The Scale Factor: A New Degree of Freedom in Phase Type Approximation*

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Abstract

This paper introduces a unified approach to phase-type approximation in which the discrete and the continuous phase-type models form a common model set. The models of this common set are assigned with a non-negative real parameter, the *scale factor*. The case when the scale factor is strictly positive results in Discrete phase-type distributions and the scale factor represents the time elapsed in one step. If the scale factor is 0, the resulting class is the class of Continuous phase-type distributions. Applying the above view, it is shown that there is no qualitative difference between the discrete and the continuous phase-type models.

Based on this unified view of phase-type models one can choose the best phase-type approximation of a stochastic model by optimizing the scale factor.

Keywords: Discrete and Continuous Phase type distributions, Phase type expansion, approximate analysis.

1 Introduction

This paper presents new comparative results on the use of Discrete Phase Type (DPH) distributions [19] and of Continuous Phase Type (CPH) distributions [20] in applied stochastic modeling.

DPH distributions of order n are defined as the time to absorption in a Discrete-State Discrete-Time Markov Chain (DTMC) with n transient states and one absorbing state. CPH distributions of order n are defined, similarly, as the distribution of the time to absorption in a Discrete-State Continuous-Time Markov Chain (CTMC) with n transient states and one absorbing state. The above definition implies that the properties of a DPH distribution are computed over the set of the natural numbers while the properties of a CPH distribution are defined as a function of a continuous time variable t. When DPH distributions are used to model timed activities, the set of the natural numbers must be

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related to a time measure. Hence, a new parameter need to be introduced that represents the time span associated to each step. This new parameter is the scale factor of the DPH distribution, and can be viewed as a new degree of freedom, since its choice largely impacts the shape and properties of a DPH distribution over the continuous time axes. When DPH distributions are used to approximate a given continuous distribution, the scale factor affects the goodness of the fit.

The paper starts discussing to what extent DPH or CPH distributions can be utilized to fit a given continuous distribution. It is shown that a DPH distribution of any order converges to a CPH distribution of the same order as the scale factor goes to zero. Even so, the DPH class contains distributions whose behavior differs substantially from the one of the corresponding distributions in the CPH class. Two main peculiar points differentiate the DPH class from the CPH class. The first point concerns the coefficient of variation: indeed, while in the continuous case the minimum coefficient of variation is a function of the order only and its lower bound is given by the well known theorem of Aldous and Shepp [1], in the discrete case the minimum coefficient of variation is proved to depend both on the order and on the mean (and hence on the scale factor) [22]. Furthermore, it is easy to see that for any order, there exist members of the DPH class that represent a deterministic value with a coefficient of variation equal to zero. Hence, for any order (greater than 1), the coefficient of variation of the DPH class spans from zero to infinity.

The second peculiar point that differentiate the DPH class is the support of the distributions. While a CPH distribution (of any order) has always an infinite support, there exist members of the DPH class of any order that have a finite support (between a minimum non-negative value and a maximum) or have a mass equal to one concentrated in a single value (deterministic distribution).

It turns out that the possibility of

- tuning the scale factor to optimize the goodness of the fit,
- having distributions with coefficient of variation spanning from 0 to infinity,
- representing deterministic values exactly,
- coping with finite support distributions,

makes the DPH class a very interesting and challenging class of distributions to be explored in applied stochastic models. The purpose of this paper is to show how these favorable properties can be exploited in practice, and to provide guidelines to the modeler to a reasonably good choice of the distributions to be used. Indeed, since a DPH distribution tends to a CPH distribution as the scale factor approaches zero, considering the scale factor as a new decision variable in a fitting experiment, and finding the value of the optimal scale factor (with respect to some error measure) provides a valuable tool to decide whether to use a discrete or a continuous approximation to the given problem.

The fitting problem for the CPH class has been extensively studied and reported in the literature by resorting to a variety of structures and numerical techniques [2, 3, 9, 15, 21]. Conversely, the fitting problem for the DPH class has received very little attention [6].

In recent years, a considerable effort has been devoted to define models with generally distributed timings [10, 7] and to merge in the same model random variables and deterministic duration [18]. Analytical solutions are possible in special cases, and the approximation of the original problems by means of CPH distributions is a rather well known technique [12, 8]. This paper is aimed at emphasizing that DPH approximation may provide a more convenient alternative with respect to CPH approximation, and also to provide a way to quantitatively support this choice. Furthermore, the use of DPH approximation can be extended from stochastic models to functional analysis where time intervals with nondeterministic choice are considered [5, 4]. Finally, discretization techniques for continuous problems [13, 14] can be restated in terms of DPH approximations.

The rest of the paper is organized as follows. After defining the notation to be used in the paper in Section 2, Section 3 discusses the peculiar properties of the DPH class with respect to the CPH class. Some guidelines for bounding the parameters of interest and extensive numerical experiments to show how the goodness of the fit is influenced by the optimal choice of the scale factor are reported in Section 4. Section 5 discusses the quality of the approximation when passing from the analysis of a single distribution to the analysis of performance measures in complete non-Markovian stochastic models. The paper is concluded in Section 6.

2 Definition and Notation

A DPH distribution [19, 20] is the distribution of the time to absorption in a DTMC with n transient states, and one absorbing state numbered $(n+1)$. The one-step transition probability matrix of the corresponding DTMC can be partitioned as:

$$\widehat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix} \tag{1}$$

where $\mathbf{B} = [b_{ij}]$ is the $(n \times n)$ matrix collecting the transition probabilities among the transient states, $\mathbf{b} = [b_{i,n+1}]^T$ is the column vector of length n grouping the probabilities from any state to the absorbing one, and $\mathbf{0} = [0]$ is the zero vector. The initial probability vector $\widehat{\alpha} = [\alpha, \alpha_{n+1}]$ is of length $(n+1)$, with $\sum_{j=1}^{n} \alpha_j = 1 - \alpha_{n+1}$. In the present paper, we consider only the class of DPH distributions for which $\alpha_{n+1} = 0$, but the extension to the case when $\alpha_{n+1} > 0$ is straightforward. The tuple $(\alpha, \mathbf{B})$ is called the representation of the DPH distribution, and n the order.

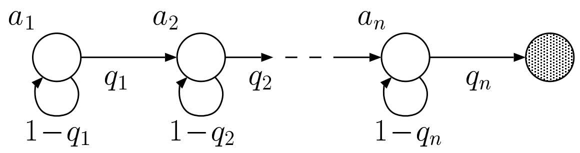
Similarly, a CPH distribution [20] is the distribution of the time to absorption in a CTMC with n transient states, and one absorbing state numbered $(n+1)$. The infinitesimal generator $\widehat{\mathbf{Q}}$ of the CTMC can be partitioned in the following way:

$$\widehat{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q} & \mathbf{q} \\ \mathbf{0} & 1 \end{bmatrix} \tag{2}$$

where, $\mathbf{Q}$ is a $(n \times n)$ matrix that describes the transient behavior of the CTMC and $\mathbf{q}$ is the column vector grouping the transition rates to the absorbing state. Let $\hat{\alpha} = [\, \alpha,\ \alpha_{n+1} \,]$ be the $(n+1)$ initial probability (row) vector with $\sum_{i=1}^{n} \alpha_i = 1 - \alpha_{n+1}$. The tuple $(\alpha, \mathbf{Q})$ is called the representation of the CPH distribution, and n the order.

