a_{ik}}{s + v\,r_i - a_{ii}}\, D_k^{\sim *}(s,v) \tag{3.38}$$

which becomes

$$(s + v\,r_i) D_i^{\sim *}(s,v) = \frac{1}{v} \sum_{k\in R^c} a_{ik} + \sum_{k\in R} a_{ik}\, D_k^{\sim *}(s,v) \tag{3.39}$$

Introducing the following column vectors $\underline{D}^{\sim *}(s,v) = \{D_i^{\sim *}(s,v)\}$, $\underline{h} = \{1\}$ and the diagonal matrix of the reward rates $\mathbf{R} = \text{Diag}\{r_i\}$ equation (3.39) can be organized into matrix form:

$$(s\,\mathbf{I} + v\,\mathbf{R})\underline{D}^{\sim *}(s, v) = \frac{1}{v}\,\mathbf{C}\underline{h} + \mathbf{B}\underline{D}^{\sim *}(s, v) \tag{3.40}$$

whose solution is

$$\underline{D}^{\sim *}(s, v) = \frac{1}{v}\,(s\,\mathbf{I} + v\,\mathbf{R} + \mathbf{B})^{-1}\mathbf{C}\underline{h} \tag{3.41}$$

Chapter 4

Timed Petri Nets

The designer and the analyst of a system are interested in the solution of the modelling problem at first instance, and do not care so much how this solution is actually derived. They should be able to describe their system in a natural and easy to use manner. The *modeller's representation* should include enough information to build up an *analytical representation* suitable for numerical solution, and should also contain the specification of the required measures. The modeller's representation should then automatically be transformed into the analytical representation. Finally the numerical results should be again automatically mapped back into the modeller's representation, so that the user of the tool can interpret them in that context. For Markovian systems several tools have been developed in recent years, based on various specification paradigms, as surveyed in [46].

There are, however, situations that are not covered by these tools. One typical situation occurs when the random time characteristic of the system cannot be conveniently approximated by exponential random variables. A second situation occurs when the analyst requires the computation of stochastic measures (like the measures derived in the previous section and in [80, 11]) whose numerical evaluation cannot be performed in the framework of the standard linear first order equations typical of Markovian systems.

In recent years several classes of Stochastic Petri Net (*SPN)* models have been elaborated which incorporate some non exponential characteristics in their definition. The semantics of *SPN*'s with generally distributed transition times has been discussed in [1]. We refer to this model as *Generally Distributed Transition_SPN (GDT_SPN)*. In general, the stochastic process underlying a *GDT_SPN* does not have a numerically tractable analytical formulation, while a simulative approach has been investigated in [43].

Various restrictions of the general *GDT_SPN* model have been discussed in the literature to provide the *analytical representation* of problems to be generated automatically based on their *modeller's representation.*

4.1 Generally Distributed Transition_SPN

A marked Petri Net (*PN*) is a four tuple $PN = (P, T, A, M)$, where:

- $P = \{p_1, p_2, \dots, p_{np}\}$ is the set of places (drawn as circle);
- $TR = \{tr_1, tr_2, \dots, tr_{nt}\}$ is the set of transitions (drawn as bars);
- A is the set of directed arcs from places to transitions and from transitions to places; I, O and H are the input, the output and the inhibitor arcs, respectively. The input function I provides the multiplicities of the input arcs from places to transitions; the output function O provides the multiplicities of the output arcs from transitions to places; the inhibitor function H provides the multiplicity of the inhibitor arcs from places to transitions. (Input and output arcs have an arrowhead on their destination, inhibitor arcs have a small circle.)
- $M = \{m_1, m_2, \dots, m_{np}\}$ is the marking. The generic entry m_i is the number of tokens (drawn as black dots) in place p_i, in marking M.

A transition is enabled in a marking if each of its ordinary input places contains at least as many tokens as the multiplicity of the input function I and each of its inhibitor input places contains fewer tokens than the multiplicity of the inhibitor function H. An enabled transition fires by removing as many tokens as the multiplicity of the input function I from each ordinary input place, and adding as many tokens as the multiplicity of the output function O to each output place. The number of tokens in an inhibitor input place is not affected.

The reachability set $\mathcal{R}(M_0)$ is the set of all the markings that can be generated from an initial marking M_0 by repeated application of the above rules. If the set T comprises both timed and immediate transitions, $\mathcal{R}(M_0)$ is partitioned into tangible (no immediate transitions are enabled) and vanishing markings, according to [2]. Let Ω be the tangible subset of $\mathcal{R}(M_0)$.

Marking M', obtained from M by firing tr_k, is said to be *immediately reachable* from M, and the firing operation is denoted by the symbol $(M - tr_k \rightarrow M')$.

An *execution sequence* $\mathcal{E}$ in a marked *PN*, is a sequence of legal markings obtained by firing a sequence of enabled transitions [1]:

$$\mathcal{E} \;=\; \{\,(M_0)\,;\,(tr_1, M_1)\,;\,\dots\,;\,(tr_j, M_j)\,;\,\dots\}$$

An execution sequence $\mathcal{E}$ is a connected path in the reachability graph $\mathcal{R}(M_0)$ of the net. A timed execution sequence $\mathcal{T}_E$ is an execution sequence $\mathcal{E}$ augmented by a non-decreasing sequence of real non-negative values representing the epochs of firing of each transition, such that consecutive transitions $(t_i\,;\,t_{i+1})$ in $\mathcal{E}$ correspond to ordered epochs $t_i \leq t_{i+1}$ in $\mathcal{T}_E$.

$$\mathcal{T}_E \;=\; \{\,(t_0, M_0)\,;\,(tr_1, t_1, M_1)\,;\,\dots\,;\,(tr_i, t_i, M_i)\,;\,\dots\} \tag{4.1}$$

The time interval $t_{i+1} - t_i$ between consecutive epochs represents the period of time that the *PN* sojourns in marking M_i.

Definition 4.1 *A Timed PN (TPN) is a marked PN in which a set of specifications are provided and a set of rules are defined such that to each legal execution sequence $\mathcal{E}$ a timed execution sequence $\mathcal{T}_E$ can be univocally associated.*

A variety of timing mechanisms have been proposed in the literature. The distinguishing features of the timing mechanisms are whether the duration of the events is modelled by deterministic variables or random variables, and whether the time is associated to the *PN* places, transitions or tokens. If a probability measure is assigned to the ensemble of all the possible execution sequences, a timed execution sequence $\mathcal{T}_E$ is mapped into a right continuous stochastic process $Z_{\mathcal{T}}(t)$, $(t \geq 0)$, called the *Marking Process*. *TPN*s in which the timing mechanism is stochastic are referred to as Stochastic *PN* (*SPN*).

Definition 4.2 *An execution sequence $\mathcal{T}_E$ truncated at t_i ($t_0 = 0$) is called a history of the* PN *and is denoted by H_i.*

Assumption 4.3 *Let H_i be the history and M_i the marking entered by firing tr_i at t_i. We assume that for all i, H_i and M_i, and for all the transitions tr_k enabled in M_i the following firing distributions are uniquely defined:*

$$\mathcal{D}_k(t \,|\, H_i, M_i) \;=\; Pr\,\{\, tr_k \textit{ fires},\;\; t_{i+1} \,-\, t_i \,\leq t \,|\, H_i, M_i\} \tag{4.2}$$

The firing distribution function $\mathcal{D}_k(t \,|\, H_i, M_i)$ is called the kernel of the marking process $Z_{\mathcal{T}}(t)$ and completely characterizes the stochastic realization of a timed execution sequence $\mathcal{T}_E$.

From the Assumption 4.3 follows that the unconditioned probability $p_k(H_i, M_i)$ of selecting tr_k to be the next transition to fire in M_i is:

$$p_k(H_i, M_i) \;=\; \lim_{t\to\infty}\; \mathcal{D}_k(t \,|\, H_i, M_i) \;=\; Pr\,\{\, tr_k \text{ fires } |\, H_i, M_i\} \tag{4.3}$$

with
$$\sum_{\forall tr_k \textit{ enabled in } M_i} p_k(H_i, M_i) \;=\; 1 \quad ,$$

and that the distribution of the sojourn time in M_i before the next transition is:

$$\begin{aligned} \mathcal{D}\,(\,t \,|\, H_i, M_i) \;\;&=\;\; \sum_{\forall tr_k \textit{ enabled in } M_i} \mathcal{F}_k(\,x \,|\, H_i, M_i) \\ &=\;\; Pr\,\{\, t_{i+1} \,-\, t_i \,\leq t \,|\, H_i, M_i\} \end{aligned} \tag{4.4}$$

A *SPN* with stochastic timing associated to the *PN* transitions and with generally distributed firing times was defined in [1], with particular emphasis to the semantical interpretation of the model. We refer to this model as *Generally Distributed Transition_SPN (GDT_SPN)*.

Definition 4.4 *A stochastic GDT_SPN is a marked SPN in which:*

- *To any transition $tr_k \in Tr$ a random variable γ_k is associated, modelling the time needed by the activity represented by tr_k to complete, when tr_k is considered in isolation.*
- *Each random variable γ_k is characterized by the (possibly marking dependent) cumulative distribution function $\mathcal{G}_k(t|M_i)$.*

- *A set of specifications are given for calculating the kernel $\mathcal{D}_k(t \mid H_i, M_i)$ over the ensemble of all the timed execution sequences $\mathcal{T}_E$. This set of specifications is called the* execution policy.

- *An initial probability is given on the reachability set.*

An *execution policy* is a set of specifications for deriving an analytical formulation of the stochastic process associated to a marked *PN*, given the net structure and the set of Cdf's $\mathcal{G}_k(t|M_i)$. The semantics of different execution policies has been discussed in [1]. The *execution policy* comprises two specifications: a criterion to choose the next transition to fire (the *firing policy*), and a criterion to keep memory of the past history of the process (the *memory policy*). The most widely used choice to select the next transition to fire is according to a *race policy*: if more than one transition is enabled in a given marking, the transition fires whose associated random delay is statistically the minimum.

The *memory policy* is the part of the set of specifications of the *execution policy* that defines how the process is conditioned upon the past. We associate to each transition tr_k an *age variable* a_k. The way in which a_k is related to the past history H_i determines the different memory policies. We consider three alternatives:

- *Age memory* - The age variable a_k accounts for the work performed by the activity corresponding to tr_k from its last firing up to the current epoch. The firing distribution depends on the residual time needed for this activity to complete given a_k.

- *Enabling memory* - The age variable a_k accounts for the work performed by the activity corresponding to tr_k from the last epoch in which tr_k has been enabled. The firing distribution depends on the residual time needed for this activity to complete a given a_k. When transition tr_k is disabled (even without firing) the corresponding enabling age variable is reset.

- *Resampling* - The age variable a_k is reset to zero at any change of marking. The firing distribution depends only on the time elapsed in the present marking.

There are environments in which the age and the enabling, or the enabling and the resampling, or all of the memory policies have the same effects. If a timed transition is only exclusively enabled in any reachable tangible marking its memory policy can be either age, enabling or resampling with the same effect. A transition, which can not become disabled after it become enabled without firing, can be either of age or enabling memory policy type with the same properties. The resampling and the enabling memory policies provide the same features for the transitions which become disabled or fire at the next transition in any tangible marking in which the transition is enabled.

It is easy to prove that, when all the *PN* transitions are assigned a resampling policy, the associated stochastic marking process $Z_{\mathcal{T}}(t)$ becomes a SMP. However, the resampling policy very rarely might have a practical application in system modelling.

Transition tr_k is called immediate transition if $Pr(\gamma_k = 0) = 1$. The markings of a GDT_SPN with immediate transitions can be divided into the set of vanishing markings, in which at least one immediate transition is enabled, and the set of tangible markings, in which no immediate marking is enabled. The right continuous stochastic marking process contains only tangible markings. In the rest of this work we restrict the considered random variables (γ_k) associated to the transitions ($tr_k \in TR$) to be positive, thus we exclude the opportunity of more than one transitions in a time instant.

4.2 Overview of the existing GDT_SPN models

The stochastic marking process $Z_{\mathcal{T}}(t)$ does not have, in general, an analytically tractable formulation, while a simulative approach has been described in [43, 44]. Various restrictions of the general model have been discussed in the literature such that the underlying marking process $Z_{\mathcal{T}}(t)$ is confined to belong to a known class of analytically tractable problems.

4.2.1 Exponentially Distributed SPN

When the random variables γ_k associated to the *PN* transitions are exponentially distributed, the dynamic behaviour of the net can be mapped into a time continuous homogeneous Markov chain *(CTMC)*, with state space isomorphic to the reachability graph of the net. This restriction is the most popular in the literature [71, 36], and a number of packages are built on this model [23, 28, 70, 62]. Ajmone Marsan et al. showed that the right continuous marking process of a GDT_SPN with exponentially distributed and immediate transitions is a CTMC as well [2], and the class is called Generalized Stochastic Petri Net.

4.2.2 Semi Markov SPN

When a resampling policy is assigned to all the *PN* transitions the marking process becomes a SMP. This restriction has been studied in [73, 7] but is of little interest in applications where it is difficult to imagine a situation where the firing of each transition of the *PN* has the effect of forcing a resampling resetting to all the other transitions. Only the case in which each transition is competing with all the other ones seems to be appropriate for this model.

A more interesting Semi Markov *SPN* model has been discussed in [33]. In this definition, the transitions are partitioned into three classes: exclusive, competitive and concurrent. Provided that the firing time of all concurrent transitions is exponentially distributed and that competitive transitions are resampled at the time the transition is enabled, the associated marking process becomes a semi Markov process.

4.2.3 Phase Type SPN (PHSPN)

A numerically tractable realization of the *GDT_SPN*, is obtained by restricting the firing time random variables γ_k to be *PH* distributed [74], according to the following:

Definition 4.5 *A PHSPN is a GDT_SPN in which:*

- *To any transition $tr_k \in T$ a PH random variable γ_k is associated with Cdf $\mathcal{G}_k(t|M_i)$. The PH model assigned to transition tr_k has ν_k stages with a single initial stage numbered stage 1 and a single final stage numbered stage ν_k.*

- *To any transition $tr_k \in T$ a memory policy is assigned from the three defined alternatives: age, enabling or resampling memory.*

The distinguishing feature of this model is that it is possible to design a completely automated tool that responds to the requirements stated in [46], and at the same time includes all the issues listed in Definition 4.5. The non-Markovian process generated by the *GDT_SPN* is converted into a *CTMC* defined over an expanded state space. The measures pertinent to the original process can be evaluated by solving the expanded *CTMC*.

The program package *ESP* [31] realizes the *PHSPN* model according to Definition 4.5. The program allows the user to assign a specific *memory policy* to each *PN* transition so that the different execution policies can be put to work. In the *ESP* tool, the expanded *CTMC* is generated from the model specifications (the *PN* topology, and the *PH* models assigned to each timed transition). The generation algorithm is driven by the different execution policies that the user assigns to each transition.

4.2.4 Deterministic SPN

The *Deterministic and Stochastic PN* model has been introduced in [3], with the aim of providing a technique for considering stochastic systems in which the duration of some activities assume a constant value. In [3] only the steady state solution has been addressed. An improved algorithm for the evaluation of the steady state probabilities has been successively presented in [64]. Recently, the *DSPN* model has been revisited by Choi et al. [25]. In [25], the stochastic process associated to the *DSPN* model is proved to be a MRP and an analytical method for the transient solution is provided.

Definition 4.6 *A DSPN is a GDT_SPN in which:*

- *At most, a single deterministic transition (DET) is allowed to be enabled in each tangible marking and the firing time of the deterministic transition is marking independent.*

- *All the other timed transitions $tr_k \in TR$ an exponentially distributed random variable γ_k is associated.*

- *The time elapsed in a* DET *cannot be remembered when the transition becomes disabled; the only allowed execution policy is the* race policy *with* enabling memory.