It has been shown in [6] for the discrete case and in [11] for the continuous case that the representations in (1) and (2), because of their too many free parameters, do not provide a convenient form for running a fitting algorithm. Instead, resorting to acyclic phase-type distributions, the number of free parameters is reduced significantly since both in the discrete and the continuous case a canonical form can be used. The canonical form and its constraints for the discrete case [6] is depicted in Figure 1. Figure 2 gives the canonical form and associated constraints for the continuous case.

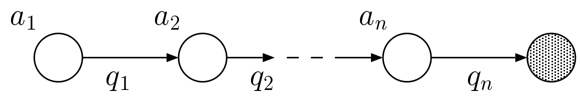
A fitting algorithm that provides acyclic CPH, acyclic DPH distributions has been provided in [3] and [6], respectively. Experiments suggests (an exhaustive comparison of fitting algorithms can be found in [17]) that, from the point of view of applications, the Acyclic phase-type class is as flexible as the whole phase-type class.



$\sum_{i=1}^{n} a_i = 1$

$0 < q_i \leq q_{i+1} \leq 1, \; 1 \leq i \leq n-1$

Figure 1: Canonical representation of acyclic DPH distributions and its constraints



$\sum_{i=1}^{n} a_i = 1$

$0 < q_i \leq q_{i+1}, \; 1 \leq i \leq n-1$

Figure 2: Canonical representation of acyclic CPH distributions and its constraints

3 Comparing properties of CPH and DPH distributions

CTMC are defined as a function of a continuous time variable t, while DTMC are defined over the set of the natural numbers. In order to relate the number of jumps in a DTMC with a time measure, a time span must be assigned to each step. Let δ be (in some arbitrary units) the scale factor, i.e. the time span assigned to each step. The value of δ establishes an equivalence between the sentence "probability at the k-th step" and "probability at time $k\,\delta$", and hence, defines the time scale on which the properties of the DTMC are measured. The consideration of the scale factor δ introduces a new parameter, and consequently a new degree of freedom, in the DPH class with respect to the CPH class. In the following, we discuss how this new degree of freedom impacts the properties of the DPH class and how it can be exploited in practice.

Let u be an "unscaled" DPH distributed random variable (r.v.) of order n with representation $(\alpha, \mathbf{B})$, defined over the set of the non-negative natural numbers. Let us consider a scale factor δ; the scaled r.v. $\tau = \delta\, u$ is defined over the discrete set of time points $(0, 1\,\delta, 2\,\delta, 3\,\delta, \ldots, k\,\delta, \ldots)$, being k a non-negative natural number. For the unscaled and the scaled DPH r.v. the following equations hold.

$$\begin{aligned} &F_u(k) = Pr\{u \leq k\} = 1 - \alpha \mathbf{B}^k\, \mathbf{e} \quad ; \quad F_\tau(\delta k) = Pr\{\tau \leq \delta k\} = 1 - \alpha \mathbf{B}^k\, \mathbf{e} \\ &m_u^i = E(u^i) \quad ; \quad m_\tau^i = E(\tau^i) = \delta^i\, E(u^i) \qquad i \geq 1, \end{aligned} \tag{3}$$

where $\mathbf{e}$ is the column vector of ones, and $E(u^i)$ is the i-th moment calculated from the factorial moments of u: $E(u(u-1)\ldots(u-i+1)) = i!\, \alpha(\mathbf{I}-\mathbf{B})^{-i}\mathbf{B}^{i-1}\mathbf{e}$. It is evident from (3) that the mean m_τ of the scaled r.v. τ is δ times the mean m_u of the unscaled r.v. u. While m_u is an invariant of the representation $(\alpha, \mathbf{B})$, δ is a free parameter; adjusting δ, the scaled r.v. can assume any mean value $m_\tau \geq 0$. On the other hand, one can easily infer from (3) that the coefficients of variation of τ and u are equal. A consequence of the above properties is that one can easily provide a scaled DPH of order ≥ 2 with arbitrary mean and arbitrary coefficient of variation with an appropriate scale factor. Or more formally: the unscaled DPH r.v. u of any order $n > 1$ can exhibit a coefficient of variation between $0 \leq cv_u^2 \leq \infty$. For $n = 1$ the coefficient of variation ranges between $0 \leq cv_u^2 \leq 1$.

As mentioned earlier, an important property of the DPH class with respect to the CPH class is the possibility of exactly representing a deterministic delay. A deterministic distribution with value a can be realized by means of a scaled DPH distribution with n phases with scale factor δ if $n = a/\delta$ is integer. In this case, the structure of the DPH distribution is such that phase i is connected with probability 1 only to phase $i+1$ $(i = 1, \ldots, n)$, and with an initial probability concentrated in state 1. If $n = a/\delta$ is not integer for the given δ, the deterministic behavior can only be approximated.

3.1 First order discrete approximation of CTMCs

Given a CTMC with infinitesimal generator $\widetilde{\mathbf{Q}}$, the transition probability matrix over an interval of length δ can be written as:

$$e^{\widetilde{\mathbf{Q}}\delta} = \sum_{i=0}^{\infty} (\widetilde{\mathbf{Q}}\delta)^i / i! = \mathbf{I} + \widetilde{\mathbf{Q}}\delta + \sigma(\delta),$$

hence the first order approximation of $e^{\widetilde{\mathbf{Q}}\delta}$ is matrix $\mathbf{\Pi}(\delta) = \mathbf{I} + \widetilde{\mathbf{Q}}\delta$. $\mathbf{\Pi}(\delta)$ is a proper stochastic matrix if $\delta < 1/q$, where $q = \max_{i,j} |\widetilde{\mathbf{Q}}_{ij}|$. $\mathbf{\Pi}(\delta)$ is the exact transition probability matrix of the CTMC assumed that at most one transition occurs in the interval of length δ.

We can approximate the behavior of the CTMC at time $(0, \delta, 2\delta, 3\delta, \ldots, k\delta, \ldots)$ using the DTMC with transition probability matrix $\mathbf{\Pi}(\delta)$. The approximate transition probability matrix at time $t = k\delta$ is:

$$\mathbf{\Pi}(\delta)^k = (\mathbf{I} + \widetilde{\mathbf{Q}}\delta)^{\frac{t}{\delta}}$$

The following theorem proves the intuitive property that the above first order approximation becomes exact as $\delta \to 0$.

Theorem 1 *As the length of the interval of the first order approximation, δ, tends to 0, such that $t = k\delta$ the approximate transition probability matrix tends to the exact one.*

Proof: The scalar version of the applied limiting behavior is well-known in the following form $\lim_{x \to 0}(1 + ax)^{\frac{1}{x}} = e^a$. Since matrices $\mathbf{I}$ and $\mathbf{Q}$ are commutative we can obtain the matrix version of the same expression as follows

$$\lim_{\delta \to 0} (\mathbf{I} + \widetilde{\mathbf{Q}}\delta)^{\frac{t}{\delta}} = \lim_{k \to \infty} \mathbf{\Pi}(t/n)^k = \lim_{k \to \infty} \left(\mathbf{I} + \frac{\widetilde{\mathbf{Q}}t}{k} \right)^k = \lim_{k \to \infty} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{\widetilde{\mathbf{Q}}t}{k} \right)^j =$$

$$= \lim_{k \to \infty} \sum_{j=0}^{k} \frac{(\widetilde{\mathbf{Q}}t)^j}{j!} \; \frac{k!}{k^j \; (k-j)!} = \sum_{j=0}^{\infty} \frac{(\widetilde{\mathbf{Q}}t)^j}{j!} = e^{\widetilde{\mathbf{Q}}t},$$

since $\lim_{k\to\infty} \frac{k!}{k^j\ (k-j)!} = \lim_{k\to\infty} \frac{k\cdot(k-1)\cdot\ldots\cdot(k-j+1)}{k^j} = 1.$ □

An obvious consequence of Theorem 1 for PH distributions is given in the following corollary.