In order to prove that the marking process associated to a *DSPN* is a MRP, Choi et al. [25] have introduced the following modified execution sequence:

$$\mathcal{T}_E^* = \{ (T_0, M_{(0)}) \, ; \, (tr_{(1)}, T_1, M_{(1)}) \, ; \, \ldots \, ; \, (tr_{(k)}, T_k, M_{(k)}) \, ; \, \ldots \} \tag{4.5}$$

Epoch T_{k+1} is derived from T_k as follows:

1. If no *DET* transition is enabled in marking $M_{(k)}$, define T_{k+1} to be the first time after T_k that a state change occurs. If no such time exists, set $T_{k+1} = \infty$.
2. If a *DET* transition is enabled in marking $M_{(k)}$, define T_{k+1} to be the time when the *DET* transition fires or is disabled as a consequence of the firing of a competitive exponential transition.

According to case 2) of the above definition, during $[T_k,\, T_{k+1})$, the *PN* can evolve in the subset of $\mathcal{R}(M_0)$ reachable from $M_{(k)}$, through exponential transitions concurrent with the given *DET* transition. The marking process during this time interval is a *CTMC* called the *subordinated CTMC* of marking $M_{(k)}$. Therefore, if a *DET* transition is enabled in $M_{(k)}$, the regeneration period is given by the minimum between the first passage time of the *subordinated CTMC* to any of states in which the DET transition is disabled and the constant firing time associated to the *DET* transition.

Choi et al. show that, with Definition 4.6, the time execution sequence $\mathcal{T}_E^*$ is mapped into a MRP. Hence, the kernel of the modified execution sequence satisfies the following condition:

$$\begin{aligned} \mathcal{D}_{(k+1)}(t \mid H_{(k)}, M_{(k)}) &= Pr\,\{ tr_{(k+1)} \text{ fires},\;\; T_{k+1} - T_k \le t \mid H_{(k)}, M_{(k)}\} \\ &= Pr\,\{ tr_{(k+1)} \text{ fires},\;\; T_{k+1} - T_k \le t \mid M_{(k)}\} \\ &= Pr\,\{ tr_{(k+1)} \text{ fires},\;\; T_1 \le t \mid M_{(0)} = M_{(k)}\} \end{aligned} \tag{4.6}$$

The first equality expresses the Markov property (i.e. the condition on the history is condensed in the present state); the second equality expresses the time homogeneity (i.e. the same probability measure holds even if translated in the time axis). Hence $T_k\,;\, k \ge 0$ are RTPs and $\{M_{(k)}, T_k\,;\, k \ge 0\}$ is a Markov renewal sequence.

We simplify the notation by setting: $M_{(k)} = i$ and $M_{(k+1)} = j$ (where marking j is reached by firing $tr_{(k+1)}$). The kernel (4.6) can be written in matrix form:

$$\forall i,\, j \in \mathcal{R}(M_0)$$

$$\mathcal{D}_{(k+1)}(t \mid H_{(k)}, M_{(k)}) = \mathcal{D}_{i\,j}\,(t) = Pr\,\{ M_{(1)} = j;\, T_1 \le t \mid M_{(0)} = i\} \tag{4.7}$$

According to (4.3), we have:

$$p_{i\,j} = \mathcal{D}_{i\,j}\,(\infty) = Pr\,\{ M_{(1)} = j \mid M_{(0)} = i\} \tag{4.8}$$

showing that the sequence $M_{(i)}$ is an *embedded Markov chain.* Furthermore, the Cdf of the time interval T_1 starting from $M_{(0)} = i$ is derived by combining (4.4) with (4.7):

$$\mathcal{F}_i\,(t) = \sum_{\forall tr_{(k+1)}\ of\ M_{(k)}} \mathcal{D}_{ij}(t) \tag{4.9}$$

The kernel defined in (4.6) can be obtained from the specifications given in Definition 4.6. A transient solution for the marking probability has been derived in [25]. The solution is in the form of an integral convolution equation, that can be solved numerically in the time domain. An alternative approach suggested by the authors consists in transforming the transient solution in the Laplace transform domain, and then deriving the time solution by a numerical inversion technique. The paper proposes to use the Jagerman's method [50], as adapted by Chimento and Trivedi [22].

4.2.5 The class MRSPN*

A further extension of DSPN called MRSPN*, has been developed in [24]:

Definition 4.7 *A MRSPN* is a GDT_SPN in which:*

- *At most, a single timed transition with generally distributed firing time is allowed to be enabled in each marking.*
- *All the other timed transitions $tr_k \in TR$ an exponentially distributed random variable γ_k is associated.*
- *The only allowed execution policy is the* race policy *with* enabling memory. *This means that the firing time of the generally distributed transition is sampled at the time the transition is enabled and cannot change until the transition either fires or is disabled.*
- *The firing time distribution may depend upon the marking at the time the transition is enabled.*

The kernel equations from (4.6) to (4.9) still hold; however, the analytic kernel expressions depend on the specific Cdf's assumed in the model. In [24], closed form expressions are derived when the Cdf of the generally distributed transitions is the uniform distribution.

4.3 Modelling Power

The considered models differ because of the different classes of distribution functions they are able to support, and the way in which the history of the process is taken into account to condition the future evolution of the net.

Under the *enabling memory* policy the time accumulated by a *PN* transition is reset as soon as the transition is disabled, while under the *age memory* policy the time accumulates whenever the transition is enabled before firing. The *enabling memory* policy is suited to realize the interaction mechanism among tasks in service that in queueing theory or fault-tolerant systems is called a *preemptive repeat different (prd)* policy. Whenever the task in service is preempted a corresponding *PN* transition is disabled resetting the accumulated time. Hence, when the preempted task restarts its work requirement should be resampled from the same distribution [9]. On the other hand, the *age memory* policy is suited to represent an interaction mechanism usually referred to as *preemptive resume* policy: the server does not lose memory of the work already done even if the task is preempted (and the corresponding *PN* transition disabled). When the task is enabled again the execution restarts from the point it was interrupted.

Between the above introduced models only the *PHSPN* model fully supports the *age memory* policy and then allows the modeller to represent in a natural way *prs* interaction mechanisms. Moreover, if the random variables of the system to be modelled are really of *PH* type, the *PHSPN* provides exact results. Otherwise, a preliminary step is needed in which the random times of the system are approximated by *PH* random variables resorting to a suitable estimation technique [12, 14, 16]. The *DSPN* model, on the other hand, combining constant times with exponential random times, offers an innovative approach in many practical applications.

The main limitation of the DSPN and the MRSPN* models discussed in [3, 65, 27, 25, 24, 39, 26] is that the deterministic (or generally distributed) transitions must be assigned a firing policy of enabling memory type [1]. The memory of the underlying stochastic process cannot extend beyond a single cycle of enable/disable of the non-exponential transition[1]. In the language of queueing systems this assumption implies that the server should work on the job up to completion (the non-exponential transition fires), or if the job is interrupted before completion (due, for instance, to failure or preemption), the work already done is lost [37, 22].

The next chapter proposes a semantical generalization of the DSPN and the MRSPN* models, by the consideration of the stochastic process between two consecutive RTPs to be more general than CTMC on one hand, and by including age memory policy (which allows the modelling of preemptive resume mechanisms) on the other hand. This modelling extension is crucial in connection with fault tolerant and parallel computing systems, where a single task may be interrupted either during a fault/recovery cycle or for the execution of a higher priority task, but when the reason causing the interruption is ceased, the dormant task is resumed from the point it was interrupted. Even if a *prs* execution policy is the main goal of a dependable fault tolerant design, its analytical modelling was not possible in the framework of the available DSPN and MRSPN* tools.

[1]The enabling memory assumption is relaxed in [27] where a deterministic transition can be disabled in vanishing markings only. Since vanishing markings are transversed in zero time, this assumption does not modify the behavior of the marking process versus time.

Chapter 5

Analysis of Markov Regenerative Stochastic Petri Nets

This chapter introduces a general class of the SPNs, called Markov Regenerative SPN (MRSPN), which includes the former introduced classes (i.e.: Exponentially Distributed SPN, Semi-Markov SPN, DSPN and MRSPN*, except PHSPN), but includes other subclasses which have not been studied before. We propose a new approach to the analysis of MRSPNs, by which the analysis of further subclasses (and the studied ones as well) becomes possible. Finally we discuss some restrictions on the language of the SPNs which provide subclasses of MRSPNs that can be analyzed by the proposed method.

5.1 Markov Regenerative Stochastic Petri Net

Choi et al. defined the general class of the Markov Regenerative Stochastic Petri Nets in [24], as:

Definition 5.1 *A SPN is called a Markov Regenerative Stochastic Petri Net (MRSPN)*[1] *if its marking process is a MRP.*

The class of MRSPNs is indeed very large as it is mentioned in [24][2].

Based on the results introduced in Chapter 2 the transient and the steady state analysis of MRSPN is possible by the completion of the following steps:

1. identification of the RTPs of the life time of the stochastic marking process;
2. characterization of the process between two consecutive RTPs (which provides the element of the $\mathbf{E}(t)$ matrix);
3. characterization of the occurrence of the next RTP (which provides the element of the $\mathbf{K}(t)$ matrix);

These steps are analyzed below.

[1] Based on the alternative name of the MRP, which is Semi Regenerative Process, the MRSPNs are referred as Semi Regenerative SPNs (SR_SPN) in [26].

[2] In the same paper the studied subclass of this general class is called MRSPN* which can result confusion in the notations in the subsequent papers ([17]).

5.2 Regenerative time points of MRSPNs

The introduction of the age variables associated to the transitions in Chapter 4 allows us to give a natural definition of the RTPs of MRSPNs.

Definition 5.2 *A regeneration time point T_n in the marking process $Z_{\mathcal{T}}(t)$ is the epoch of entrance in a tangible marking $M_{(n)}$ in which all the age variables a_d; $tr_d \in TR$ are equal to 0.*

Let us denote by $\{T_n; n \geq 0\}$ the strictly monotone sequence of the regeneration time points embedded into a realization $\mathcal{R}$. The tangible marking $M_{(n)}$ entered at a regeneration time point T_n is called a regeneration marking. The embedded sequence of regeneration time points $\{T_n,\, M_{(n)};\, n \geq 0\}$ is a (homogeneous) Markov renewal sequence and the marking process $Z_{\mathcal{T}}(t)$ is a Markov regenerative process as discussed above and in [30, 24, 26, 18]. By the memoryless property of the MRPs in the RTPs the analysis of a MRSPN can be divided into the analysis of evolution between the consecutive RTPs, called regeneration periods.

The memory of a MRSPN is based on the values stored in the age variables and a RTP is the epoch (of a state transition) in which the MRSPN resets its memory.

In accordance with Definition 5.2 the execution policy, that is composed by the firing and the memory policy, can be interpreted as:

memory policies:

- A transition of resampling type restarts at every change in the marking,
- a transition of enabling type restarts if it becomes disable or fires,
- a transition of age type restarts only if it fires,

where the restart of transition tr_k means that its age variable a_d is reset to 0 and its firing time w_d is resampled from the same distribution.

race policy: At the entrance in a new tangible marking, the residual firing time is computed for each enabled timed transition given its age variable a_d and firing time w_d (the amount of work required to fire the transition), so that the next marking is determined by the minimal residual firing time $w_d - a_d$ among the enabled transitions. The age variable a_d is reset to 0 and the firing time w_d is resampled from the same distribution at each restart of transition tr_d.

Because of the memoryless property, the three mentioned policies are equivalent if the firing distribution is exponential. Hence, for a transition tr_d with exponentially distributed firing time, we assume, conventionally, that the corresponding memory policy is of resampling type, so that the age variable a_d is reset at each transition.

5.3 Characterization of the regeneration periods of MRSPNs

The elaboration of the 2nd and 3rd step of the analysis process of MRSPNs requires some further investigations on the stochastic process subordinated to a regeneration period.

Definition 5.3 *The stochastic process subordinated to a regeneration marking* $i \in \{M_{(n)}\}$ *(denoted by* $Z^i_{\mathcal{T}}(t)$*) is the restriction of the marking process* $Z_{\mathcal{T}}(t)$ *for* $t \leq T_1$ *given* $Z_{\mathcal{T}}(T_0) = i; T_0 = 0$ *:*

$$Z^i_{\mathcal{T}}(t) \;=\; [Z_{\mathcal{T}}(t) \,:\, t \leq T_1,\, Z_{\mathcal{T}}(0) = i]$$

According to Definition 5.3, $Z^i_{\mathcal{T}}(t)$ describes the evolution of the *PN* starting at the RTP $T_0 = 0$ in the regeneration marking i, up to the next regeneration time point T_1. Therefore, $Z^i_{\mathcal{T}}(t)$ includes all the markings that can be reached from the regeneration marking i before the next regeneration time point. The entries of the i-th row of the matrix $\mathbf{E}(t)$ are determined by $Z^i_{\mathcal{T}}(t)$ as:

$$E_{ij}(t) = Pr\{Z^i_{\mathcal{T}}(t) = j, T_1 \geq t \,|\, Z^i_{\mathcal{T}}(0) = i\} \tag{5.1}$$

Let us define the branching probability matrix of transition tr_d ($\mathbf{\Delta}^d = \{\Delta^d_{k\ell}\}$) as

$$\Delta^d_{k\ell} = Pr\{\text{next tangible marking is } \ell \,|\, tr_d \text{ fires in tangible marking } k\} \;.$$

The $\Delta^d_{k\ell}$ element of the branching probability matrix describe the effect of the firing of transition tr_d in marking k by defining the probability that the next tangible marking is marking ℓ.

Based on the semantics of the GDT_SPNs described by the branching probability matrices $\mathbf{\Delta}^d = \{\Delta^d_{k\ell}\}$ the subordinated process $Z^i_{\mathcal{T}}(t)$ also determines the entries of the i-th row of the $\mathbf{K}(t)$ matrix, if the transition tr_d whose firing completes the regeneration period is known. Denoting the event that the firing of transition tr_d completes the regeneration by $\mathcal{A}_d$ the elements of the external kernel is defined as:

$$K_{ij}(t) = \sum_{tr_d \in TR} \sum_{k \in \Omega} Pr\left\{Z^i_{\mathcal{T}}(T_1^-) = k, T_1 \geq t \,|\, Z^i_{\mathcal{T}}(0) = i,\, \mathcal{A}_d\right\} \,\Delta^d_{kj}\, Pr\left\{\,\mathcal{A}_d\,\right\} \tag{5.2}$$

Equations (5.1) and (5.2) together with Equation (2.10) form a general framework for the analysis of the MRSPNs.

5.4 Characterization of the life cycles of transitions

The life time of a transition is naturally divided into intervals by its consecutive resets. Inside these intervals there can be a period of time while the transition is not enabled and its age variable is equal to 0 continuously. We will refer to the period of time during which the age variable is greater than 0 as the life cycle of the transition.

The main features of the life cycles of a transition are affected by its memory policy as summarized in Table 1. (The enabling subset of transition tr_d is the subset of markings in which the transition is enabled.)

Memory policy	Resampling	Enabling	Age
end of the life cycle	firing of any transition	firing or disabl. of the tagged tr.	firing of the tagged tr.
during the life cycle	no state transition	moving inside the enabling subset	moving without restriction
age variable inside the l.c.	continuously increasing	continuously increasing	increasing or constant

Table 1. Features of the life cycle of transitions

In the life cycle of an age type transition the increase of the age variable of the transition is marking dependent. It means that it is increasing in the markings in which the transition is enabled, and it is constant in the markings in which the transition is disabled. An indicator variable can be assigned to every marking to indicate whether the accumulation is on or off. A variable of this kind is generally called **reward rate**. Hence the stochastic process subordinated to a regeneration period is a **Stochastic Multi Reward Process** with binary reward rates ($r_{id} = 0, 1; i \in \Omega, d \in TR$), where $r_{id} = 1$ means that the transition tr_d is enabled in marking M_i and $r_{id} = 0$ means that the transition tr_d is disabled in marking M_i. The different transitions have different effects on the accumulation process. There are as many kinds of reward as many transitions have age variable. The reward accumulation process and the lost of accumulated reward as well as the preemption type of the state transitions can be different for the different kind of rewards. For example a firing of a transition is of *prd* type for the reward, represent the age variable of the transition.

The main problem in the analysis of the MRSPNs is the existence of transitions whose life cycle overlap each other. Any transition with generally distributed firing time (GEN)[3] whose memory policy is of resampling type can not have overlapping life cycles and since the transitions with exponentially distributed firing time (EXP) are supposed to be resampling type, they can neither have overlapping periods.