Corollary 1 *Given a scaled DPH distribution of order n, representation* $(\alpha, \mathbf{I}+\mathbf{Q}\delta)$ *and scale factor* δ*, the limiting behavior as* $\delta \to 0$ *is the CPH distribution of order n with representation* $(\alpha, \mathbf{Q})$.

3.2 The minimum coefficient of variation

It is known that one of the main limitation in approximating a given distribution by a PH one is the attainable coefficient of variation. In order to discuss this point, we recall two theorems that state the minimum coefficient of variation for the class of CPH and DPH distributions, and we discuss some implications.

Theorem 2 (Aldous and Shepp [1]) *The minimal squared coefficient of variation,* cv^2_{min}*, of a* CPH *distributed r.v. of order n is* $cv^2_{min} = 1/n$ *and is attained by the Erlang(n) distribution independent of its mean* m_c *or of its parameter* $\lambda = n/m_c$.

The corresponding theorem for the unscaled DPH class has been proved in [22]. However, before stating the theorem, the following notation needs to be introduced. Given a real number x, $\lfloor x \rfloor$ denotes the integer part of x and $\langle x \rangle$ denotes the fractional part of x, i.e. $x = \lfloor x \rfloor + \langle x \rangle$, such that $\lfloor x \rfloor$ is an integer and $0 \leq \langle x \rangle < 1$.

Theorem 3 *The* cv^2_{min} *of an unscaled DPH r.v. of order n and mean* m_u *is:*

$$\begin{cases} \dfrac{\langle m_u\rangle(1-\langle m_u\rangle)}{m_u^2} & \text{if } m_u \leq n\ , \\[2ex] \dfrac{1}{n} - \dfrac{1}{m_u} & \text{if } m_u > n\ , \end{cases} \tag{4}$$

The unscaled DPH r.v. which exhibits this cv^2_{min} *is referred to as* MDPH*, and has the following canonical structure:*

- *if* $m_u \leq n$: *each state is connected to the next with probability 1 and the nonzero initial probabilities are* $\alpha_{n-\lfloor m_u\rfloor} = \langle m_u\rangle$ *and* $\alpha_{n-\lfloor m_u\rfloor+1} = 1-\langle m_u\rangle$ *(Figure 3);*
- *if* $m_u > n$: *each state is connected to the next with probability* n/m_u *and the only nonzero initial probability is* $\alpha_1 = 1$ *(Figure 4).*

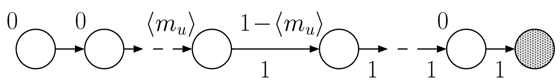


Figure 3: MDPH with $m_u \leq n$

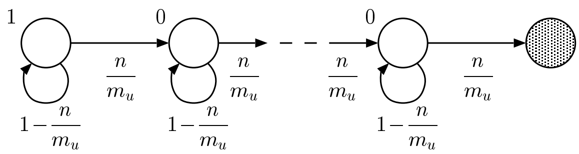


Figure 4: MDPH with $m_u > n$

The MDPH structure is uniquely specified given the order n and the mean m_u if $m_u > n$. If $m_u < n$ the MDPH structure is the mixture of two deterministic DPH distributions, but there are several DPH structures that exhibit this behavior. The canonical representation of the MDPH is presented in the theorem (Fig. 3). If $m_u < n$ the MDPH is the mixture of two deterministic DPH distributions with length $\lfloor m_u \rfloor + 1$ and initial probability $\langle m_u \rangle$ and with length $\lfloor m_u \rfloor$ and initial probability $1 - \langle m_u \rangle$. This structure derives from the following identity: if x is real, $x = \langle x \rangle (\lfloor x \rfloor + 1) + (1 - \langle x \rangle) \lfloor x \rfloor$. Hence, for $m_u \leq n$, the corresponding MDPH structure has an effective order $\lfloor m_u \rfloor + 1$, being the initial probabilities from state 1 to $n - \lfloor m_u \rfloor$ equal to 0. Hence, in contrast with the continuous case, increasing the order beyond $n > \lfloor m_u \rfloor + 1$ does not affect cv^2_{min}. Furthermore, if m_u is any integer value less or equal to n, $\langle m_u \rangle = 0$ and $cv^2_{min} = 0$. The case $m_u \geq n$ is closer to the continuous case, since the cv^2_{min} is attained by the discrete Erlang(n) distribution.

An important consequence of Theorem 3 is that the cv^2_{min} of DPH distributions depends both on the order n and on the mean m_u. Instead, cv^2_{min} of CPH distributions depends only on the order n. This point deserves further discussion and we consider two cases:

<u>Case 1</u> - cv^2_{min} as a function of the order n for fixed m_u;

<u>Case 2</u> - cv^2_{min} as a function of the mean m_u for fixed n.

<u>Case 1</u>: fixed m_u - As the order n increases beyond $n > m_u$ the cv^2_{min} of the DPH class remains unchanged, while the cv^2_{min} of the CPH class decreases like $1/n$. Hence, given m_u a value $n = n_C$ can always be found, such that the cv^2_{min} of the CPH class of order n_C is less or equal than the cv^2_{min} of the DPH class of the same order. Recalling (4), the value of n_C is the smallest positive integer which satisfies:

$$\frac{1}{n_C} < \frac{\langle m_u \rangle (1 - \langle m_u \rangle)}{m_u^2} \, . \tag{5}$$

It is clear from (5) that if m_u is integer,then $\langle m_u \rangle = 0$ and $n_C \to \infty$. Using the relation $m_u = \lfloor m_u \rfloor + \langle m_u \rangle$, in the denominator of (5), we can find the value of $\langle m_u \rangle$ that minimizes (5), for any positive integer $\lfloor m_u \rfloor$, and the corresponding minimal value of n_C. Equating to 0 the derivative of n_C with respect to $\langle m_u \rangle$, with $\lfloor m_u \rfloor = const$, we get:

$$\frac{\partial \, n_C}{\partial \, \langle m_u \rangle} = 0 \quad \text{iff} \quad \langle m_u \rangle = \frac{\lfloor m_u \rfloor}{1 + 2 \lfloor m_u \rfloor}$$

From which the minimal value of n_C, corresponding to any mean with integer part equal to $\lfloor m_u \rfloor$, is given by:

$$n_{Cmin} = 4 \lfloor m_u \rfloor (1 + \lfloor m_u \rfloor) \tag{6}$$

From (6) we derive Table 1 which gives us the minimal order n_{Cmin} as a function of the integer part of the mean $\lfloor m_u \rfloor$ for which the CPH class provides a cv^2_{min} less than the cv^2_{min} of the DPH class of the same order. Since for an unscaled DPH distribution the lowest significant value of $\lfloor m_u \rfloor$ is $\lfloor m_u \rfloor = 1$, we obtain from Table 1 that for an order $n < 8$ no member of the CPH class reaches a cv^2_{min} less than the cv^2_{min} of the DPH class.