In the following we discuss special cases in which the overlapping of the life cycles of the transitions are excluded. In the next two subsections we consider regeneration periods without internal transition and in the subsequent sections we discuss the analysis of regeneration periods with internal transitions.

[3]Transitions with deterministic firing time (DET) are considered to be GEN transitions.

5.4.1 Markovian regeneration period

If in the regeneration marking M_i only EXP transitions are enabled the next regeneration time point is the epoch of jump into any one of the immediately reachable states. A regeneration period of this kind is referred to as Markovian regeneration period.

Let S^i be the set of transitions enabled in marking M_i ($S^i = \{tr_d : tr_d \,\text{enabled in}\, M_i\}$), λ_d be the transition rate of transition $tr_d \in S^i$, and $\lambda^i = \sum_{tr_d \in S^i} \lambda_d$.

The entry $K_{ij}(t)$ of the external kernel provides the probability of reaching the successive regeneration state j before time t, while the entry $E_{ij}(t)$ of the internal kernel gives the probability of jumping from i to j before the next regeneration time point. Since, in this case, any firing provides a new regeneration time point, the only nonzero entry of the i-th row of matrix $\mathbf{E}(t)$ corresponds to $j = i$. It follows:

$$K_{ij}(t) \;=\; \sum_{tr_d \in S^i} \frac{\lambda_d}{\lambda^i} \, (1 \;-\; e^{-\lambda^i t}) \, \Delta_{ij}^d \qquad\qquad E_{ij}(t) \;=\; \delta_{ij} \, e^{-\lambda^i t} \tag{5.3}$$

5.4.2 Semi-Markovian regeneration period

A regeneration period is called semi-Markovian, if any transition completes a tangible marking completes the regeneration period as well. It may happens if:

- in the regeneration marking M_i only one transition is enabled,
- only resampling type transitions are enabled,
- the enabled transitions are enabling type and competitive (i.e. the firing of one of them disables the others)

Let $S^i = \{tr_d \,:\, tr_d \,\text{enabled in}\, M_i\}$ be the set of enabled transitions, and $F_e(t)$ the probability distribution function of the firing time of transition tr_e.

Since, in this case, any firing provides a new regeneration time point, the only nonzero entry of the i-th row of matrix $\mathbf{E}(t)$ corresponds to $j = i$ and the transition with minimal firing time completes the regeneration periods and provides the next regeneration state. The distribution of the first firing time $F^i(t)$ is the minimum of the firing time of the enabled transitions:

$$F^i(t) = 1 - \prod_{tr_e \in S^i} \; (1 - F_e(t)) \quad ,$$

and the elements of the i-th row of matrices $\mathbf{K}(t)$ and $\mathbf{E}(t)$ are:

$$\begin{aligned} K_{ij}(t) \;&=\; \sum_{tr_d \in S^i} \int_{u=0}^{t} \prod_{tr_e \in S^i ; tr_e \neq tr_d} \; (1 - F_e(t)) \;\; d\, F^i(t) \;\; \Delta_{ij}^d \\ E_{ij}(t) \;&=\; \delta_{ij} \; (1 - F^i(t)) \end{aligned} \tag{5.4}$$

5.5 Classification of regeneration periods with internal state transitions

The marking process of a SPN is indeed a discrete random walk on its reachability graph according to the directed arcs. The first step of the analysis of a SPN is the generation of this graph for which standard software tools are available [23, 28, 70, 62]. The different realizations of the marking process mean different walks on the reachability graph, and for the purpose of the analysis the value of the age variable has to be evaluated for the possible realizations. In the following, we give some restrictions for the number of the enabled GEN transitions[4] and for the value of the age variables for all the possible realizations of the marking process between the consecutive regeneration time points to exclude the opportunity of the overlapping life cycles of transitions. These restrictions make it possible to evaluate the regeneration periods by a single reward model.

Definition 5.4 *A regeneration period of a MRSPN belongs to the* **Class A** *if in any tangible marking, that can be reached during the given regeneration period, the following conditions are imposed:*

- *at most one GEN transition is enabled,*
- *at most one disabled transition can have memory (i.e. age variable with positive value),*
- *no GEN transition can be enabled while an other disabled transition has memory.*

The subordinated process of a Class A type regeneration period is CTMC.

Definition 5.5 *A regeneration period of a MRSPN belongs to the* **Class B** *if in any tangible marking, that can be reached during the given regeneration period, the following conditions are imposed:*

- *at most one GEN transition is enabled,*
- *at most one disabled transition can have memory,*
- *at most one exclusively enabled GEN transition can exist while another disabled transition has memory.*

The subordinated process of a Class B type regeneration period is SMP.

Definition 5.6 *A regeneration period of a MRSPN belongs to the* **Class C** *if in any tangible marking, that can be reached during the given regeneration period, the following conditions are imposed:*

[4] The restriction on the number GEN transitions do not mean any restiction on the number of EXP transitions.

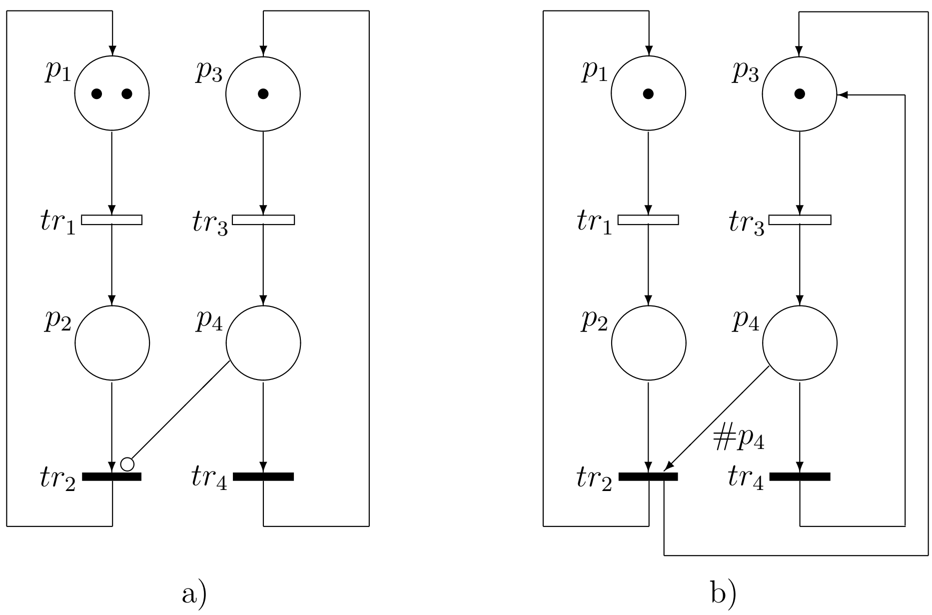

Figure 5.1: Petri nets with Class C and D regeneration periods.

- *at most one GEN transition is enabled,*
- *at most one disabled transition can have memory,*

The subordinated process of a Class C type regeneration period is MRP whose subordinated process is CTMC[5]. The regeneration period of the Petri net on Figure 5.1a starting from marking $M = (1\ 1\ 1\ 0)$ is Class C type, when the GEN transitions (tr_2 and tr_4) are age memory type.

A further generalization is the allowance of at most two GEN transitions to be enabled in a tangible marking but in this case there is no simple way to describe the restrictions against overlapping.

Definition 5.7 *A regeneration period of a MRSPN belongs to the* **Class D** *if the following conditions are imposed:*

- *in any tangible marking, that can be reached during the given regeneration period, at most two GEN transitions can be enabled,*
- *in any tangible marking, that can be reached during the given regeneration period, no disabled transition can have memory*[6]

[5]Without the detailed study of this class we mention that the subordinated process fulfills the restrictions discussed in subsection 3.1.2, hence the analysis of a regeneration period of this kind can be performed based on the external kernel of the subordinated process by applying the same series of steps as required for subordinated SMPs.

[6]This requirement practically means that the transitions are restricted to be enabling type or different type but with the same effect as an enabling type transition has in the given environment.

- *there is no other transition enabled, in any tangible marking, that can be reached during the given regeneration period, when two GEN transitions are enabled,*
- *there is no overlapping in the life cycles of the transitions,*
- *in every regeneration marking, at most one GEN transition is enabled.*

The subordinated process of a Class D type regeneration period is SMP. The regeneration period of the Petri net on Figure 5.1b starting from marking $M = (0\ 1\ 1\ 0)$ is Class D type, when the GEN transitions (tr_2 and tr_4) are enabling memory type. ($\#p_4$ means that the multiplicity of the input arc from p_4 to tr_2 equals to the number of tokens in p_4.)

To univocally relate the type of the published subclasses of MRSPNs to the classes of the regeneration periods let us further define a restriction of Class A.

Definition 5.8 *A regeneration period of a MRSPN belongs to the* **Class A1** *if the following conditions are imposed:*

- *in any tangible marking, at most one GEN transition is allowed to be enabled,*
- *in any tangible marking, no disabled transition can have memory.*

The subordinated process of a Class A1 type regeneration period is CTMC.

5.6 Analysis of the reward SMP subordinated to MRSPNs

The overlapping of transition life cycles is excluded in the above defined subclasses of the MRSPNs every regeneration period can be related to the transition whose life cycle coincides with the given regeneration period, hence Subclasses A - D can be analyzed by single reward models.

The opportunity of exiting from the enabling subset of markings of an enabling type transition can be considered as the existence of an absorbing group of markings. A movement of the marking process into the absorbing subset of markings completes the regeneration period.

The superscript i refers to the appropriate quantity of the regeneration period starting from marking M_i, and from the point of view of the underlying stochastic process marking M_i is referred to as state i. In this section, the analysis of the single reward models of regeneration periods is elaborated supposing that the subordinated processes are SMPs.

At $t = T_0 = 0$ a single GEN transition tr_g (with age memory variable a_g and duration w_g) starts its firing process in marking M_i ($a_g = 0$). The successive regeneration time point is T_1.

Let Ω^i be the subset of the tangible markings (Ω) including the states of the subordinated process $Z^i_T(t)$ (i.e. the markings reachable from M_i before T_1). For the rest of this chapter we take into consideration the MRSPNs without immediate transitions, which means that all markings are tangible and we can avoid the handling of the evolution of the vanishing markings, which allows the application of simpler notations. For notational convenience we do not renumber the markings in Ω^i so that all the subsequent matrix functions have the dimensions $(\#\Omega \times \#\Omega)$, but with the significant entries located in position $(k,\, \ell)$ only, with $k, \ell \in \Omega^i$.

Let $Z^i_{\mathcal{T}}(t)$ $(t \geq 0)$ be the SMP defined over Ω^i, $\underline{r}^i$ the corresponding binary reward vector and R^i the subset of markings from which an exit to the complement subset $R^{i\,c} = \Omega - R^i$ completes the regeneration period. The age variable a_d increases at a rate r^i_j (which is equal to 0 or 1) when $Z^i_{\mathcal{T}}(t) = j$. The subordinated process $Z^i_{\mathcal{T}}(t)$ starts in marking M_i $(Z^i_{\mathcal{T}}(0) = i)$, so that the initial probability vector is $\underline{V}^i_0 = [0,\, 0,\, \ldots,\, 1_i,\, \ldots,\, 0]$ (a vector with all the entries equal to 0 but entry i equals to 1).

Let $\mathbf{Q}^i(t) = [Q^i_{k\ell}(t)]$ be the kernel of the subordinated SMP $(Z^i_{\mathcal{T}}(t))$. We denote by H the time duration until the first embedded time point in the SMP if it starts from state k at time 0 ($Z^i_{\mathcal{T}}(0) = k$).

Let us introduce the following matrix functions: $\mathbf{P}^i(t,w)$, $\mathbf{F}^i(t,w)$ and $\mathbf{D}^i(t,w)$ so defined:

$$P^i_{k\ell}(t,w) \;=\; Pr\{Z^i_{\mathcal{T}}(t) = \ell \in R^i\,,\, T_1 > t \,|\, Z^i_{\mathcal{T}}(0) = k \in R^i\,,\, \text{firing time} \,=\, w\} \quad (5.5)$$

$$F^i_{k\ell}(t,w) \;=\; Pr\{Z^i_{\mathcal{T}}(T_1^-) = \ell \in R^i\,,\, T_1 \leq t,\, \text{GEN tr. fires} \,|\, Z^i_{\mathcal{T}}(0) = k \in R^i\,,\, \text{fr. time} \,=\, w\} \quad (5.6)$$

$$D^i_{k\ell}(t,w) \;=\; Pr\{Z^i_{\mathcal{T}}(T_1) = \ell \in R^{i\,c}\,,\, T_1 \leq t \,|\, Z^i_{\mathcal{T}}(0) = k \in R^i\,,\, \text{firing time} \,=\, w\} \quad (5.7)$$

and $P^i_{k\ell}(t,w)$, $F^i_{k\ell}(t,w)$ and $D^i_{k\ell}(t,w)$ are equal to 0 otherwise (i.e. $P^i_{k\ell}(t,w)$ and $F^i_{k\ell}(t,w)$ are 0 if $k \in \Omega - R^i$ or $\ell \in \Omega - R^i$; and $D^i_{k\ell}(t,w)$ is 0 if $k \in \Omega - R^i$ or $\ell \in R^i$). In this definition the dependence on the GEN transition (tr_g) characterizes regeneration period is neglected, and the firing time (i.e. the absorbing barrier) is denoted by w in stead of w_g.

$P^i_{k\ell}(t,w)$ is the probability of being in state $\ell \in R^i$ at time t before absorption either at the barrier w or in the absorbing subset $R^{i\,c}$, starting in state $k \in R^i$ at $t = 0$. $F^i_{k\ell}(t,w)$ is the probability of firing of tr_d from state $\ell \in R^i$ (hitting the absorbing barrier w in state ℓ) before t starting in state $k \in R^i$ at $t = 0$. $D^i_{k\ell}(t,w)$ is the probability of leaving the state group R^i to state $\ell \in R^{i\,c}$ before hitting the barrier w starting in state $k \in R^i$ at $t = 0$.

From (5.5), (5.6) and (5.7), it follows for any t:

$$\sum_{\ell \in \Omega^i} [\, P^i_{k\ell}(t,w) \,+\, F^i_{k\ell}(t,w) \,+\, D^i_{k\ell}(t,w)\,] \,=\, 1$$

Due to Equation (5.1) and (5.2) the elements of the i-th row of matrices $\mathbf{K}(t)$ and $\mathbf{E}(t)$ can be expressed based on the matrices $\mathbf{P}^i(t,w)$, $\mathbf{F}^i(t,w)$ and $\mathbf{D}^i(t,w)$ as follows:

$$\begin{aligned} K_{ij}(t) \;&=\; \sum_{k \in R^i} F^i_{ik}(t,w)\,\Delta^i_{kj} \,+\, D^i_{ik}(t,w) \\ E_{ij}(t) \;&=\; P^i_{ij}(t,w) \end{aligned} \quad (5.8)$$

In the definition of matrices $P^i_{k\ell}(t,w)$, $F^i_{k\ell}(t,w)$ and $D^i_{k\ell}(t,w)$ we maintain the explicit dependence on the barrier level w, since this dependence will be exploited in the subsequent analytical treatment.

In order to avoid unnecessarily cumbersome notation in the following subsection, we neglect the explicit dependence on the particular subordinated process $Z^i_T(t)$, by eliminating the superscript (i). It is however tacitly intended, that all the quantities $\underline{r}$, $\mathbf{Q}(t)$, $\mathbf{P}(t,w)$, $\mathbf{F}(t,w)$, $\mathbf{D}(t,w)$, $\mathbf{\Delta}$, R and R^c refer to the specific process subordinated to the regeneration period starting from state i.

5.6.1 Derivation of the matrix functions $\mathbf{P}(t,w)$, $\mathbf{F}(t,w)$ and $\mathbf{D}(t,w)$

These derivations follow the same pattern as appeared in Chapter 3 but the quantities differ from the former ones due to their dependence not only on the starting state but on the present (in case of matrix $\mathbf{P}(t)$) or final (in case of matrices $\mathbf{F}(t)$ and $\mathbf{D}(t)$) states as well.