Figure 5 shows the cv^2_{min} as a function of the order n for both the unscaled DPH class and the CPH class in the case when the mean is $m_u = 4.5$. (Note that in Figure 5, $m_u < n$

$\lfloor m_u \rfloor$	n_{Cmin}
1	8
2	24
3	48
4	80
5	120

Table 1: Values of n_{Cmin} as a function of the integer part of the mean $\lfloor m_u \rfloor$

when $n \geq 5$.) According to Theorem 3 the cv^2_{min} for the DPH class remains unchanged ($cv^2_{min} = 1/81$) for $n \geq 5$, while the cv^2_{min} for the CPH class ($cv^2_{min} = 1/n$) decreases to 0 as $n \to \infty$. Application of (6) tells us that if $\lfloor m_u \rfloor = 4$ (i.e., the mean is any value $4 \leq m_u < 5$), the minimal number of phases for which cv^2_{min} of the CPH class is lower than that of the DPH class is 82.

Case 2: fixed n - We already know from Table 1 that if $n < n_{Cmin}$, cv^2_{min} is lower for the DPH class than for the CPH class. However, if m_u increases with fixed n, we arrive in a situation in which $m_u > n$, and applying the second part of (4) we see that $cv^2_{min} \to 1/n$ as $m_u \to \infty$. Hence, as m_u increases, the behavior of the DPH class tends to be similar to the one of the CPH class.

Figure 6 shows cv^2_{min} as a function of the mean for the DPH class of order $n = 5$. Note that for $m_u \leq n (= 5)$, cv^2_{min} equals zero for any m_u integer, and cv^2_{min} tends to the value of the CPH class $(1/n)$ as $m_u \to \infty$.

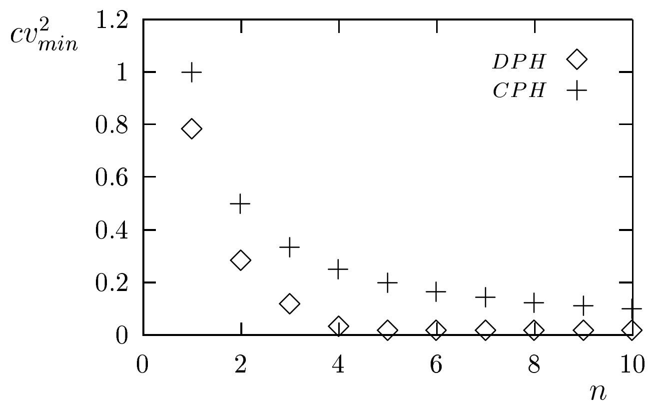


Figure 5: cv^2_{min} for $m_u = 4.5$

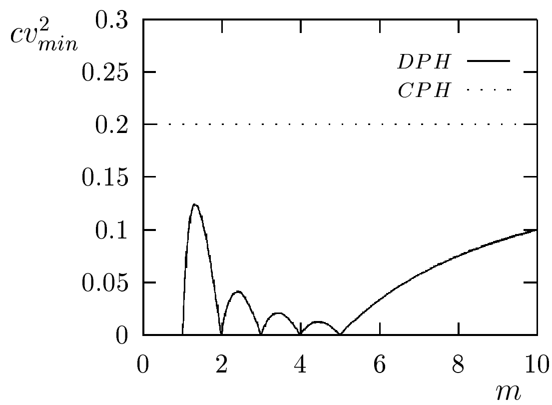


Figure 6: cv^2_{min} for $n = 5$

3.3 The minimum coefficient of variation of scaled DPH distributions

For scaled DPH distribution Theorem 3 can be restated as follows.

Theorem 4 *The cv^2_{min} of a scaled DPH r.v. of order n with scale factor δ and mean $m_\tau = \delta\, m_u$ is:*

$$
\begin{cases}
\dfrac{\left\langle \frac{m_\tau}{\delta} \right\rangle \left(1 - \left\langle \frac{m_\tau}{\delta} \right\rangle\right)}{\left(\frac{m_\tau}{\delta}\right)^2} & \text{if } m_\tau \le n\,\delta\ , \\[2ex]
\dfrac{1}{n} - \dfrac{\delta}{m_\tau} & \text{if } m_\tau > n\,\delta\ ,
\end{cases}
\tag{7}
$$

The scaled DPH r.v. which exhibits the cv^2_{min} has the same MDPH structure of Figures (3) and (4), as in the unscaled case (see Theorem 3).

Corollary 2 *For finite mean m_τ, as $\delta \to 0$ only the second part of (7) remains effective, and $cv^2_{min} \to 1/n$ as $\delta \to 0$.*

Corollary 2 proves that the cv^2_{min} of the DPH class converges to the cv^2_{min} of the CPH class of the same order as $\delta \leftarrow 0$. The following corollary presents a much stronger convergence result for the case of approximating distributions with low coefficient of variation. It is about the convergence of the distributions.

Corollary 3 *The* best fitting *scaled DPH approximation of distributions with low coefficient of variation converges, in distribution, to the* best fitting *CPH approximation of the same distribution as δ tends to 0, where the* best fitting *PH approximation is defined as the one which exhibits the same mean and provides the closest approximation for the 2nd moment.*

Proof: Both the CPH and the DPH classes have limits in approximating distributions with low coefficient of variation. The best approximation of a distribution with coefficient of variation less than these limits is the Erlang(n) distribution in both the discrete and the continuous case (Theorem 2 and 3).

The representation (α, $\mathbf{Q}$) of the continuous Erlang(n) with mean m_τ and the representation (α, $\mathbf{B}$) of the discrete Erlang(n) with mean m_τ, scale factor δ are:

$$\alpha = \{1, 0, \ldots, 0\} \qquad\qquad \alpha = \{1, 0, \ldots, 0\}$$

$$
\mathbf{Q} = \begin{bmatrix}
-\frac{n}{m_\tau} & \frac{n}{m_\tau} & 0 & \ldots & 0 \\
0 & -\frac{n}{m_\tau} & \frac{n}{m_\tau} & \ldots & \\
 & & \ddots & & \\
0 & & & \ldots & -\frac{n}{m_\tau}
\end{bmatrix}
\qquad
\mathbf{B} = \begin{bmatrix}
1-\frac{n\delta}{m_\tau} & \frac{n\delta}{m_\tau} & 0 & \ldots & 0 \\
0 & 1-\frac{n\delta}{m_\tau} & \frac{n\delta}{m_\tau} & \ldots & \\
 & & \ddots & & \\
0 & & & \ldots & 1-\frac{n\delta}{m_\tau}
\end{bmatrix}
$$

Note that $\mathbf{B} = \mathbf{I} - \mathbf{Q}\delta$ and Corollary 3 follows from Corollary 1. □

In this particular case, when the structure of the best fitting scaled DPH and CPH distributions are known, we can show that the distribution of the best fitting scaled DPH distribution converges to the distribution of the best fitting CPH distribution when $\delta \to 0$. Unfortunately, this same convergence property cannot be proved in general, since the structural properties of the best fitting PH distributions are not known and they depend on the chosen (arbitrary) optimization criterion. Instead, in Section 4 we provide an extensive experimental study on the behavior of the best fitting scaled DPH and CPH distributions as a function of the scale factor δ .

3.4 DPH distributions with finite support

Another peculiar characteristic of the DPH class is to contain distributions with finite support. A DPH distribution has finite support if its structure does not contain cycles and self-loops (any cycle or self loop implies an infinite support).

Let $[a, b]$ be the finite support of a given distribution, with $a, b \geq 0$ and $a \leq b$ (when $a = b$ the finite support distribution reduces to a deterministic distribution with mass 1 at $a = b$). If a/δ and b/δ are both integers, it is possible to construct a scaled DPH of order b/δ for which the probability mass function has non-zero elements only for the values $a, a + \delta, a + 2\delta, ..., b$. As an example, the discrete uniform distribution between $a = 2$ and $b = 6$ is reported in Figure 7, for scale factor $\delta = 1$.