Theorem 5.9 *For the firing probability $F_{k\ell}(t,w)$ the following double transform equation holds:*

$$F^{\sim *}_{k\ell}(s,v) \;=\; \delta_{k\ell}\,\frac{r_k\,[1 \,-\, Q^{\sim}_k(s \,+\, v\,r_k)\,]}{s \,+\, v\,r_k} \,+\, \sum_{u\in R} \,Q^{\sim}_{ku}(s \,+\, v\,r_k)\,F^{\sim *}_{u\ell}(s,v) \tag{5.9}$$

Proof:

Conditioning on $H = h$, let us define:

$$F_{k\ell}(t,w\,|\,H=h) \;=\; \begin{cases} \delta_{k\ell}\,U\left(t - \dfrac{w}{r_k}\right) & \text{if}: h\,r_k \geq w \\[2ex] \displaystyle\sum_{u\in R}\frac{dQ_{ku}(h)}{dQ_k(h)}\cdot F_{u\ell}(t-h, w-hr_k) & \text{if}: h\,r_k < w \end{cases} \tag{5.10}$$

In (5.10), two mutually exclusive events are identified. If $r_k \neq 0$ and $h\,r_k \geq w$, a sojourn time equals to w is accumulated before leaving state k, so that the firing time (next regeneration time point) is $T_1 = w/r_k$. If $h\,r_k < \, w$ then a transition occurs to state u with probability $dQ_{ku}(h)/dQ_k(h)$ and the residual service $(w \,-\, hr_k)$ should be accomplished starting from state u at time $(t \,-\, h)$. Taking the LST transform with respect to t (denoting the transform variable by s), the LT transform with respect to w (denoting the transform variable by v) of (5.10) and then evaluating the mean of the conditional expression with respect to H, (5.10) becomes (5.9).

□

Corollary 5.10 *The state probability $P_{k\ell}(t,w)$ satisfies the following double transform equation:*

$$P^{\sim *}_{k\ell}(s,v) \;=\; \delta_{k\ell}\,\frac{s\,[1 \,-\, Q^{\sim}_k(s \,+\, v\,r_k)\,]}{v(s \,+\, v\,r_k)} \,+\, \sum_{u\in R} \,Q^{\sim}_{ku}(s \,+\, v\,r_k)\,P^{\sim *}_{u\ell}(s,v) \tag{5.11}$$

Proof: Conditioning on $H = h$, let us define:

$$P_{k\ell}(t, w \mid H = h) = \begin{cases} \delta_{k\ell} \left[U(t) - U\left(t - \dfrac{w}{r_k}\right) \right] & \\ & \text{if} : h\, r_k \geq w \\ \delta_{k\ell} \left[U(t) - U(t-h) \right] + \displaystyle\sum_{u \in R} \frac{dQ_{ku}(h)}{dQ_k(h)} P_{u\ell}(t-h, w - h r_k) & \\ & \text{if} : h\, r_k < w \end{cases} \tag{5.12}$$

The derivation of the matrix function $\mathbf{P}(t, w)$ based on (5.12) follows the same pattern as for the function $\mathbf{F}(t, w)$.

□

Corollary 5.11 *The probability of entering to R^c ($D_{k\ell}(t, w)$) satisfies the following double transform equation:*

$$D^{\sim *}_{k\ell}(s, v) = \frac{1}{v} Q^{\sim}_{kl}(s + v\, r_k) + \sum_{u \in R} Q^{\sim}_{ku}(s + v\, r_k)\, D^{\sim *}_{u\ell}(s, v) \tag{5.13}$$

Proof: Conditioning on $H = h$, $D_{k\ell}(t, w)$ can be defined us:

$$D_{k\ell}(t, w \mid H = h) = \begin{cases} 0 & \text{if} : h\, r_k \geq w \\ \dfrac{dQ_{k\ell}(h)}{dQ_k(h)} U(t-h) + \displaystyle\sum_{u \in R} \frac{dQ_{ku}(h)}{dQ_k(h)} D_{u\ell}(t-h, w - h r_k) & \\ & \text{if} : h\, r_k < w \end{cases} \tag{5.14}$$

The derivation of the matrix function $\mathbf{D}(t, w)$ based on (5.14) follows the same pattern as for the function $\mathbf{F}(t, w)$.

□

5.6.2 The subordinated process is a Reward CTMC

Let us consider the particular case in which the subordinated process $Z_{\mathcal{T}}(t)$ is a reward *CTMC* with infinitesimal generator $\mathbf{A} = \{a_{k\ell}\}$. Let us suppose that the states numbered $1, 2, \ldots, m$ belong to R ($1, 2, \ldots, m \in R$) and the states numbered $m+1, m+2, \ldots, n$ belong to R^c ($m+1, m+2, \ldots, n \in R^c$). By this ordering of states $\mathbf{A}$ can be partitioned into the following submatrices $\mathbf{A} = \begin{array}{|c|c|} \hline \mathbf{B} & \mathbf{C} \\ \hline \mathbf{D} & \mathbf{E} \\ \hline \end{array}$ where B contains the intensity of the transitions inside R, and C contains the intensity of the transitions from R to R^c, the other submatrices are irrelevant in our model.

Corollary 5.12 *The entries of the matrix functions ($P_{k\ell}(t,w)$, $F_{k\ell}(t,w)$ and $D_{k\ell}(t,w)$) are as follows:*

$$(s+vr_k)\, F^{\sim *}_{k\ell}(s,v) \;=\; \delta_{k\ell}\;\; r_k \;\; + \sum_{u\in R} a_{ku}\, F^{\sim *}_{u\ell}(s,v) \tag{5.15}$$

$$(s+vr_k)\, P^{\sim *}_{k\ell}(s,v) \;=\; \delta_{k\ell}\;\; \frac{s}{v} \;\; + \sum_{u\in R} a_{ku}\, P^{\sim *}_{u\ell}(s,v) \tag{5.16}$$

and

$$(s+vr_k)\, D^{\sim *}_{k\ell}(s,v) \;=\; \frac{a_{k\ell}}{v} \;\; + \sum_{u\in R} a_{ku}\, D^{\sim *}_{u\ell}(s,v) \tag{5.17}$$

Proof: The entries of the matrix functions $P_{k\ell}(t,w)$, $F_{k\ell}(t,w)$ and $D_{k\ell}(t,w)$ can be obtained from (5.11), (5.9) and (5.13) by substituting the proper kernel describing the given *CTMC*:

$$Q_{k\ell}(t) \;=\; \begin{cases} \dfrac{a_{k\ell}}{-a_{kk}}\,(1\;-\;e^{a_{kk}\,t}) & \text{if} : k \;\neq\; \ell \\[2ex] 0 & \text{if} : k \;=\; \ell \end{cases} \tag{5.18}$$

and in *LST* domain:

$$Q^{\sim}_{k\ell}(s) \;=\; \begin{cases} \dfrac{a_{k\ell}}{s\;-\;a_{kk}} & \text{if} : k \;\neq\; \ell \\[2ex] 0 & \text{if} : k \;=\; \ell \end{cases} \tag{5.19}$$

Keeping in mind that $a_{kk} = -\sum_{\ell\in\Omega} a_{k\ell}$

□

Equations (5.16), (5.15) and (5.17) can be rewritten in matrix form:

$$\mathbf{F}^{\sim *}(s,v) = (s\mathbf{I} + v\mathbf{R} - \mathbf{B})^{-1}\mathbf{R}$$

$$\mathbf{P}^{\sim *}(s,v) = \frac{s}{v}\;(s\mathbf{I} + v\mathbf{R} - \mathbf{B})^{-1}$$

$$\mathbf{D}^{\sim *}(s,v) = \frac{1}{v}\;(s\mathbf{I} + v\mathbf{R} - \mathbf{B})^{-1}\mathbf{C}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{R}$ is the diagonal matrix of the reward rates (r_k); the dimension of $\mathbf{I}$, $\mathbf{R}$, $\mathbf{B}$, $\mathbf{F}$ and $\mathbf{P}$ is $(m\times m)$, and the dimension of $\mathbf{C}$ and $\mathbf{D}$ is $(m\times(n-m))$.

5.7 Subclasses of MRSPNs

An Exponentially Distributed SPN can have only Markovian regeneration periods, and a Semi-Markov SPN can have only semi-Markovian regeneration periods. (The second one naturally includes the Markovian case as well). The life time of a DSPN and a MRSPN* can contain only Class A1 type regeneration periods ([25, 24]), and Age Memory DSPN (defined and analyzed in [18]) can contain regeneration period of Class A and B.

The above introduced method opens the possibility of the analysis of the regeneration periods of Class A, B, D, as well as of Class A1. In the definition of the introduced classes of MRSPNs (Exponentially Distributed SPN, Semi-Markov SPN, DSPN, MRSPN*, Age memory DSPN) the restrictions are general for the whole life time of the PNs, but due to the independence of the regeneration periods different restrictions can be considered in different regeneration period.

A more general class of MRSPN can be defined and analyzed by restricting the regeneration periods to belong to one of the mentioned classes.

Definition 5.13 *A MRSPN is called* **Single Reward MRSPN** *if all of its regeneration period are one of the following type: Markovian, Semi-Markovian, Class A, Class B, Class C, Class D, Class A1.*

The analysis of a Single Reward MRSPN is composed by the analysis of its regeneration periods.

The introduced classes of the regeneration periods can be divided basically into two groups considering the type of the GEN transition whose life cycle coincides with the regeneration period.

- In Class A, B, C the disabled GEN transition can have memory (positive value stored in its age variable) This means that the GEN transition is age type. In this case the subordinated process contains states, in which the age type GEN transition is enabled (and the corresponding reward rate is 1) as well as states in which it is disabled (and the corresponding reward rate is 0). The completion of the life cycle of an age type GEN transition can occur only when it fires. Hence the state space of the subordinated process does not contain any absorbing state which could complete the regeneration period without completion ($\#R^i = \#\Omega^i$).

- In Class D and A1 no disabled transition can have memory, thus the GEN transition whose life cycle coincides with the regeneration period accumulates reward in its age variable throughout the regeneration period, i.e. reward rate is 1 in every tangible marking of the subordinated process R^i. The regeneration period of Class D, E can complete without the firing of the GEN transition which refers to enabling and resampling type transitions. The opportunity of completing the regeneration period without the firing of the GEN transition is considered in our method by mean of absorbing states ($\#R^i \leq \#\Omega^i$).

Chapter 6

Example of application

6.1 Preemptive repair system - The M/D/1/2/2 preemptive queue

A *PN* model for the non-preemptive M/D/1/2/2 queue has been introduced in [3], where the steady state solution was derived. The transient analysis for the same system was carried on in [25]. In the following, we examine two different mechanisms of preemptive service with a reliability interpretation of two machines and one repairman[1].

A. - Preemptive M/D/1/2/2 with identical machines.

The M/D/1/2/2 queue has a preemptive service with the same kind of machines. The repair in execution is preempted as soon as a new demand for repair eventually arrives to the repairman. The preempted repair is restarted as soon as the repairman becomes free again. Two different recovery policies can be considered depending on whether the repairman is able to remember the work already performed on the machine before preemption or not. In the latter case, the prior work is lost due to the interruption and the recovered repair must be repeated from scratch with a service time resampled from the original cdf (*prd policy).* In the former case, the prior work is not lost and the service time of the recovered repair equals the residual service time given the work already executed before preemption (*prs policy).* Figure 6.1a shows a *PN* which describes an M/D/1/2/2 system containing only two machines and in which any new failure preempts the repair eventually in progress. Place p_1 contains the machines working without failure, while place p_2 contains the number of failed machines (including the one under repair). Starting from the initial marking $M_1 = (2\ 0\ 0\ 1)$ (Figure 6.1b), tr_1 is the only enabled transition. Firing of tr_1 represents the failure of the first machine and leads to state $M_2 = (1\ 1\ 1\ 0)$. In M_2 transitions tr_2 and tr_3 are competing. tr_2 represents the repair of the failed machine and its firing returns the system to the initial state M_1. tr_3 represents the failure of the second machine and its firing disables tr_2 by removing one token from p_3 (the first repair becomes dormant). In $M_3 = (0\ 2\ 0\ 1)$ one machine is under repair and one repair is dormant, and the only enabled activity is the repair of the actual machine. Firing of tr_4 leads the system again in M_2, where the dormant

[1]This problem was mentioned in Example 1.

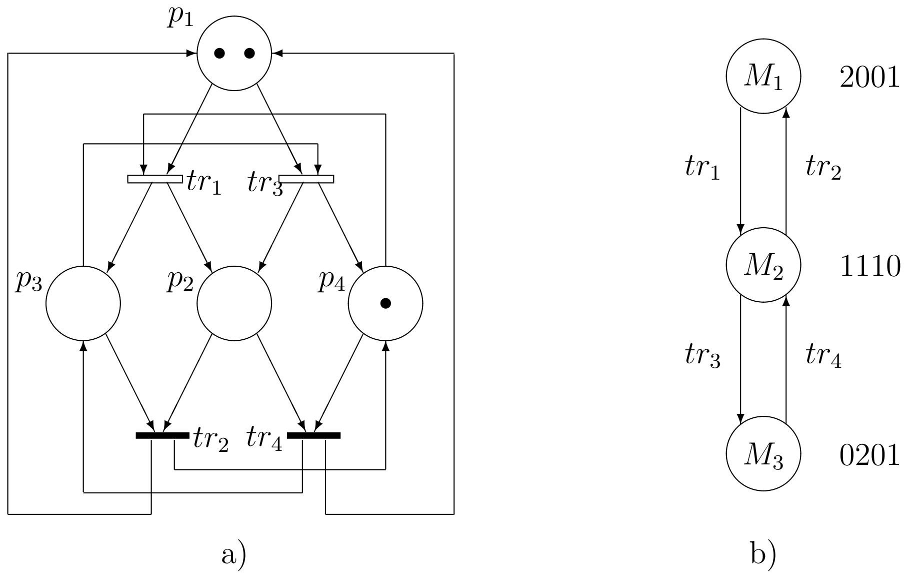

Figure 6.1: Preemptive M/D/1/2/2 queue with identical machines

repair is recovered. Assuming the failure time of both machines to be exponentially distributed with parameter λ, tr_1 is associated an exponential firing rate equal to $(2\,\lambda)$ and tr_3 a firing rate equal to λ. Deterministic repair time of duration w is assigned to transitions tr_2 and tr_4.
A.1 - enabling memory policy is assigned to tr_2 *and* tr_3*.* - Each time tr_2 is disabled by the failure of the second machine (tr_3 fires before tr_2), the corresponding enabling age variable a_2 is reset. As soon as tr_2 becomes enabled again (the second repair completes and tr_4 fires) no memory is kept of the prior service, and the execution restarts from scratch. This behaviour corresponds to a *prd* service policy, and is covered by the model definition in [25, 27].
A.2 - age memory policy is assigned to tr_2 *and* tr_3*.* - Each time tr_2 is disabled without firing (tr_3 fires before tr_2) the age variable a_2 is not reset. Hence, as the second repair completes (tr_4 fires), the system returns to M_2 keeping the value of a_2, so that the time to complete the interrupted repair can be evaluated as the residual service time given a_2. a_2 counts the total time during which tr_2 is enabled before firing, and is equal to the cumulative sojourn time in M_2. The assignment of the age memory policy to tr_2 realizes a *prs* service mechanism. This behavior is not compatible with the definition of *DSPN* given in [25] and requires a new analysis methodology. The regeneration time points in the marking process $Z_{\mathcal{T}}(t)$ correspond to the epochs of entrance to markings in which the age variables associated to all the transitions are equal to zero. By inspecting Figure 6.1b, the regeneration time points result to be the epochs of entering M_1 and of entering M_2 from M_1. The process $Z^1_{\mathcal{T}}(t)$ subordinated to state M_1 is a single step *CTMC* (being the only enabled transition tr_1 exponential) and includes the only immediately reachable state M_2 (Markovian regeneration period).

The process $Z^2_{\mathcal{T}}(t)$ subordinated to state M_2 includes all the states reachable from

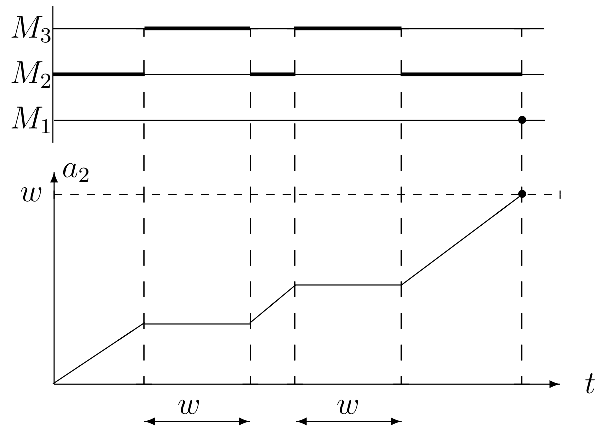

Figure 6.2: A possible realization of the subordinated marking process $Z_T^2(t)$

M_2 before firing of tr_2: these states are M_3, M_2. Since M_2 is the only state in which tr_2 is enabled, the corresponding reward rate vector is $\underline{r}^2 = [0\ 1\ 0]$. Firing of tr_2 can only occur from state M_2 leading to state M_1; it turns out that the only relevant nonzero entry in the branching probability matrix is $\Delta_{21}^2 = 1$.

A possible realization of the subordinated marking process $Z_T^2(t)$ is shown in Figure 6.2. Notice that $Z_T^2(t)$ is semi-Markovian since tr_4 is deterministic (Class B type regeneration period). The age variable a_2 growths whenever $Z_T^2(t) = M_2$, and the firing of tr_2 occurs when a_2 reaches the value w (the deterministic duration assigned to tr_2). Considering w as an absorbing barrier for the accumulation functional represented by the age variable a_2, the firing time of tr_2 is determined by the first passage time of a_2 across the absorbing barrier w.

In the present example, M_3 can never be a regeneration marking, since a_2 is not reset at the entrance to M_3.

B. - Preemptive M/D/1/2/2 with different machines

The two machines are of different classes, and the failure of machine of class 2 (later machine 2) preempts the repair of machine of class 1 (later machine 1) but not vice versa. Two possible preemption policies are again possible depending whether the repairman is able to remember the work done before the interruption. A *PN* modelling the M/D/1/2/2 queue in which the failures of machine 2 have higher priority over the repairs of machine 1 is reported in Figure 6.3a. Place p_1 (p_3) represents machine 1 (2) working without failure, while place p_2 (p_4) represent machine 1 (2) under repair. Transition tr_1 (tr_3) means the failure of a machine 1 (2), while transition tr_2 (tr_4) denotes the completion of repair of a machine 1 (2). The inhibitor arc from p_4 to tr_2 models the described preemption mechanism: as soon as one machine 2 joins the queue the machine 1 eventually under repair is interrupted. The reachability graph of the *PN* of Figure 6.3a is in Figure 6.3b. Under *prs* repair policy, after completion of the repair of machine 2, the interrupted repair of machine 1 is resumed continuing the new service period from the point reached just before the last interruption. In the *PN* of Figure 6.3a this service policy is realized by assigning to transitions tr_2

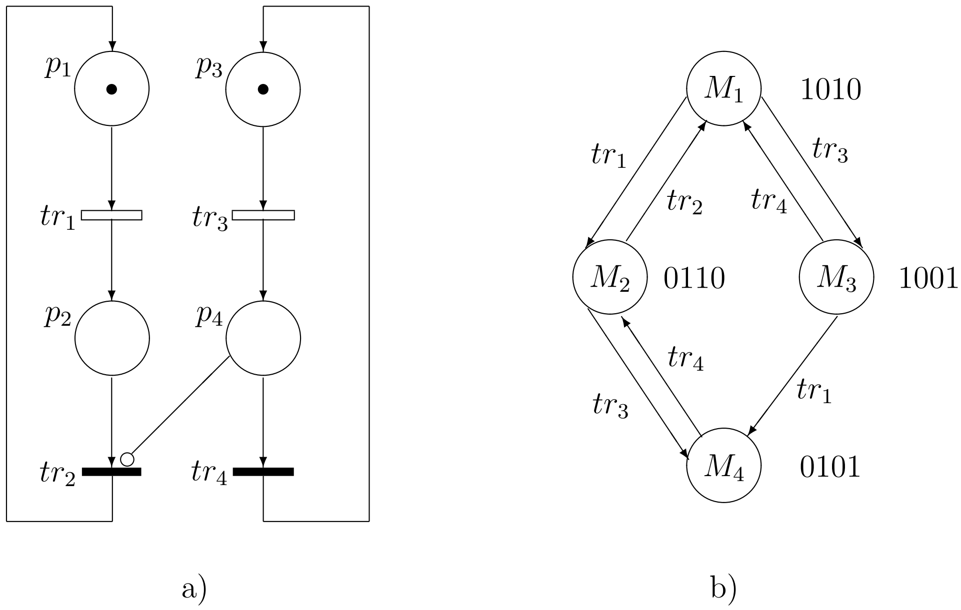

Figure 6.3: Preemptive M/D/1/2/2 queue with different machines

and tr_4 an age memory policy. The failure times (transitions tr_1 and tr_3) are exponentially distributed with parameters λ, while the repair times (transitions tr_2 and tr_4) are deterministic with duration w.

From Figure 6.3b, it is easily recognized that M_1, M_2 and M_3 can all be regeneration states, while M_4 can never be a regeneration state (in M_4 a machine 2 is always in execution so that its corresponding age variable a_2 is never 0). Only exponential transitions are enabled in M_1 and the next regeneration states can be either M_2 or M_3 depending whether tr_1 or tr_3 fires first (Markovian regeneration period). From M_2 the next regeneration state can be only M_1, but multiple cycles (M_2 - M_4) can occur depending whether machines 2 arrive to interrupt the repair of the machine 1 (Class B type regeneration period). From state M_3 the next regeneration marking can be either state M_1 or M_2 depending whether during the execution of the machine 2 a machine 1 does require repair (but remains blocked until completion of the machine 2) or does not (Class A type regeneration period).

6.1.1 Numerical results

The closed form *LST* expressions of $\mathbf{K}(t)$ and $\mathbf{E}(t)$ for the two *prs* M/D/1/2/2 queuing systems are derived in detail, applying the technique developed in the previous chapter. The time domain values are obtained by performing an analytical inversion with respect to the transform variable v, and a numerical inversion with respect to the transform variable s.

A. - prs preemptive M/D/1/2/2 with identical machines - Let us build up the $\mathbf{K}^{\sim}(s)$ and $\mathbf{E}^{\sim}(s)$ matrices row by row by considering separately all the states that can be regeneration states and can originate a subordinated process. Since M_3 can never

be a regeneration state the third row of the above matrices is irrelevant. The fact that M_3 is not a regeneration marking, means that the process can stay in M_3 only between two successive regeneration time points (Figure 6.2).
A.1) - The starting regeneration state is M_1 - (Markovian regeneration period) No deterministic transition is enabled and the next regeneration state can only be state M_2. Applying equation (5.3) we obtain:

$$
\begin{array}{lll}
K^{\sim}_{11}(s) \;=\; 0 & K^{\sim}_{12}(s) \;=\; \dfrac{2\lambda}{s\,+\,2\lambda} & K^{\sim}_{13}(s) \;=\; 0 \\
\text{and} & & \\
E^{\sim}_{11}(s) \;=\; \dfrac{s}{s\,+\,2\lambda} & E^{\sim}_{12}(s) \;=\; 0 & E^{\sim}_{13}(s) \;=\; 0
\end{array}
\tag{6.1}
$$

A.2) - The starting regeneration state is M_2 - (Class B type regeneration period) Transition tr_2 is deterministic so that the next regeneration time point is the epoch of firing of tr_2. The subordinated process $Z^2_{\mathcal{T}}(t)$ (Figure 6.2) comprises states M_2 and M_3 and is a SMP (since tr_4 is deterministic) whose kernel is:

$$
Q^{\sim}(s) = \left| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & \dfrac{\lambda}{s\,+\,\lambda} \\ 0 & e^{-ws} & 0 \end{array} \right|
$$

The reward vector is $\underline{r}^2 \;=\; [0,\,1,\,0]$, and the only nonzero entry of the branching probability matrix is $\Delta^2_{21} \,=\, 1$. Applying Equations (5.9) and (5.11) we obtain the following results for the nonzero entries:

$$
F^{\sim *}_{22}(s,v) = \frac{1}{s+v+\lambda-\lambda e^{-sw}}
$$

$$
P^{\sim *}_{22}(s,v) = \frac{s/v}{s+v+\lambda-\lambda e^{-sw}}
$$

$$
P^{\sim *}_{23}(s,v) = \frac{\lambda(1-e^{-sw})/v}{s+v+\lambda-\lambda e^{-sw}}
$$

Applying (5.8), and after inverting the *LT* transform with respect to v, the *LST* matrix functions $\mathbf{K}^{\sim}(s)$ and $\mathbf{E}^{\sim}(s)$ become:

$$
K^{\sim}(s) = \left| \begin{array}{ccc} 0 & \dfrac{2\lambda}{s\,+\,2\lambda} & 0 \\ e^{-w(s\,+\,\lambda-\,\lambda e^{-ws})} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right|
\tag{6.2}
$$

and

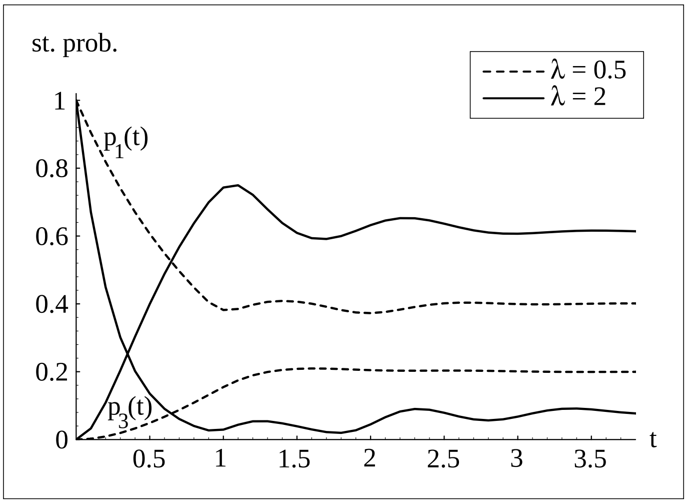

Figure 6.4: Transient behavior of the state probabilities for the preemptive M/D/1/2/2 system with identical machines.

$$E^{\sim}(s) = \left| \begin{array}{ccc} \dfrac{s}{s\,+\,2\lambda} & 0 & 0 \\ 0 & \dfrac{s[1-e^{-w(s+\lambda-\lambda e^{-ws})}]}{s+\lambda-\lambda e^{-ws}} & \dfrac{\lambda(1-e^{-ws})\,[1-e^{-w(s+\lambda-\lambda e^{-ws})}]}{s+\lambda-\lambda e^{-ws}} \\ 0 & 0 & 0 \end{array} \right| \tag{6.3}$$

The *LST* of the state probabilities are obtained by solving (2.15). The time domain probabilities are calculated by numerically inverting (2.15) by resorting to the Jagerman method [50]. The plot of the state probabilities versus time for states M_1 and M_3 is depicted in Figure 6.4, for $w = 1$ and for two different values of the failure rate $\lambda = 0.5$ and $\lambda = 2$.

B. - prs preemptive M/D/1/2/2 with different machines - The reachability graph in Figure 6.3b comprises 4 states. Let us build up the $\mathbf{K}^{\sim}(s)$ and $\mathbf{E}^{\sim}(s)$ matrices row by row, taking into consideration that state M_4 can never be a regeneration marking since a machine 2 with nonzero age memory is always active.

B.1) - The starting regeneration state is M_1 - (Markovian regeneration period) No deterministic transitions are enabled: the state is markovian and the next regeneration state can be either state M_2 or M_3. The nonzero elements of the 1st row of matrices $\mathbf{K}^{\sim}(s)$ and $\mathbf{E}^{\sim}(s)$ are from (5.3):

$$K^{\sim}_{12}(s) = \frac{\lambda}{s\,+\,2\lambda} \qquad ; \qquad K^{\sim}_{13}(s) = \frac{\lambda}{s\,+\,2\lambda} \qquad ; \qquad E^{\sim}_{11}(s) = \frac{s}{s\,+\,2\lambda}$$

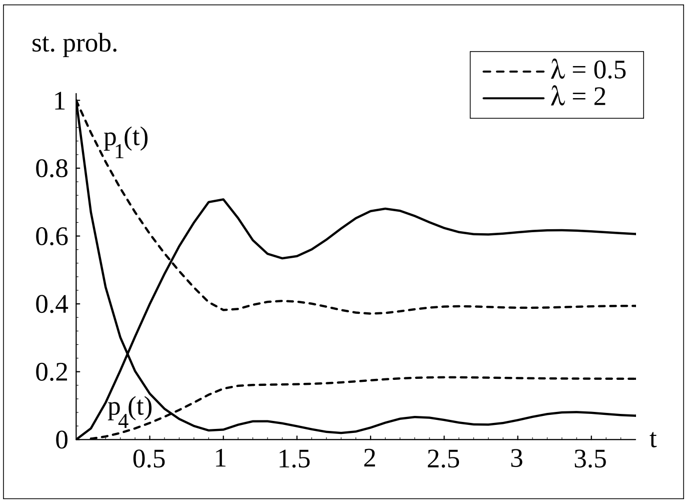

Figure 6.5: Transient behavior of the state probabilities for the preemptive M/D/1/2/2 system with different machines.

B.2) - The starting regeneration state is M_2 - (Class B type regeneration period) The subordinated process coincides, in this case, with the subordinated process $Z^2_{\overline{T}}(t)$ of the previous example (see Figure 6.2), but with state M_4 in Figure 6.3b, playing the role of state M_3 in Figure 6.1b. Thus, with an obvious permutation of pieces, we can derive the nonzero entries $K^{\sim}_{21}(s)$, $E^{\sim}_{22}(s)$ and $E^{\sim}_{24}(s)$ from the 2nd row in (6.2) and (6.3), respectively.
B.3) - The starting regeneration state is M_3 - (Class A type regeneration period) The subordinated process is a *CTMC*, hence Equation 5.16 and 5.15 can be applied. The infinitesimal generator of the *CTMC* is:

$$\mathbf{A} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

and the reward vector is $\underline{r}^3 = [0,\, 0,\, 1,\, 1]$. The branching probabilities arising from the firing of tr_4 are $\Delta^3_{31} = 1$ and $\Delta^3_{42} = 1$. Applying (5.15), (5.16) and solving the sets of equations, the nonzero entries take the form:

$$F^{\sim *}_{33}(s,v) = \frac{1}{s+\lambda+v} \qquad ; \qquad F^{\sim *}_{34}(s,v) = \frac{\lambda}{(s+v)(s+\lambda+v)}$$

$$P^{\sim *}_{33}(s,v) = \frac{s}{v(s+\lambda+v)} \qquad ; \qquad P^{\sim *}_{34}(s,v) = \frac{\lambda s}{v(s+v)(s+\lambda+v)}$$

Inverting the above equations with respect to v, taking into account the branching probabilities, yields:

$$K^{\sim}_{31}(s) = e^{-w(s+\lambda)} \quad ; \quad K^{\sim}_{32}(s) = e^{-ws}(1 - e^{-w\lambda})$$

$$E^{\sim}_{33}(s) = \frac{s}{s+\lambda}(1 - e^{-w(s+\lambda)}) \quad ; \quad E^{\sim}_{34}(s) = \frac{\lambda}{s+\lambda} - (1 - \frac{s}{s+\lambda}e^{-w\lambda})e^{-ws}$$

Finally, the complete $\mathbf{K}^{\sim}(s)$ and $\mathbf{E}^{\sim}(s)$ matrices become:

$$\mathbf{K}^{\sim}(s) = \left| \begin{array}{cccc} 0 & \frac{\lambda}{s\,+\,2\lambda} & \frac{\lambda}{s\,+\,2\lambda} & 0 \\ e^{-w(s+\lambda-\,\lambda e^{-ws})} & 0 & 0 & 0 \\ e^{-w(s+\lambda)} & e^{-ws}(1 - e^{-w\lambda}) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| \tag{6.4}$$

and

$$\mathbf{E}^{\sim}(s) = \left| \begin{array}{cccc} \frac{s}{s\,+\,2\lambda} & 0 & 0 & 0 \\ 0 & \frac{s[1 - e^{-w(s+\lambda-\,\lambda e^{-ws})}]}{s+\lambda-\lambda e^{-ws}} & 0 & \frac{\lambda - \lambda e^{-ws}[1 - e^{-w(s+\lambda-\,\lambda e^{-ws})}]}{s+\lambda-\lambda e^{-ws}} \\ 0 & 0 & \frac{s(1 - e^{-w(s+\lambda)})}{s+\lambda} & \frac{\lambda}{s+\lambda} - (\frac{s+\lambda-s\,e^{-w\lambda}}{s+\lambda})e^{-ws} \\ 0 & 0 & 0 & 0 \end{array} \right| \tag{6.5}$$

As in the previous example, the time domain probabilities are calculated by numerically inverting (2.15). The plot of the state probabilities versus time for states M_1 and M_4 is reported in Figure 6.5, for $w = 1$ and for two different values of the failure rate $\lambda = 0.5$ and $\lambda = 2$.