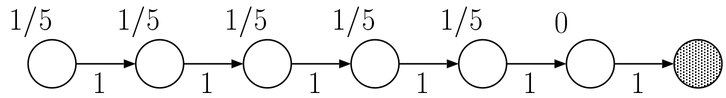


Figure 7: DPH representation of the discrete uniform distribution $[a = 2, b = 6]$

4 The optimal δ in PH fitting

The scale factor δ provides a new degree of freedom in fitting, and, furthermore, since the limit of a DPH distribution for $\delta \to 0$ is a CPH distribution, the optimization of the scale factor in a fitting problem provides a quantitative way to decide whether a continuous or a discrete approximation performs better in the given problem. Hence, assuming δ as a decision variable, we can consider the CPH and the DPH class as a unique model set in which the choice among DPH or CPH classes is given by the optimal value of δ.

Let X be the continuous r.v. to be fit by a PH distribution, and let $F_X(x)$ be its cdf, $E(X^i)$ the i-th moment and $cv^2(X)$ the squared coefficient of variation. In order to define a fitting procedure, a distance measure between X and the approximating PH distribution needs to be defined. Then, the fitting algorithm provides the PH distribution which minimizes the chosen distance measure. In order to compare, in a unified framework, the goodness of the approximation reached by CPH and DPH distributions, we need to chose a distance measure that is meaningful and applicable both in the continuous as well as in the discrete setting. The selected distance measure is the squared area difference between the original cdf $F(\cdot)$ and the approximating cdf $\hat{F}(\cdot)$:

$$\mathcal{D} = \int_x (F(x) - \hat{F}(x))^2 dx \tag{8}$$

The distance measure $\mathcal{D}$ is easily applicable for any combination of discrete and continuous distributions. All the numerical experiments reported in the sequel are based on the minimization of the area difference given in (8).

4.1 Fitting distributions with low cv^2

The following considerations provide practical upper and lower bounds to guide in the choice of a suitable scale factor δ, and are mainly based on the dependence of the minimal coefficient of variation of a scaled DPH distribution on the order n and on the mean m_τ.

Since we only consider DPH distributions with no mass at zero, the mean of any unscaled DPH distribution is greater than 1. This means that δ should be less than $E(X)$. However, a more convenient upper bound that exploits the flexibility associated with the n phases, is given by:

$$\delta \leq \frac{E(X)}{n-1} \ . \tag{9}$$

If the squared coefficient of variation of the distribution to be approximated is less than $1/n$, δ should satisfy the following relation (see Theorem 3):

$$\delta > \left(\frac{1}{n} - cv^2(X)\right) E(X) \tag{10}$$

Let X be a Lognormal r.v. with parameters $(1, 0.2)$, whose mean is $E(X) = 1$ and $cv^2(X) = 0.0408$ (this distribution is the distribution L3 taken from the benchmark examined in [9, 6], hence we refer to it as L3). Table 2 reports the lower and upper bounds of δ, with $n = 2, 4, 8, 12$, computed from (10) and (9).

The cdf and pdf of the best fitting CPH and DPH distributions of order $n = 10$, with different scale factors δ, are presented in Figure 8. When considering the approximate DPH distribution, the $f(x)$ values are calculated at the discrete points $(\delta, 2\delta, 3\delta, \ldots, k\delta, \ldots)$ to which the following mass is assigned:

$$f(k\delta) = 1/\delta(F(k\delta) - F((k-1)\delta)) \tag{11}$$

For the ease of visual interpretation the points are connected with a line.

When δ is less than its lower bound the required cv^2 cannot be attained; when δ becomes too large the wide separation of the discrete steps increases the approximation error; when δ is in the proper range (e.g. $n = 10$; $\delta = 0.06$) a reasonably good fit is achieved. This example also suggests that an optimal value of δ exists that minimizes the chosen distance measure $\mathcal{D}$ in (8).

In order to display the goodness of fit for the L3 distribution, Figure 9 shows the distance measure $\mathcal{D}$ as a function of δ for various values of the order n. A minimum value of δ is attained in the range where the parameters fit the bounds of Table 2. Notice also that, as δ increases, the advantage of having more phases disappears, according to Theorem 3. The circles in the left part of this figure (as well as in all the successive figures) indicate the corresponding distance measure $\mathcal{D}$ obtained from CPH fitting. The figure (and the subsequent ones as well) suggests that the distance measure obtained from DPH fitting converges to the distance measure obtained by the CPH approximation as δ tends to 0.

n	lower bound of δ equation (10)	upper bound of δ equation (9)
4	0.2092	0.333
8	0.0792	0.1428
12	0.0425	0.0909
16	0.0217	0.0666

Table 2: Upper and lower bound of δ for fitting distribution L3

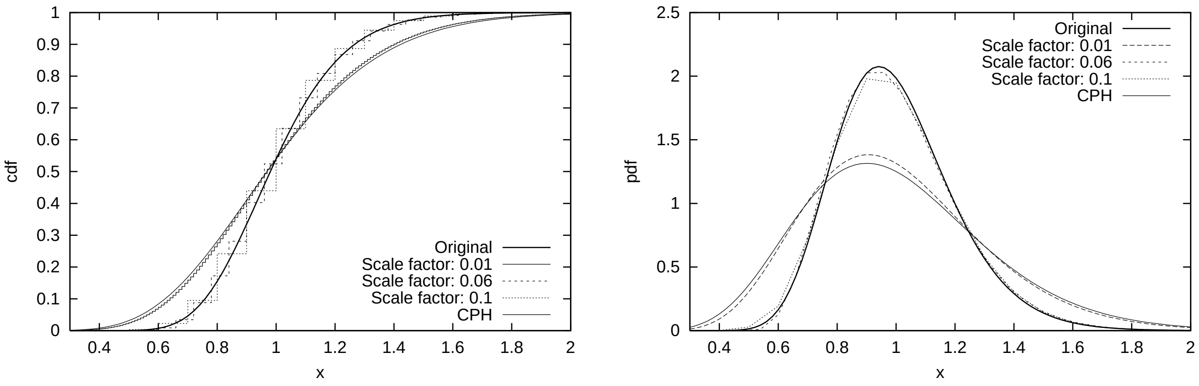


Figure 8: Approximating the L3 distribution with 10-phase PH approximations

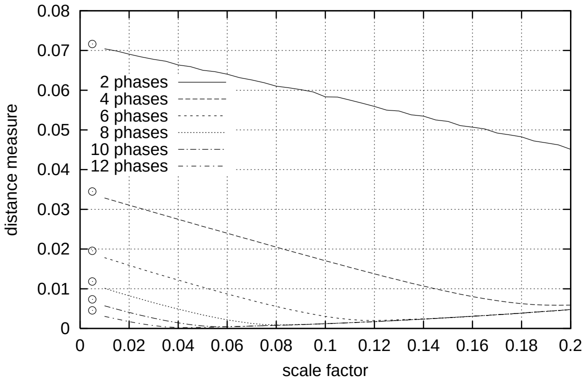


Figure 9: Distance measure as the function of the scale factor δ for low cv^2 (L3)

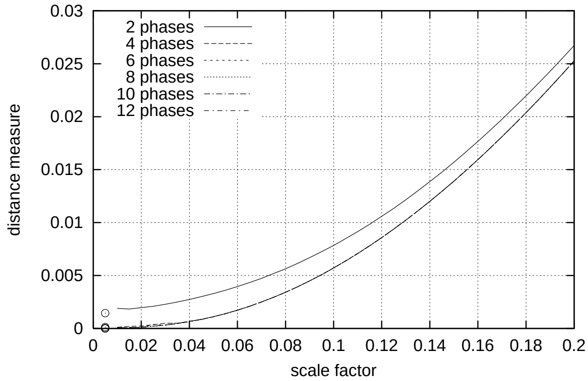


Figure 10: Distance measure as the function of the scale factor δ for high cv^2 (L1)

4.2 Fitting distributions with high cv^2

We have seen in the previous subsections that it is beneficial to approximate distributions with a low coefficient of variation by means of a DPH distributions. In this subsection, we investigate the optimal value of δ when fitting distributions with a high coefficient of variation.

Let X be a Lognormal r.v. with parameters $(1, 1.8)$ (this is the distribution L1 taken from the benchmark in [9, 6]). For X we have $E(X) = 1$ and $cv^2(X) = 24.534$. Figure 10 shows the measure of the goodness of fit as a function of δ for various orders n (the cases when the number of phases are greater than 2 result in practically the same goodness of fit). The distance measures $\mathcal{D}$ decreases as $\delta \to 0$ indicating that the optimal fitting is achieved by applying CPH distribution. This example suggests that, for distributions with infinite support and $cv^2(X) > 1/n$, the optimal value of δ tends to 0, implying that the best fit is obtained by a CPH. However, this conclusion might not be true for distributions with finite support, as it is explored in the next subsection.

4.3 Fitting distributions with finite support

In this case, two features must be considered, namely the cv^2 and the maximum value of the finite support. It should be stressed that the chosen distance measure $\mathcal{D}$ in (8) can be

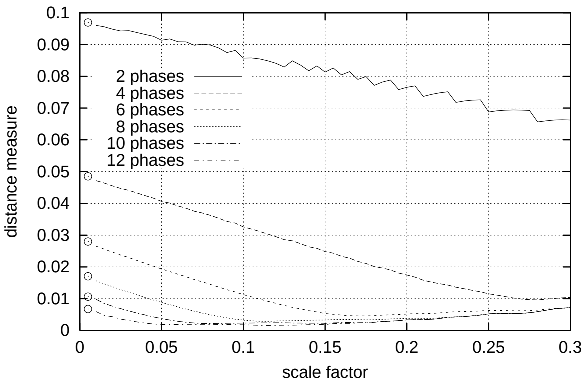


Figure 11: Distance measure as the function of the scale factor δ for Uniform(1,2) (U2)

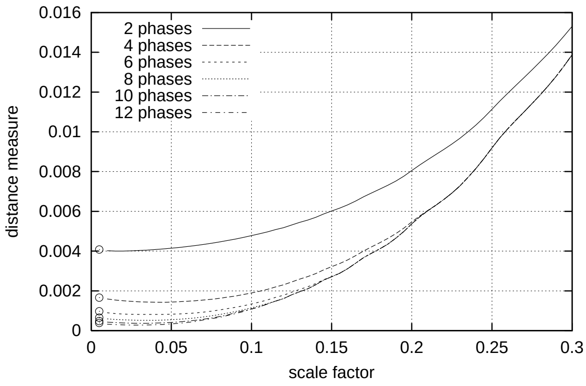


Figure 12: Distance measure as the function of the scale factor δ for Uniform(0,1) (U1)

considered as not completely appropriate in the case of finite support, since it does not force the approximating PH to have its mass confined in the finite support and 0 outside.

Let X be a uniform r.v. over the interval [1, 2], with $E(X) = 1.5$ and $cv^2(X) = 0.0370$ (this is the distribution U2 taken from the benchmark in [9, 6]). Figure 11 shows the distance measure as a function of δ for various orders n. It is evident that, for each n, a minimal value of δ is obtained, that provides the best approximation according to the chosen distance measure.

As a second example, let X be a uniform r.v. over the interval [0, 1], with $E(X) = 0.5$ and $cv^2(X) = 0.333$ (this is the distribution U1 taken from the benchmark in [9, 6]). Figure 12 shows the distance measure as a function of δ for various orders n. Since, in this example $cv^2(X) = 0.333$, an order $n = 3$ is large enough for a CPH to attain the coefficient of variation of the distribution. Nevertheless, the optimal δ in Figure (12), which minimizes the distance measure $\mathcal{D}$ for high order PH ($n > 2$), ranges between $\delta = 0.02$ and $\delta = 0.05$, thus leading to the conclusion that a DPH provides a better fit. This example evidences that the coefficient of variation is not the only factor which influences the optimal δ value. The shape of the distribution plays an essential role as well. Our experiments show that a discontinuity in the pdf (or in the cdf) is hard to approximate with CPH, hence in the majority of these cases DPH provides a better approximation.

Figure 13 shows the cdf and the pdf of the U1 distribution, compared with the best fit PH approximations of order $n = 10$, and various scale factors δ. In the case of DPH approximation, the $f(x)$ values are calculated as in (11). With respect to the chosen distance measure, the best approximation is obtained for $\delta = 0.03$, which corresponds to a DPH distribution with infinite support . When $\delta = 0.1$ the approximate distribution has a finite support. Hence, the value $\delta = 0.1$ (for $n = 10$) provides a DPH able to represent the logical property that the random variable is less than 1. Another fitting criterion may, of course, stress this property.

5 Approximating non-Markovian models

Section 4 has explored the problem of how to find the best fit among either a DPH or a CPH distribution by tuning the scale factor δ. When dealing with a stochastic model of

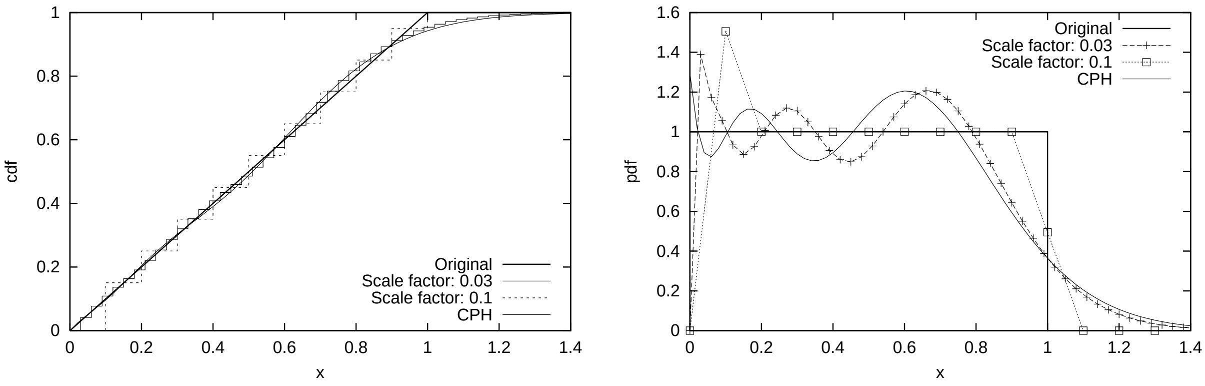


Figure 13: Approximating the Uniform $(0,1)$ distribution (U1)

a system that incorporates non exponential distributions, a well know solution technique consists in a markovianization of the underlying non-Markovian process by substituting the non exponential distribution with a best fit PH distribution, and then expanding the state space. A natural question arises also in this case, on how to decide among a discrete (using DPH) or a continuous (using CPH) approximation, in order to minimize the error in the performance measures we are interested in for the overall model.