Chapter 7

Summary

The aim of this chapter is to summarize briefly the most important results and to mention the potential further steps in the research fields of the study.

At first sight the main results of this study are related to the analysis of different stochastic processes including the marking process of timed Petri nets, but they are obviously close to the request of the applied reliability modelling and some other research fields as well. Apart from the overviews of available results (Chapter 2 and 4) and the application example (Chapter 6) the new content of this study can be divided into three groups:

- investigation of stochastic processes concerning the objectives of the considered analysis problems (Chapter 2, Appendix B),
- analysis of stochastic reward models, (Chapter 3),
- analysis of Markov regenerative stochastic Petri nets. (Chapter 5),

Investigation of stochastic processes

This group of results is hopefully useful for understanding and probably new as a uniform approach. Some results belongs to this group was mentioned by other authors, and there are results which have been developed during this research work independently, but their appearance in other works can not be excluded.

The main characteristic of this group of results seems to be the unified approach in the introduction of stochastic processes by the definition of regeneration time points and conditional equations; which is the same approach applied in deriving all of the later results. The idea of the conditional Equation 2.11 appeared in [24], but it was not elaborated there.

An other feature of this approach is that it requires the definition of the switching probability conditioned on the switching time ($p_{ij}(t)$). The derivation published in [61, 13] is not correct because of the absence of this switching probability.

There is an other shortage in the mentioned papers ([61, 13]), since both of them supposed that the distribution of the sojourn time of a state is $Q_i(t)$, which is not true in general, only if the diagonal of the kernel does not contain positive valued functions. Appendix B shows how can be transformed a general kernel (with positive valued functions in the diagonal) to a canonical one (without positive valued function in the diagonal). The derivation of [61, 13] are true only for SMPs given by canonical kernels.

Analysis of stochastic reward models

In Chapter 3 homogeneous (regarding to the preemption policies) state spaces with *prs*, *prd* and *pri* states are considered for the evaluation of reward measures of MRPs, and the three main ideas the results are based on can be summarized as follows.

Firstly, the reward analysis of MRPs with subordinated SMPs, can be performed by the introduction of a supplementary variable which is the elapsed time from the last RTP.

Secondly, a description of the stochastic processes with the opportunity of virtual state transitions ($i \to i$) in RTPs causes difficulties in the analysis of models with *prd* and *pri* states. The analysis of these processes can be carried out by the introduction of an additional supplementary variable which is the amount of the accumulated reward of the process[1].

Thirdly, the existence of an absorbing group of states is allowed where the considered life time of the examined processes is completed by reaching the work requirement or by jumping into the absorbing group of states. Relevant measures are introduced and evaluated for this kind of models.

Analysis of MRSPNs

Chapter 5 gives a general approach to the evaluation of MRSPN models, which allows the analysis of more general classes of MRSPN. The three main elements of this approach are

- the introduction of the age and firing time variables associated to every timed transition, and the definition of the regenerative time points on this base,
- the definition of a general framework of the analysis of MRSPN by Equation 5.1 and 5.2,
- the recognition of the relation between the behaviour of MRSPNs with their age and firing time variables and the stochastic multi reward processes[2].

An important consequence of this general approach is that the semantical requirements of Petri nets which provide the analysis of the model, have to be fulfilled

[1] This supplementary variable must be introduced for the analysis of SMPs given by a kernel with at least one positive valued function in the diagonal without applying the canonical representation of the process.

[2] The results of Chapter 5 are based one a continuous cooperation with Andrea Bobbio since 1991, hence it is very difficult and almost impossible to name the owner of each one. However, since in this special situation the owner of the results is required to be defined, with the kind allowance of Andrea Bobbio the results concerning the MRSPN analysis can be considered as the own results of the author with the exception of this last and probably most important one. This result is declared to be common with Andrea Bobbio.

only regeneration period by regeneration period, not throughout the lifetime of the process as defined in the former publications ([25, 24, 26]).

On the line of the general approach two further steps have been elaborated.

A classification of the regeneration periods is given by semantical restrictions provided that the evolution of the subordinated process in a regeneration period can be described by a single reward stochastic process with the allowed existence of an absorbing group of states. Based on this classification the new class of Single Reward MRSPNs is introduced. The class of Singe Reward MRSPNs contains the former introduced classes of MRSPNs (DSPN, MRSPN*, Age Memory DSPN) and it contains other MRSPN whose analysis was not possible by the former published methods. The matrix functions $\mathbf{P}(t, w)$, $\mathbf{F}(t, w)$ and $\mathbf{D}(t, w)$ are defined and evaluated for the subordinated SMPs by which the regeneration periods can be described by Equation 5.8.

Results are derived for a general single reward model and they can be used for the evaluation of different kinds of regeneration periods (subordinated to either AGE or ENABLING type GEN transitions) on the one hand, and they give the opportunity of a more detailed analysis of SRMs which can be called state dependent analysis, on the other hand. The introduction and the analysis of some new state dependent measures of SRMs is probably a further new issue of this study, which hopefully provides some more practically interesting measures. For example the state where the task completion occurs can be evaluated in this manner and the former studied measures such as the completion time can be derived by the sum of them.

Between the perceptible epilogue of the research work concluded in this study it is worth mentioning that a very important fact will determine how useful this results are. It is still an open question how the ideas can be implemented for automatic evaluation of models by computer programs, and how complicated models can be evaluated with acceptable computational complexity. An opportunity of less difficult analysis seems to be for the cases when the work requirement or the firing time is Phase type random variable.

Chapter 8

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Bibliography

[1] M. Ajmone Marsan, G. Balbo, A. Bobbio, G. Chiola, G. Conte, and A. Cumani. The effect of execution policies on the semantics and analysis of stochastic Petri nets. *IEEE Transactions on Software Engineering*, SE-15:832–846, 1989.

[2] M. Ajmone Marsan, G. Balbo, and G. Conte. A class of generalized stochastic Petri nets for the performance evaluation of multiprocessor systems. *ACM Transactions on Computer Systems*, 2:93–122, 1984.

[3] M. Ajmone Marsan and G. Chiola. On Petri nets with deterministic and exponentially distributed firing times. In *Lecture Notes in Computer Science*, volume 266, pages 132–145. Springer Verlag, 1987.

[4] R.E. Barlow and F. Proschan. *Statistical Theory of Reliability and Life Testing.* Holt, Rinehart and Winston, New York, 1975.

[5] M.D. Beaudry. Performance-related reliability measures for computing systems. *IEEE Transactions on Computers*, C-27:540–547, 1978.

[6] K. Begain, G. Farkas, L. Jereb, and M. Telek. Step-by-step state space generation for the reliability of complex markovian systems. In *Proceedings 7-th Symposium on Reliability in Electronics (RELECTRONIC '88)*, pages 314–321, Budapest, 1988.

[7] A. Bertoni and M. Torelli. Probabilistic Petri nets and semi Markov processes. In *Proceedings 2-nd European Workshop on Petri Nets*, 1981.

[8] A. Bobbio. The effect of an imperfect coverage on the optimum degree of redundancy of a degradable multiprocessor system. In *Proceedings RELIABILITY'87*, Paper 5B/3, Birmingham, 1987.

[9] A. Bobbio. Petri nets generating Markov reward models for performance/reliability analysis of degradable systems. In R. Puigjaner and D. Poitier, editors, *Modeling Techniques and Tools for Computer Performance Evaluation*, pages 353–365. Plenum Press, 1989.

[10] A. Bobbio. A multi-reward stochastic model for the completion time of parallel tasks. In A. Jensen and V.B. Iversen, editors, *Teletraffic and Datatraffic (Proceedings 13-th International Teletraffic Congress, ITC-13)*, pages 577–582. Elesevier Science Publisher, 1991.

[11] A. Bobbio. Stochastic reward models in performance/reliability analysis. *Journal on Communications*, XLIII:27–35, January 1992.

[12] A. Bobbio and A. Cumani. ML estimation of the parameters of a PH distribution in triangular canonical form. In G. Balbo and G. Serazzi, editors, *Computer Performance Evaluation*, pages 33–46. Elsevier Science Publishers, 1992.

[13] A. Bobbio and L. Roberti. Distribution of the minimal completion time of parallel tasks in multi-reward semi-Markov models. *Performance Evaluation*, 14:239–256, 1992.

[14] A. Bobbio and M. Telek. Parameter estimation of Phase type distributions. Technical report, R.T. 423, Istituto Elettrotecnico Nazionale Galileo Ferraris, 20-th European Meeting of Statisticians, August 1992.

[15] A. Bobbio and M. Telek. Task completion time. In *Proceedings PMCCS2 - 2nd International Workshop on Performability Modelling of Computer and Communication Systems*, Le Mont Sain Michel, France, 1993.

[16] A. Bobbio and M. Telek. A benchmark for PH estimation algorithms: results for acyclic-PH. *Stochastic Models*, 10:3, 1994.

[17] A. Bobbio and M. Telek. Computational restriction for SPN with generally distributed transition times. In *Proceedings 1-st European Dependable Computing Conference (EDCC-1)*, 1994. Berlin, Germany, Oct. 4-6, 1994, accepted paper.

[18] A. Bobbio and M. Telek. Transient analysis of a preemptive resume M/D/1/2/2 through Petri nets. Technical report, Dept. of Telecommunication, TUB., Apr. 1994.

[19] A. Bobbio and K.S. Trivedi. Computation of the distribution of the completion time when the work requirement is a PH random variable. *Stochastic Models*, 6:133–149, 1990.

[20] X. Castillo and D.P. Siewiorek. A performance reliability model for computing systems. In *Proceedings 10-th International Symposium on Fault-Tolerant Computing*, pages 187–192, 1980.

[21] P.F. Chimento and K.S. Trivedi. The performance of block structured programs on processors subject to failure and repair. In E. Gelenbe, editor, *High Performance Computer Systems*, pages 269–280. North Holland, 1988.

[22] P.F. Chimento and K.S. Trivedi. The completion time of programs on processors subject to failure and repair. *IEEE Transactions on Computers*, 42:1184–1194, 1993.

[23] G. Chiola. *GreatSPN* 1.5 Software architecture. In G. Balbo and G. Serazzi, editors, *Computer Performance Evaluation*, pages 121–136. Elsevier Science Publishers, 1992.

[24] Hoon Choi, V.G. Kulkarni, and K. Trivedi. Markov regenerative stochastic Petri nets. In G. Iazeolla and S.S. Lavenberg, editors, *Proceedings International Conference PERFORMANCE'93*, pages 339–356, 1993.

[25] Hoon Choi, V.G. Kulkarni, and K. Trivedi. Transient analysis of deterministic and stochastic Petri nets. In *Proceedings of the 14-th International Conference on Application and Theory of Petri Nets*, Chicago, June 1993.

[26] G. Ciardo, R. German, and C. Lindemann. A characterization of the stochastic process underlying a stochastic Petri net. In *Proceedings International Workshop on Petri Nets and Performance Models - PNPM93*, pages 170–179. IEEE Computer Society, 1993.

[27] G. Ciardo and C. Lindemann. Analysis of deterministic and stochastic Petri nets. In *Proceedings International Workshop on Petri Nets and Performance Models - PNPM93*, pages 160–169. IEEE Computer Society, 1993.

[28] G. Ciardo, J. Muppala, and K.S. Trivedi. On the solution of GSPN reward models. *Performance Evaluation*, 12:237–253, 1991.

[29] E. Cinlar. Markov renewal theory. *Advances in Applied Probability*, 1:123–187, 1969.

[30] E. Cinlar. *Introduction to Stochastic Processes.* Prentice-Hall, Englewood Cliffs, 1975.

[31] A. Cumani. Esp - A package for the evaluation of stochastic Petri nets with phase-type distributed transition times. In *Proceedings International Workshop Timed Petri Nets*, pages 144–151, Torino (Italy), 1985. IEEE Computer Society Press no. 674.

[32] L. Donatiello and B.R. Iyer. Analysis of a composite performance reliability measure for fault tolerant systems. *Journal of the ACM*, 34:179–199, 1987.

[33] J. Bechta Dugan, K. Trivedi, R. Geist, and V.F. Nicola. Extended stochastic Petri nets: applications and analysis. In *Proceedings PERFORMANCE '84*, Paris, 1984.

[34] L.C. Dunbar. A mathematical expression describing the failure probability of a system of redundant components with finite maximum repair time. *Reliability Engineering*, 7:169–179, 1984.

[35] E. De Souza e Silva and H.R. Gail. Calculating availability and performability measures of repairable computer systems using randomization. *Journal of the ACM*, 36:171–193, 1989.

[36] G. Florin and S. Natkin. Les reseaux de Petri stochastiques. *Technique et Science Informatique*, 4:143–160, 1985.

[37] D.P. Gaver. A waiting line with interrupted service, including priorities. *Journal of the Royal Statistical Society*, B24:73–90, 1962.

[38] F.A. Gay and M.L. Ketelsen. Performance evaluation for gracefully degrading systems. In *Proceedings 9-th International Symposium on Fault-Tolerant Computing*, pages 51–58, 1979.

[39] R. German and C. Lindemann. Analysis of SPN by the method of supplementary variables. In G. Iazeolla and S.S. Lavenberg, editors, *Proceedings International Conference PERFORMANCE'93*, pages 320–338, 1993.

[40] A. Goyal, V.F. Nicola, A.N. Tantawi, and K.S. Trivedi. Reliability of systems with limited repair. *IEEE Transactions on Reliability*, R-36:202–207, 1987.

[41] A. Goyal and A.N. Tantawi. Evaluation of performability for degradable computer systems. *IEEE Transactions on Computers*, C-36:738–744, 1987.

[42] V. Grassi, L. Donatiello, and G. Iazeolla. Performability evaluation of multicomponent fault-tolerant systems. *IEEE Transactions on Reliability*, R-37:216–222, 1988.

[43] P.J. Haas and G.S. Shedler. Regenerative stochastic Petri nets. *Performance Evaluation*, 6:189–204, 1986.

[44] P.J. Haas and G.S. Shedler. Stochastic Petri nets with simultaneous transition firings. In *Proceedings International Workshop on Petri Nets and Performance Models - PNPM87*, pages 24–32. IEEE Computer Society, 1987.

[45] B.R. Haverkort. *Performability modelling, tools, evaluation techniques and applications*. Phd thesis, Computer Science Dept, University of Twente, Enschede (NL), 1990.

[46] B.R. Haverkort and K. Trivedi. Specification techniques for Markov Reward Models. *Discrete Event Dynamic Systems: Theory and Applications*, 3:219–247, 1993.

[47] R.A. Howard. *Dynamic Probabilistic Systems, Volume II: Semi-Markov and Decision Processes*. John Wiley and Sons, New York, 1971.

[48] R. Huslende. A combined evaluation of performance and reliability for degradable systems. In *Proceedings ACM/SIGMETRICS Conference*, pages 157–164, 1981.

[49] B.R. Iyer, L. Donatiello, and P. Heidelberger. Analysis of performability for stochastic models of fault-tolerant systems. *IEEE Transactions on Computers*, C-35:902–907, 1986.

[50] D.L. Jagerman. An inversion technique for the Laplace transform. *The Bell System Technical Journal*, 61:1995–2002, October 1982.

[51] T. Jakab, L. Jereb, and M. Telek. Reliability analysis and planning of telecommunication networks. In *Proceedings MICROCOLL workshop*, pages 85–90, Budapest, 1990.