One possible way to handle this problem could consist in finding the best PH fits for any single distribution and to plug them in the model. In the present paper, we only consider the case where the PH distributions are either all discrete (and with the same scale factor δ) or they are all continuous. Various embedding techniques have been explored in the literature for mixing DPH (with different scale factors) and CPH ([13, 16]), but these techniques are out of the scope of the paper.

In order to quantitatively evaluate the influence of the scale factor on some performance measures defined at the system level, we have considered a preemptive M/G/1/2/2 queue with two classes of customers. We have chosen this example because accurate analytical solutions are available both in transient condition and in steady-state using the methods presented in e.g.[13]. The general distribution G is taken from the set of distributions (L1, L3, U1, U2) already considered in the previous section.

Customers arrive at the queue with rate $\lambda = 0.5$ in both classes. The service time of a higher priority job is exponentially distributed with parameter $\mu = 1$. The service time distribution of the lower priority job is either L1, L3, U1 or U2. Arrival of a higher priority job preempts the lower priority one. The policy associated to the preemption of the lower priority job is preemptive repeat different (prd), i.e. after the departure of the higher priority customer the service of the low priority customer starts from the beginning with a new service time sample.

The system has 4 states (Figure 14): in state *s1* the server is empty, in state *s2* a higher priority customer is under service with no lower priority customer in the system, in state *s3* a higher priority customer is under service with a lower priority customer waiting, in state *s4* a lower priority job is under service (in this case there cannot be a higher priority job).

Let p_i $(i = 1, \ldots, 4)$ denote the steady state probability of the M/G/1/2/2 queue obtained from an exact analytical solution.

In order to evaluate the correctness of the PH approximation we have solved the model by substituting the original general distribution (either L1, L3, U1 or U2) with

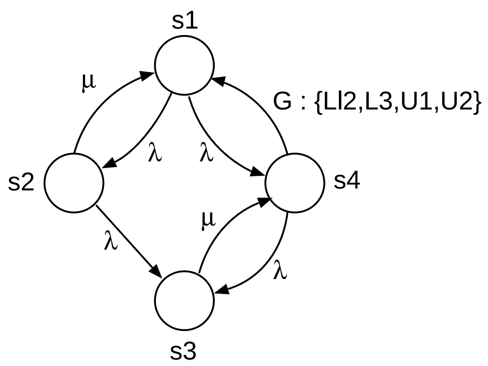


Figure 14: The state space of the considered M/G/1/2/2 queue

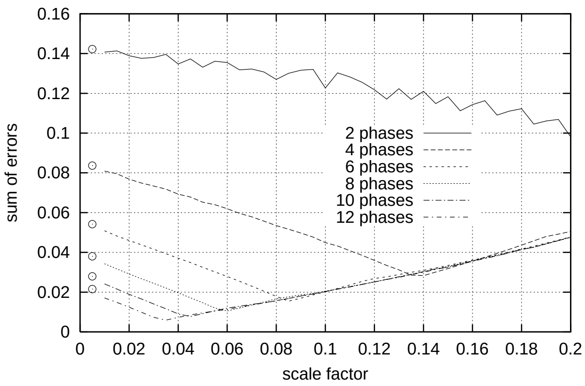


Figure 15: ϵ_{SUM} with scale factor δ and distribution L3

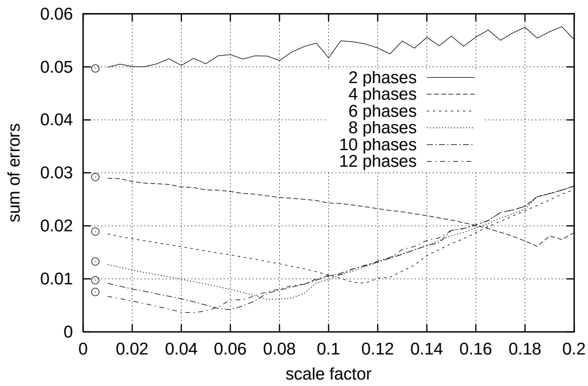


Figure 16: ϵ_{MAX} with scale factor δ and distribution L3

approximating DPH or CPH distributions. Let $\hat{p}_i$ $(i = 1, \ldots, 4)$ denote the steady state probability of the M/PH/1/2/2 queue with the PH approximation.

The overall approximation error is measured in terms of the difference between the exact steady state probabilities p_i and the approximate steady state probabilities $\hat{p}_i$. Two error measures are defined:

$$\epsilon_{SUM} = \sum_i |p_i - \hat{p}_i| \qquad \text{and} \qquad \epsilon_{MAX} = \max_i |p_i - \hat{p}_i|.$$

The evaluated numerical values for ϵ_{SUM} and ϵ_{MAX} are reported in Figures 15 and 16 for the distribution L3, in Figures 17 and 18 for the distribution L1, in Figures 19 and 20 for the distribution U1, and, finally, in Figures 21 and 22 for the distribution U2. The figures, that refer to the error measure in a performance index of a global stochastic model, show a behavior similar to the one obtained for a single distribution fitting. Depending on the coefficient of variation and on the shape of the considered non-exponential distributions an optimal value of δ is found which minimizes the approximation error. In this example, the optimal value of δ is close to the one obtained for the single distribution fitting.

Based on our experiments, we guess that the observed property is rather general. If the stochastic model under study contains a single non-exponential distribution, then the approximation error in the evaluation of the performance indices of the global model can be minimized by resorting to a PH type approximation (and subsequent DTMC or CTMC expansion) with the optimal δ of the single distribution. The same should be true if the