[52] T. Jakab, L. Jereb, and M. Telek. Computer aided reliability analysis and planning of telecommunication networks. *Communications*, XVII:14–19, March 1991. Special issue on computer aided network simulation.

[53] T. Jakab, L. Jereb, and M. Telek. Reliability analysis of telecommunication networks. *Magyar Távközlés*, II, Oct 1991. in Hungarian.

[54] T. Jakab, L. Jereb, and M. Telek. RELNET: a tool for the reliability analysis of communication networks. In *Proceedings 8th Symposium on Reliability in Electronics (RELECTRONIC '91)*, pages 513–522, Budapest, 1991.

[55] T. Jakab, L. Jereb, and M. Telek. s-t terminal reliability modeling and analysis of communication networks with multistate, dependent links. In *Proceedings NETWORKS'92*, pages 265–270, Kobe (Japan), 1992.

[56] L. Jereb. *Reliability analysis of complex Markovian systems*. PhD thesis, Technical University of Budapest, Budapest, 1987. in Hungarian.

[57] J. Keilson and S.S. Rao. A process with chain dependent growth rate. *Journal of Applied Probability*, 7:699–711, 1970.

[58] J. Keilson and S.S. Rao. A process with chain dependent growth rate. Part II: the ruin and ergodic problems. *Advances in Applied Probability*, 3:315–338, 1971.

[59] H.M. Khelalfa and A.K. von Mayrhauser. Models to evaluate trade-offs between performance and reliability. In *Proceedings CMG XV*, pages 24–29, San Francisco, 1984.

[60] V.G. Kulkarni, V.F. Nicola, and K. Trivedi. On modeling the performance and reliability of multi-mode computer systems. *The Journal of Systems and Software*, 6:175–183, 1986.

[61] V.G. Kulkarni, V.F. Nicola, and K. Trivedi. The completion time of a job on a multi-mode system. *Advances in Applied Probability*, 19:932–954, 1987.

[62] R. Lepold. PEPNET: A new approach to performability modelling using stochastic Petri nets. In *Proceedings 1st International Workshop on Performability Modelling of Computer and Communication Systems*, pages 3–17, University of Twente - Enschede (NL), 1991.

[63] Y. Levy and P.E. Wirth. A unifying approach to performance and reliability objectives. In *Proceedings 12-th International Teletraffic Congress, ITC-12*, pages 4.2B2.1–4.2B2.7, Torino, 1988.

[64] C. Lindemann. An improved numerical algorithm for calculating steady-state solutions of deterministic and stochastic Petri net models. In *Proceedings of the 4-th International Workshop on Petri Nets and Performance Models*, pages 176–185, Melbourne, 1991.

[65] C. Lindemann. An improved numerical algorithm for calculating steady-state solutions of deterministic and stochastic Petri net models. *Performance Evaluation*, 8:79–95, 1993.

[66] R.A. McLean and M.F. Neuts. The integral of a step function defined on a Semi-Markov process. *SIAM Journal on Applied Mathematics*, 15:726–737, 1967.

[67] J.F. Meyer. On evaluating the performability of degradable systems. *IEEE Transactions on Computers*, C-29:720–731, 1980.

[68] J.F. Meyer. Closed form solution of performability. *IEEE Transactions on Computers*, C-31:648–657, 1982.

[69] J.F. Meyer. Performability: a retrospective and some pointers to the future. *Performance Evaluation*, 14:139–156, 1992.

[70] J.F. Meyer, A. Movaghar, and W.H. Sanders. Stochastic activity networks: structure, behavior and application. In *Proceedings International Workshop Timed Petri Nets*, pages 106–115, Torino (Italy), 1985. IEEE Computer Society Press no. 674.

[71] M.K. Molloy. Performance analysis using stochastic Petri nets. *IEEE Transactions on Computers*, C-31:913–917, 1982.

[72] J.K. Muppala, S.P. Woolet, and K.S. Trivedi. Composite performance and dependability analysis. In *Proceedings 1st International Workshop on Performability Modelling of Computer and Communication Systems*, pages 131–161, University of Twente - Enschede (NL), 1991.

[73] S. Natkin. Les reseaux de Petri stochastiques et leur application a l'evaluation des systemes informatiques. Technical report, These de Docteur Ingegneur, CNAM, Paris, 1980.

[74] M.F. Neuts. *Matrix Geometric Solutions in Stochastic Models*. Johns Hopkins University Press, Baltimore, 1981.

[75] V.F. Nicola. A single server queue with mixed types of interruptions. *Acta Informatica*, 23:465–486, 1986.

[76] V.F. Nicola, A. Bobbio, and K. Trivedi. A unified performance reliability analysis of a system with a cumulative down time constraint. *Microelectronics and Reliability*, 32:49–65, 1992.

[77] V.F. Nicola, V.G. Kulkarni, and K. Trivedi. Queueing analysis of fault-tolerant computer systems. *IEEE Transactions on Software Engineering*, SE-13:363–375, 1987.

[78] K.G. Ramamurthy and N.K. Jaiswal. A two-dissimilar-unit cold standby system with allowed downtime. *Microelectronics and Reliability*, 22:689–691, 1982.

[79] A. Reibman. Modeling the effect of reliability on performance. *IEEE Transactions on Reliability*, R-39:314–320, 1990.

[80] A. Reibman, R. Smith, and K.S. Trivedi. Markov and Markov reward model transient analysis: an overview of numerical approaches. *European Journal of Operational Research*, 40:257–267, 1989.

[81] P. Risztics. *Reliability-performance analysis of process control systems.* PhD thesis, Technical University of Budapest, Budapest, 1988. in Hungarian.

[82] S.M. Ross and J. Schechtman. On the first time a separately maintained parallel system has been down for a fixed time. *Naval Research Logistic Quarterly*, 26:285–290, 1979.

[83] R. Smith, K. Trivedi, and A.V. Ramesh. Performability analysis: Measures, an algorithm and a case study. *IEEE Transactions on Computers*, C-37:406–417, 1988.

[84] U. Sumita, J.G. Shanthikumar, and Y. Masuda. Analysis of fault tolerant computer systems. *Microelectronics and Reliability*, 27:65–78, 1987.

[85] L. Szirmay-Kalos. *Kvantitative analysis and optimization of computer systems.* PhD thesis, Technical University of Budapest, Budapest, 1991. in Hungarian.

[86] K. Trivedi, A.S. Sathaye, O.C. Ibe, and R.C. Howe. Should I add a processor ? In *Proceedings 23-rd Annual Hawaii International Conference on System Sciences HICSS-23*, pages 214–221, 1990.

Appendix A

Some important properties of the Laplace-Stieltjes Transform

Let us consider a function with the following properties $F(t) = 0$ if $t < 0$, is analytical at $t = 0$ and $\int_{0^-}^{\infty} |F(t)| e^{-ct} dt < \infty$ for all $c > 0$ from which follows that $\lim_{t \to \infty} F(t)\, e^{-ct} = 0$ for all $c > 0$. The Laplace transform of $F(t)$ is defined as $F^*(s) = \int_{0^-}^{\infty} F(t) e^{-st} dt$ and its Laplace-Stieltjes transform as $F^{\sim}(s) = \int_{0^-}^{\infty} e^{-st} dF(t)$. Let us consider the following function:

$$F_m(t) = \begin{cases} \dfrac{dF(t)}{dt} & \text{if} : t > 0 \\ \lim\limits_{t \to 0^+} \dfrac{dF(t)}{dt} & \text{if} : t = 0 \\ 0 & \text{if} : t < 0 \end{cases}$$

and its Laplace transform pair:

$$F_m^*(s) = \int_{0^-}^{\infty} F_m(t) e^{-st} dt = \int_{0^+}^{\infty} \frac{dF(t)}{dt} e^{-st} dt =$$

$$\int_{0^+}^{\infty} F(t)\, s\, e^{-st} dt + \int_{0^+}^{\infty} \frac{d\left(F(t)\, e^{-st}\right)}{dt}\, dt =$$

$$s \int_{0^-}^{\infty} F(t)\, e^{-st} dt + \int_{0^+}^{\infty} d\left(F(t)\, e^{-st}\right) = s\, F^*(s) - F(0)$$

But on the other hand

$$F_m^*(s) = \int_{0^-}^{\infty} F_m(t) e^{-st} dt = \int_{0^+}^{\infty} \frac{dF(t)}{dt} e^{-st} dt =$$

$$\int_{0^+}^{\infty} e^{-st} dF(t) = \int_{0^-}^{\infty} e^{-st} dF(t) - \int_{0^-}^{0^+} e^{-st} dF(t) = F^{\sim}(s) - F(0)$$

hence

$$F^{\sim}(s) = s\, F^*(s)$$

The most important advantage of the introduction of the Laplace-Stieltjes transform is that it makes possible to handle the random variables in transform domain without probability density function. In the practical application cases there are deterministic or discrete delays very often, hence there can be probability masses in the lifetime of the random process.

In the following table some important transform pairs and the main properties of the Laplace and the Laplace-Stieltjes transforms are summarized.

Time domain	LT domain	LST domain
$F(t),\, t \geq 0$	$F^*(s) = \int_{0^-}^{\infty} F(t) e^{-st} dt$	$F^\sim(s) = \int_{0^-}^{\infty} e^{-st} dF(t)$
$a\,F(t) \,+\, b\,G(t)$	$a\,F^*(s) \,+\, b\,G^*(s)$	$a\,F^\sim(s) \,+\, b\,G^\sim(s)$
$F(t/a) \quad a > 0$	$a\,F^*(as)$	$F^\sim(as)$
$F(t-a) \quad a > 0$	$e^{-as}\,F^*(s)$	$e^{-as}\,F^\sim(s)$
$\int_{0^-}^{t} F(\tau)\,G(t-\tau)\,d\tau$	$F^*(s)\,G^*(s)$	$\frac{1}{s}\,F^\sim(s)\,G^\sim(s)$
$\int_{0^-}^{t} G(t-\tau)\,d\,F(\tau)$	$s\,F^*(s)\,G^*(s)$	$F^\sim(s)\,G^\sim(s)$
$\dfrac{dF(t)}{dt}$	$sF^*(s) - F(0)$	$s\,[F^\sim(s) - F(0)]$
$\int_{0^-}^{\infty} F(t) dt$	$\lim_{s \to 0} F^*(s)$	$\lim_{s \to 0} \frac{1}{s} F^*(s)$
$\int_{0^-}^{\infty} dF(t) = \lim_{t \to \infty} F(t)$	$\lim_{s \to 0} sF^*(s)$	$\lim_{s \to 0} F^\sim(s)$
$\lim_{t \to 0} F(t)$	$\lim_{s \to \infty} sF^*(s)$	$\lim_{s \to \infty} F^\sim(s)$

Based on the general properties listed above the most important applications are as follows:

Time domain	LT domain	LST domain
$F(t),\, t \geq 0$	$F^*(s) = \int_{0^-}^{\infty} F(t)e^{-st}dt$	$F^\sim(s) = \int_{0^-}^{\infty} e^{-st}dF(t)$
$U(t)$	$\frac{1}{s}$	1
$U(t-a)$	$\frac{e^{-as}}{s}$	e^{-as}
$\delta(t)$	1	s
$\delta(t-a)$	e^{-as}	$s\,e^{-as}$
$b\,e^{-at}$	$\frac{b}{s+a}$	$\frac{b\,s}{s+a}$
$1-e^{-at}$	$\frac{a}{s(s+a)}$	$\frac{a}{s+a}$

where $U(t)$ is the unit step function and $\delta(t)$ is the unit impulse (also referred Dirac delta).

Appendix B

Canonical representation of stochastic processes with embedded Markov renewal sequence

B.1 Canonical representation of semi-Markov Processes

Let Ω (of cardinality n) be the state space of the $Z(t)$ $(t \geq 0)$ (right continuous)[1] time homogeneous semi-Markov process which is defined by its kernel $\mathbf{Q}(t) = [Q_{ij}(t)]$ over Ω. We denote by H the time duration to the first embedded time point of the semi-Markov process starting from state i at time 0 ($Z(0) = i$). The generic element (for $i, j \in \Omega$)

$$Q_{ij}(t) = Pr\left\{H \leq t, Z(H) = j | Z(0) = i\right\}$$

is the distribution of H supposed that a transition from state i to state j took place at the embedded time point.

The distribution of H is:

$$Q_i(t) = Pr\left\{H \leq t | Z(0) = i\right\} = \sum_{j \in \Omega} Q_{ij}(t) \qquad (i = 1, ..., n)$$

and, the probability of jumping from state i to state j at the first embedded time point is:

$$p_{ij} = Pr\left\{Z(H) = j | Z(0) = i\right\} = \lim_{t \to \infty} Q_{ij}(t)$$

Denoting by $H_{ij}(t)$ the distribution of the time duration to the first embedded time point given that the semi-Markov process starts from state i at time 0 and jumps to state j at the first embedded time point we have:

$$H_{ij}(t) = Pr\left\{H \leq t | Z(H) = j, Z(0) = i\right\} = \frac{Q_{ij}(t)}{p_{ij}}$$

[1] The right continuity defines the state of the process at the state transition time points.

With positive diagonal element in $\mathbf{Q}(t)$ a virtual transition from state i to state i can occur in the embedded time points, and this fact has some disadvantageous consequences:

- there is no physical meaning of the embedded time points,
- the simulation of the semi-Markov process based on $\mathbf{Q}(t)$ is not effective since internal time points are calculated without any changes in the process,
- the representation of the semi-Markov process by its kernel $\mathbf{Q}(t)$ is not unique, i.e. there can be different kernels which results the same semi-Markov process.

The different representation of a semi-Markov process by different kernels differ only in the frequency of the embedded time points. From the above mentioned point of views the representation with the rarest embedded time points (i.e. the time points of the real state transitions) has a unique importance.

Let $\mathbf{Q}^u(t) = [Q^u_{ij}(t)]$ be the kernel of the same semi-Markov process with this property. In this case the pertaining H^u gains the visual physical meaning, it becomes the sojourn time in the initial state

$$H^u = min\{t \geq 0 | Z(t) \neq Z(0)\}$$

The discrete event simulation is the most effective in this case since the number of the required time points to be calculated is the least.

By the above assumptions the kernel $\mathbf{Q}^u(t)$ could be a good choice for canonical representation of the semi-Markov process, but for this end we have to define the way of the determination of $\mathbf{Q}^u(t)$ by any $\mathbf{Q}(t)$.

Theorem B.1 *The canonical representation $\mathbf{Q}^u(t)$ of a semi-Markov process given by its kernel $\mathbf{Q}(t)$ is defined as follows:*

$$Q^{u\sim}_{ij}(s) = \begin{cases} \dfrac{Q^{\sim}_{ij}(s)}{1 - Q^{\sim}_{ii}(s)} & \text{if} : i \neq j \\ 0 & \text{if} : i = j \end{cases} \tag{B.1}$$

Proof:

Let X_n r.v. is the time duration to the first transition out of state i given that it is to state j $(j \neq i)$ and there are n internal $i \to i$ transitions at the embedded time points before leaving state i. By its definition X_n is the sum of n i.i.d. random variables with distribution $H_{ii}(t)$ plus and additional random variable with distribution $H_{ij}(t)$.

Conditioning on the number of virtual transition $N = n$ at the embedded time points from state i to state i before leaving state i we can define the distribution function of the sojourn time in state i in LST domain[2] suppose that the next state is state j $(j \neq i)$:

[2]For independent random variables (X and Y) the distribution of the sum of them can be expressed as:

$$F^{\sim}_{X+Y}(s) = F^{\sim}_X(s) F^{\sim}_Y(s)$$

$$H_{ij}^{u\sim}(s|N=n) = E\left[e^{-sX_n}\right] = H_{ii}^{\sim}(s)^n H_{ij}^{\sim}(s)$$

Unconditioning with respect to N, we get:

$$H_{ij}^{u\sim}(s) = \sum_{n=0}^{\infty} p_{ii}^n (1-p_{ii}) H_{ii}^{\sim}(s)^n H_{ij}^{\sim}(s) = \frac{(1-p_{ii})H_{ij}^{\sim}(s)}{1-p_{ii}H_{ii}^{\sim}(s)} = \frac{1-p_{ii}}{p_{ij}} \frac{Q_{ij}^{\sim}(s)}{1-Q_{ii}^{\sim}(s)}$$

Similarly [47]:

$$p_{ij}^u = \sum_{n=0}^{\infty} p_{ii}^n p_{ij} = \frac{p_{ij}}{1-p_{ii}}$$

Hence the canonical representation of a semi-Markov process $\mathbf{Q}^u(t)$ is given by (B.1).

□

B.1.1 Example for different representations of a semi-Markov Process

As a very simple example let us consider the different semi-Markov representations of a CTMC with infinitesimal generator $A = \{a_{ij}\}$. The distribution of the sojourn time in state i jumping to any other state j $(j \neq i)$ is

$$H_{ij}^u(t) = 1 - e^{a_{ii}t} \qquad H_{ij}^{u\sim}(s) = \frac{-a_{ii}}{s-a_{ii}}$$

and the probability of the transition to state j $(j \neq i)$ after staying in state i is

$$p_{ij}^u = \frac{a_{ij}}{-a_{ii}}$$

hence the kernel in LST domain is

$$Q_{ij}^{u\sim}(s) = \begin{cases} \dfrac{a_{ij}}{s-a_{ii}} & \text{if} : i \neq j \\ 0 & \text{if} : i = j \end{cases}$$

By the mean of the randomization technique we can define a different kernel for the same process. A CTMC can be divided into two independent random processes which are a Poisson arrival process with intensity $a \geq \max_{i,j} |a_{ij}|$ and a Markov chain with one step state transition matrix $P = A/a + I$. The time durations between the embedded time points of this case are exponentially distributed with parameter a (independent of the state of the process)

$$H_{ij}^{\sim}(s) = \frac{a}{s+a}$$

and the state transition probabilities are given by the P matrix. The kernel of this representation is:

$$Q_{ij}^{\sim}(s) \;=\; \left(\frac{a_{ij}}{a}+\delta_{ij}\right)\frac{a}{s+a} \;=\; \begin{cases} \dfrac{a_{ij}}{s+a} & \text{if} : i \neq j \\[2ex] \dfrac{a_{ij}+a}{s+a} & \text{if} : i = j \end{cases}$$

Let us generate the canonical representation of the process by (B.1) for $j \neq i$ elements:

$$Q_{ij}^{u\sim}(s) = \frac{Q_{ij}^{\sim}(s)}{1-Q_{ii}^{\sim}(s)} = \frac{\dfrac{a_{ij}}{s+a}}{1-\dfrac{a_{ii}+a}{s+a}} = \frac{a_{ij}}{s-a_{ii}}$$

B.2 Canonical representation of Markov Regenerative Processes

The lifetime of a MRP given by its external and internal kernels ($\mathbf{K}(t)$ and $\mathbf{E}(t)$) contains a sequence of time points (RTPs) in which the process is defined, and intervals between them in which the process is not determined, but the state probabilities are known.

The level of the uncertainty depends on the frequency of the regenerative time points. A MRP can be defined by different external and internal kernels and the one which defines the most frequent series of the regeneration time points gives the most information on the process.

But there can be situations in which an external kernel with positive diagonal element does not meet the purposes of the analysis. In this case the canonical representation of the MRP is possible by $\mathbf{K}^u(t)$ and $\mathbf{E}^u(t)$, but this representation defines less RTPs than the original one. For this end let us define the following quantities. Let H (H^u) be the time duration to the first RTP of the MRP defined by $\mathbf{K}(t)$ ($\mathbf{K}^u(t)$) and $\mathbf{E}(t)$ ($\mathbf{E}^u(t)$) starting from state i at time 0 ($Z(0)=i$). Let

$$K_i(t) = \sum_{j\in\Omega} K_{ij}(t) \qquad \text{and} \qquad K_i^u(t) = \sum_{j\in\Omega} K_{ij}^u(t) \qquad (i = 1, ..., n)$$

similarly to the former SMP case. And let

$$G_{ij}(t) = Pr\left\{Z(H) = j | H > t, Z(0) = i\right\} = \frac{E_{ij}(t)}{1-K_i(t)}$$

and

$$G_{ij}^u(t) = Pr\left\{Z(H^u) = j | H^u > t, Z(0) = i\right\} = \frac{E_{ij}^u(t)}{1-K_i^u(t)}$$

the probability of being in state j at time t supposing that there was no RTP since the process started from state i at time 0.

Theorem B.2 *The elements of the external kernel of the canonical representation can be defined as:*

$$K_{ij}^{u\sim}(s) = \begin{cases} \dfrac{K_{ij}^{\sim}(s)}{1-K_{ii}^{\sim}(s)} & \text{if} : i \neq j \\ \\ 0 & \text{if} : i = j \end{cases} \tag{B.2}$$

and $E_{ij}^{u}(t)$ *satisfies the following equation:*

$$E_{ij}^{u}(t) = G_{ij}^{u}(t)(1 - K_{i}^{u}(t)) \tag{B.3}$$

where

$$G_{ij}^{u\sim}(s) = \frac{E_{ij}^{\sim}(s)}{1-K_{i}^{u\sim}(s)} \tag{B.4}$$

and

$$K_{i}^{u\sim}(s) = \frac{K_{i}^{\sim}(s) - K_{ii}^{\sim}(s)}{1-K_{ii}^{\sim}(s)} \tag{B.5}$$

Proof:

The elements of the external kernel of the canonical representation can be similarly defined as a canonical kernel of a SMP by (B.1).

The derivation of the internal kernel of the canonical representation is a more complicated problem. Conditioning on $H^u = h$ we can define:

$$G_{ij}^{u}(t \,|\, H^u = h) = \begin{cases} G_{ij}(t) & h > t \\ \\ G_{ij}^{u}(t-h) & h \leq t \end{cases} \tag{B.6}$$

In (B.6) two mutually exclusive events are defined: if $h > t$ then $G_{ij}(t)$ equals to $G_{ij}^{u}(t)$ since the process does not reach the first RTP; if $h \leq t$ then we have a regeneration time point in $(0, t)$ and the state probabilities are defined from that time. Evaluating the mean with respect to H^u (B.6) becomes:

$$G_{ij}^{u}(t) = E_{ij}(t) + \int_{h=0}^{t} G_{ij}^{u}(t-h) dK_i(h)$$

After Laplace-Stieltjes transforming with respect to t we have (B.4).
Since $K_{i}^{u\sim}(s)$ is defined by

$$K_{i}^{u\sim}(s) = \sum_{j\in\Omega} K_{ij}^{u\sim}(s) = \sum_{j\in\Omega, j\neq i} \frac{K_{ij}^{\sim}(s)}{1-K_{ii}^{\sim}(s)} = \frac{K_{i}^{\sim}(s) - K_{ii}^{\sim}(s)}{1-K_{ii}^{\sim}(s)}$$

$E_{ij}^{u}(t)$ is calculated as the product of these two functions.

□

Appendix C

Accumulated reward and completion time

The equation (2.20) defines the relation between the distribution of the accumulated reward and the completion time when the states of the structure-state process are of *prs* type. We discuss the consequences of this relation, considering *prs* states. A simple example is introduced.

Let $B(t, w) = Pr(B(t) \leq w)$ be the distribution of the accumulated reward up to t and $C(t, w) = Pr(C(w) \leq t)$ the distribution of the completion time.

Considering *prs* states by (2.20) we have:

$$B(t, w) + C(t, w) \equiv 1 \qquad \text{for} \quad \forall\, t, w. \tag{C.1}$$

Let us suppose that the state probabilities of the structure state process are $V_i(t)$ and the reward rates are r_i.

To get closer to the relation of the accumulated reward and the completion time let us study the following questions to have an impression about them.

- How much is a discrete state continuous time stochastic process defined by the functions of the state probabilities versus time?
- How much do the state probabilities define the amount of the accumulated reward?
- What can be said about the mean of the completion time based on the mean of the accumulated reward (which is defined by the state probabilities)?

These questions gain significant importance when the stochastic process is a MRP defined by its external and internal kernel, since, in this case, the process is defined by the state probabilities between the consecutive RTPs.

A stochastic process is not defined only by the knowledge of the state probabilities. Let us consider the following two processes with the same state probability functions.

Process A:
is a 2 state CTMC with initial probabilities $V_1(0) = 1/2$, $V_2(0) = 1/2$ and with exponentially distributed state transitions (with parameter λ) from State 1 to State 2 and vice-versa.

Process B:
is basically the same 2 state stochastic process with initial probabilities $V_1(0) = 1/2$, $V_2(0) = 1/2$ and with exponentially distributed state transition (with parameter λ) from State 1 to State 2 and vice-versa, but this process can have only one state transition. If the process starts from State 1 then State 2 is absorbing, and if the process starts from State 2 then State 1 is absorbing.
Let $r_1 = 1$ and $r_2 = 0$ for both processes.

The two processes have the same state probability functions:

$$V_1(t) \equiv V_2(t) \equiv 1/2$$

This examples are to enlighten that different stochastic processes can have the same state probability functions hence a stochastic process is not defined only by the knowledge of the state probabilities.

Based on the state probabilities the distribution of the accumulated reward can not be evaluated, only its mean:

$$\begin{aligned} E\left[\, B(t)\,\right] = E\left[\sum_{i\in\Omega} r_i\, t_i\right] = \sum_{i\in\Omega} r_i\; E\left[t_i\right] = \sum_{i\in\Omega} r_i\; E\left[\int_0^t I_{\{Z(\tau)=i\}} d\tau\right] = \\ \sum_{i\in\Omega} r_i\; \int_0^t E\left[I_{\{Z(\tau)=i\}}\right] d\tau = \sum_{i\in\Omega} r_i\; \int_{\tau=0}^t V_i(\tau)\, d\tau \end{aligned} \tag{C.2}$$

where t_i is the time the process spent in state i in $(0, t)$ and $I_{\{Z(\tau)=i\}}$ is the indicator of being in state i.

One of the important differences between the accumulated reward and the completion time is characterized by the above questions. While the state probabilities define the mean of the former one, nothing can be said about the latter. For example in Process A the mean of $C(w)$ is finite for all bounded w but in Process B it is ∞ for all positive w.

Appendix D

The applied numerical inverse transform method

The following MATHEMATICA source provided the results of Figure 6.4:

```
(* +++++++++++++++++++++++++++++++++++++++++++++++ *)
(*  M/D/1/2/2 preemptive queue, PRS memory policy *)
(*  SAME KIND OF USERS !!                          *)
(*   K and E matrixes                              *)


x1[s_] :=  Exp[- tau (s + lambda)] Exp[lambda tau Exp[- tau s] ]

x2[s_] :=  1 / (s + lambda - lambda Exp[- tau s])

ks = {{0, (1/s - 1/(s+2 lambda)), 0},
      {1/s x1[s], 0, 0},
      {0, 0, 0}
     };

es = { {1/(s+2 lambda), 0, 0},
       {0, x2[s] (1 - x1[s]), (1/s - x2[s]) (1 - x1[s])},
       {0, 0, 0}
     };

i = DiagonalMatrix[{1, 1, 1}];

(* Steady state probabilities *)
ps = Inverse[ i -  s ks ] . es;
ps11[s_] = Part[ Inverse[ i -  s ks ] . es, 1,1] ;
ps12[s_] = Part[ Inverse[ i -  s ks ] . es, 1,2] ;
ps13[s_] = Part[ Inverse[ i -  s ks ] . es, 1,3] ;
```

```
(*-------------------------------------------*)
lambda =  0.5
tau = 1.

(* Jagerman numerical method *)
ee[q_, x_] := Exp[ I 2 Pi x / q ]  ;
fnt[n_, t_, q_, r_] :=
 ((n+1) / (t q r^n) *
   Sum[ ee[q, -n j]  ps11[ (n+1) / t (1 - r ee[q, j]) ], {j, 1, q}])
ont[n_, t_, q_, r_] := (2+1/n) fnt[2n,t,q,r] - (1+1/n) fnt[n,t,q,r]

Do [ pp = ont[50, xx, 251, .8] //N ; Print[{xx, Re[pp]}],
      {xx, 0.1, 4, 0.1}]
```

Appendix E

List of notations

H	Holding time of a regeneration period, r.v.
Ω	Finite state space of the structure state process
R	Up subset of states, $R \subset \Omega$
R^c	Down subset of states, $R^c \subset \Omega$
$Z(t)$	Continuous time finite state stochastic process
r_i	Reward rate in state i
W	Work requirement, r.v.
C	Completion time of the (random) work requirement W, r.v.
$\hat{C}(t)$	Distribution of the completion time of the work requirement W $\hat{C}(t) = Pr\{C \leq t\}$
$C(w)$	Completion time of the deterministic work requirement w, r.v.
$C(t,w)$	Distribution of $C(w)$ $C(t,w) = Pr\{C(w) \leq t\}$
$W(w)$	Distribution of the work requirement W $W(w) = Pr\{W \leq w\} \quad \left(\hat{C}(t) = \int_{a=0}^{\infty} C(t,w)dW(w)\right)$

$B(t)$	Collected reward up to t, r.v.
$B(t,w)$	Distribution of $B(t)$ $B(t,w) = Pr\{B(t) \leq w\}$
$\mathbf{V}(t) = \{V_{ij}(t)\}$	State transition probability matrix $V_{ij}(t) = Pr\{Z(t) = j \| Z(0) = i\}$
$\underline{V}_0$	Initial state probability vector
$\underline{V}_{ST}$	Steady state probability vector
$V(t) = \{V_i(t)\}$	State probability vector at time t
$\mathbf{A} = \{a_{ij}\}$	Infinitesimal operator of continuous time Markov chains $a_{ii} = -\sum_{j \in \Omega, j \neq i} a_{ij}$
$\mathbf{Q}(t) = \{Q_{ij}(t)\}$	Kernel of semi-Markov process $Q_{ij}(t) = Pr\{H \leq t, Z(H) = j \| Z(0) = i\}$
$\mathbf{K}(t) = \{K_{ij}(t)\}$	External kernel of Markov regenerative process $K_{ij}(t) = Pr\{H \leq t, Z(H) = j \| Z(0) = i\}$
$\mathbf{E}(t) = \{E_{ij}(t)\}$	Internal kernel of Markov regenerative process $E_{ij}(t) = Pr\{H > t, Z(t) = j \| Z(0) = i\}$
$\mathbf{H}(t) = \{H_{ij}(t)\}$	Distribution of the next regeneration time point starting from state i suppose that the next regenerative state is state j $H_{ij}(t) = Pr\{H \leq t \| Z(t) = j,\ Z(0) = i\} = \frac{K_{ij}(t)}{p_{ij}}$
$\mathbf{\Pi} = \{p_{ij}\}$	One step state transition probability matrix of the embedded Markov chain $p_{ij} = Pr\{Z(H) = j \| Z(0) = i\}$
$\mathbf{G}(t) = \{G_{ij}(t)\}$	State transition probability matrix inside a regeneration period $G_{ij}(t) = Pr\{Z(t) = j \| H > t,\ Z(0) = i\} = \frac{E_{ij}(t)}{1 - K_i(t)}$

Petri net analysis

Ω	Finite state space of the marking process
R	Enabling subset of states
$\mathbf{F}(t) = \{F_{ij}(t)\}$	State dependent firing time $F_{ij}(t) = Pr\{firing\,in\,state\,j\,at\,H \leq t \mid Z(0) = i\}$
$\mathbf{P}(t) = \{P_{ij}(t)\}$	State probabilities before firing $P_{ij}(t) = Pr\{Z(t) = j, firing\,time > t \mid Z(0) = i\}$
$\mathbf{D}(t) = \{D_{ij}(t)\}$	Probabilities of disabling states $D_{ij}(t) = Pr\{Z(t) = j \in R^c \mid Z(0) = i\}$

Abbreviations

RTP	regeneration time point
CTMC	continuous time Markov chain
SMP	semi-Markov process
MRP	Markov regenerative process
SRM	stochastic reward model
prs	preemptive resume
prt	preemptive repeat
prd	preemptive repeat different
pri	preemptive repeat identical
PN	Petri net
TPN	timed Petri net
SPN	stochastic Petri net
GDT_SPN	generally distributed transition stochastic Petri net
DSPN	deterministic and stochastic Petri net
PHSPN	Phase type stochastic Petri net
PH	Phase type (distribution)
MRSPN	Markov regenerative stochastic Petri net