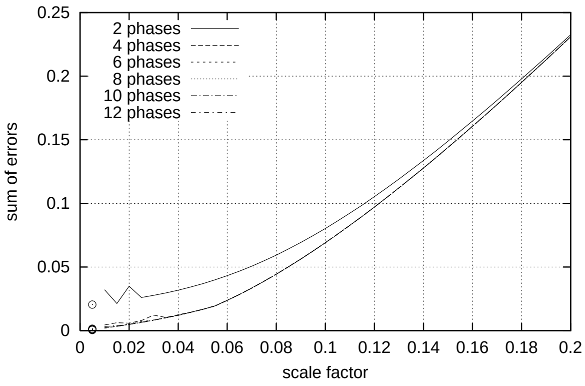


Figure 17: ϵ_{SUM} with scale factor δ and distribution L1

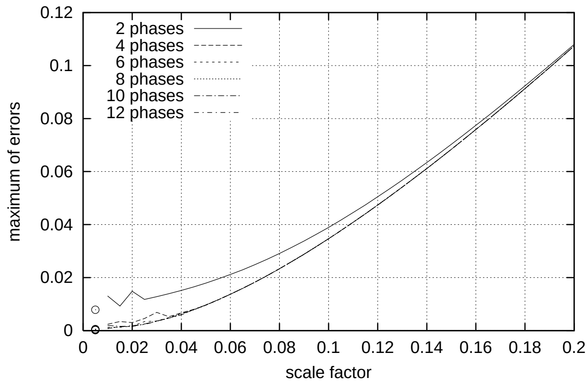


Figure 18: ϵ_{MAX} with scale factor δ and distribution L1

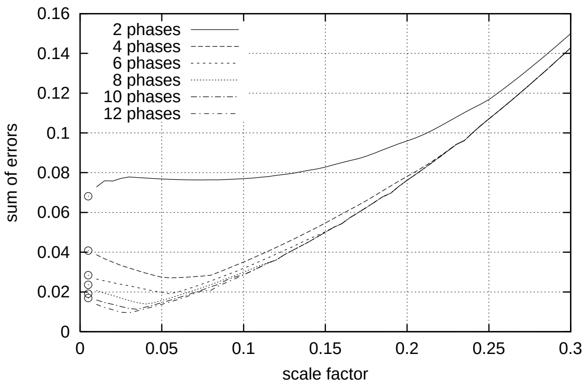


Figure 19: ϵ_{SUM} with scale factor δ and distribution U1

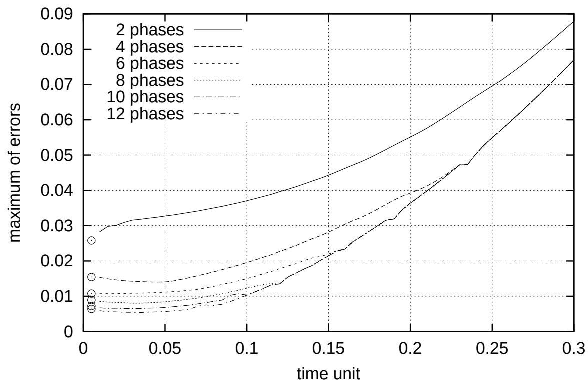


Figure 20: ϵ_{MAX} with scale factor δ and distribution U1

stochastic model under study contains more than one general distribution, whose best PH fit provides the same optimal δ.

In order to investigate the approximation error in the transient behavior, we have considered distribution U2 for the service time and we have computed the transient probability of state s_1 with two different initial conditions. Figure 23 depicts the transient probability of state s_1 with initial state s_1. Figure 24 depicts the transient probability of the same state, s_1, when the service of a lower priority job starts at time 0 (the initial state is s_4). All approximations are with DPH distributions of order $n = 10$. Only the DPH approximations are depicted because the CPH approximation is very similar to the DPH one with scale factor $\delta = 0.03$. In the first case, (Figure 23), the scale factor $\delta = 0.03$, which was the optimal one from the point of view of fitting the single distribution in isolation, provides the most accurate results for the transient analysis as well. Instead, in the second case, the approximation with a scale factor $\delta = 0.2$ captures better the sharp change in the transient probability. Moreover, this value of δ is the only one among the values reported in the figure that results in 0 probability for time points smaller than 1. In other words, the second example depicts the advantage given by DPH distributions to model durations with finite support. This example suggests also that DPH approximation can be of importance when preserving reachability properties is crucial (like in modeling time-critical systems) and, hence, DPH approximation can be seen as a bridge between the world of stochastic modeling and the world of functional analysis and model checking

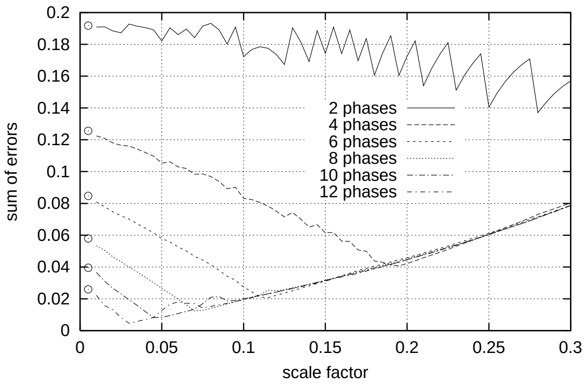


Figure 21: ϵ_{SUM} with scale factor δ and distribution U2

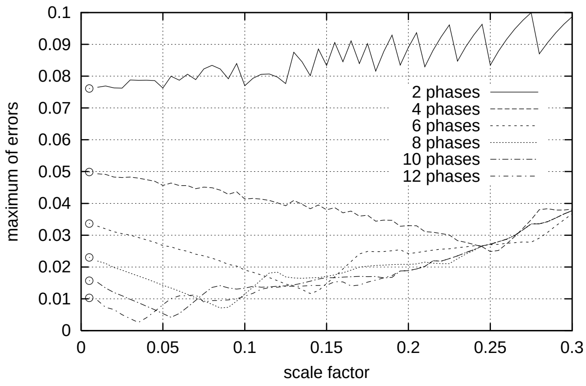


Figure 22: ϵ_{MAX} with scale factor δ and distribution U2

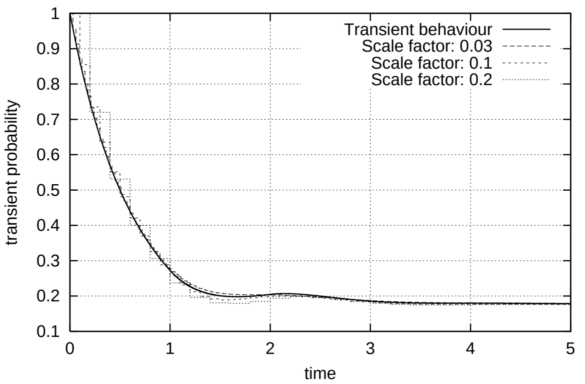


Figure 23: Approximating transient probabilities

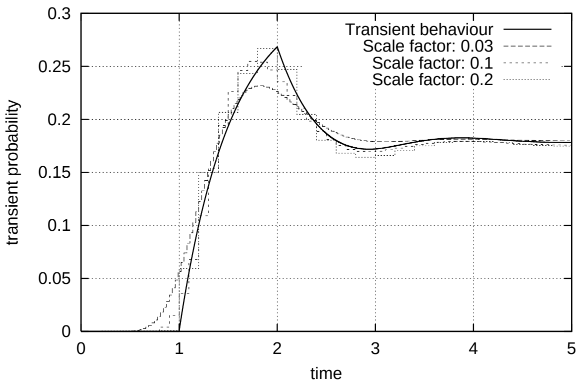


Figure 24: Approximating transient probabilities

[5].

6 Concluding remarks

The main result of this paper has been to show that the DPH and CPH classes of distributions of the same order can be considered a single model set as a function of a scale factor δ. The optimal value of δ, δ_{opt}, determines the best distribution in a fitting experiment. When $\delta_{opt} = 0$ the best choice is a CPH distribution, while when $\delta_{opt} > 0$ the best choice is a DPH distribution. This paper has also shown that the transition from DPH class to CPH class is continuous with respect to several properties, like the distance (denoted by $\mathcal{D}$ in 8) between the original and the approximate distributions. The paper presents limit theorems for special cases; however, extensive numerical experiments show that the limiting behavior is far more general than the special cases considered in the theorems.

The numerical examples have also evidenced that for very small values of δ, the diagonal elements of the transition probability matrix become very close to 1, rendering numerically unstable the DPH fitting procedure.

A deep analytical and numerical sensitivity analysis is required to draw more general conclusions for the model level "optimal δ value" and its dependence on the considered performance measure than the ones presented in this work. It is definitely a field of further research.

Finally, we summarize the advantages and the disadvantages of applying approximate DPH models (even with optimal δ value) with respect to using CPH approximations.
Advantages of using DPH: An obvious advantage of the application of DPH distributions is that one can have a closer approximate of distributions with low coefficient of variation. An other important quantitative property of the DPH class is that it can capture distributions with finite support and deterministic values. This property allows to capture the periodic behavior of a complex stochastic model, while any CPH based approximation of the same model tends to a steady state.

Numerical experiments have also shown that DPH can better approximate distributions with some abrupt or sharp changes in the CDF or in the PDF.
Disadvantages of using DPH: There is a definite disadvantage of discrete time approximation of continuous time models. In the case of CPH approximation, coincident events do not have to be considered (they have zero probability of occurrence). Instead, when applying DPH approximation coincident events have to be handled, and their consideration may burden significantly the complexity of the analysis.

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