Non-Markovian queues, matrix analytic methods

exam tests and their solutions

Telek Miklós

Budapesti Műszaki Egyetem

Exam 1

1/ A MAP is characterized by D_0, D_1*. Let* $T_0 = 0, T_1, \ldots$ *denote the arrival instances and* $X_0, X_1, \ldots$ *the phase right after arrivals. Define the following probabilities:*

- $Pr(X_1 = j | X_0 = i)$,
- $Pr(T_1 < t | X_0 = i)$,
- $Pr(X_1 = j, T_1 < t | X_0 = i)$,
- $Pr(X_1 = j | X_0 = i)$ *based on* $p_k(t)$,
- $Pr(T_1 < t | X_0 = i)$ *based on* $p_k(t)$,

where $p_k(t) = Pr(X(t) = k, T_1 > t | X_0 = i)$*, and* $X(t)$ *denotes the phase process.*

<u>Solution:</u>

- $Pr(X_1 = j | X_0 = i) = [(-D_0)^{-1} D_1]_{ij}$
- $Pr(T_1 < t | X_0 = i) = [e^{D_0 t} \mathbb{I}]_i$
- the arrival rate to phase j at time t is $[e^{D_0 t} D_1]_{ij}$, from which

$$Pr(X_1 = j, T_1 < t | X_0 = i) = \int_{\tau=0}^{t} [e^{D_0 \tau} D_1]_{ij} d\tau$$

- $Pr(X_1 = j | X_0 = i) = p_j(0)$
- $Pr(T_1 < t | X_0 = i) = p_j(0) - p_j(t)$

2/ Transient description of Semi-Markov process based on the analysis of the first state transition.

<u>Solution:</u> The required measure is the transition probability $\pi_{ij}(t) = Pr(X(t) = j \mid X(0) = i)$. Assuming that the first state transition takes place at time h we introduce

$$\pi_{ij}(t | T_1 = h) = Pr(X(t) = j \mid X(0) = i, T_1 = h).$$

For that we have

$$\pi_{ij}(t|T_1=h) = \begin{cases} \delta_{ij} & h \geq t \\ \sum_{k\in S} Pr(X(T_1)=k \mid X(0)=i, T_1=h)\ \pi_{ij}(t-h) & h < t, \end{cases}$$

where $Pr(X(T_1)=j \mid X(0)=i, T_1=h)$ is the probability that the process moves from state i to state j if $T_1=h$.

$$Pr(X(T_1)=j \mid X(0)=i, T_1=h) = \frac{dQ_{ij}(h)}{dQ_i(h)}.$$

Based on the law of total probability $\pi_{ij}(t)$ is computed as

$$\pi_{ij}(t) = \int_{h=0}^{\infty} \pi_{ij}(t|T_1=h)\ dQ_i(t),$$

where $Q_i(t) = \sum_j Q_{ij}(t)$ is the sojourn time distribution of state i. Evaluating the integral we get

$$\pi_{ij}(t) = \delta_{ij}\ (1-Q_i(t)) + \int_{h=0}^{t} \sum_{k\in S} \pi_{kj}(t-h)\ dQ_{ik}(h).$$

3/ Transient analysis of M/M/1 queue: Let $\gamma_n = min(t : X(t)=n)$, the first time to visit level n. Compute the following probability:

- $\check{P}_n(t) = Pr(X(t)=n, \gamma_{n-1} > t | X(0)=n)$
  using that the distribution of the time to reach level $n-1$ starting from level n is $G_n(t)$ and its density if $g_n(t)$.

- *Compute its Laplace transform.*

Solution:

- The time spent at level n is exponentially distributed with parameter $\lambda+\mu$. Up to the first transition the process stays at level n. The first transition goes to level $n+1$ with probability $\frac{\lambda}{\lambda+\mu}$ and to level $n-1$ with probability $\frac{\mu}{\lambda+\mu}$. As a result

$$\check{P}_n(t) = e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu} \int_{\tau=0}^{t} (\lambda+\mu) e^{-(\lambda+\mu)t} \int_{\gamma=0}^{t-\tau} g_n(\gamma)\ \check{P}_n(t-\tau-\gamma) d\gamma d\tau.$$

- We can transform the expression element wise:

$$\check{P}_n^*(s) = \frac{1}{\lambda+\mu+s} + \frac{\lambda}{\lambda+\mu}\,\frac{\lambda+\mu}{\lambda+\mu+s}\, g_n^*(s)\ \check{P}_n^*(s)$$

4/ The regular matrix blocks of a QBD are $B = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$, $L = \begin{bmatrix} \bullet & \mu_3 \\ 0 & \bullet \end{bmatrix}$, $F = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$. *What is the condition of stability for this QBD?*

Solution:

The phase process with generator $Q_J = B + L + F$ is not irreducible. Phase 1 is transient and phase 2 is absorbing. As a result the stationary phase distribution is $(0, 1)$ and the condition of stability is $\mu_2 > \lambda$.

Exam 2

1/ A MAP is defined by D_0, D_1. Compute

- *its stationary arrival rate,*
- *the stationary correlation of consecutive inter-arrival periods, ha ismert az érkezési id?köz, és az érkezés utáni fázis együttes eloszlása ($K_{ij}(t) = Pr(X_1 = j, T_1 < t | X_0 = i)$)!*

Solution:

- the stationary arrival rate is $\lambda = \alpha D_1 \mathbb{1}$, where α is the stationary distribution of the phase process ($\alpha(D_0 + D_1) = 0$, $\alpha \mathbb{1} = 1$).
- $$R(T_1, T_2) = \frac{E(T_1 T_2) - E(T_1)E(T_2)}{\sqrt{(E(T_1^2) - E^2(T_1))(E(T_2^2) - E(T_2))}},$$

  where $E(T_1) = E(T_2) = \pi(-D_0)^{-1}\mathbb{1}$, $E(T_1^2) = E(T_2^2) = 2\pi(-D_0)^{-2}\mathbb{1}$ and

  $$E(T_1 T_2) = \int_{t=0} t \int_{\tau=0} \tau \pi e^{D_0 t} D_1 e^{D_0 \tau} D_1 \mathbb{1} \, d\tau \, dt = \pi(-D_0)^{-1} P (-D_0)^{-1} \mathbb{1},$$

  where $P = (-D_0)^{-1} D_1$ and π is the stationary distribution of the phase process at arrival instances ($\pi P = \pi$, $\pi \mathbb{1} = 1$).

2/ Define the elements of the G matrix of continuous time QBD processes and summarize the numerical procedures for computing G.

Solution: Let $\{N(t), J(t)\}$ be the level and the phase process and szint és fázis folyamatban γ_n be the first time of reaching level n ($\gamma_n = \min(t > 0 | N(t) = n)$). With this notation the ij element of G is

$$G_{ij} = Pr(J(\gamma_{n-1} = j) | N(0) = n, J(0) = i)$$

Matrix G is the minimal non-negative solution of the matrix quadratic equation $0 = B + LG + FG^2$. In case of a stable QBD matrix G is a stochastic matrix, which is computable by the following procedures.

- Linear "fill up" algorithm:

$$\begin{aligned}&\mathbf{G} := \mathbf{0};\\&\mathbf{REPEAT}\\&\quad \mathbf{G} := \left(-\mathbf{A_1} - \mathbf{A_0}\mathbf{G}\right)^{-1}\mathbf{A_2};\\&\mathbf{UNTIL}\|\mathbb{1} - \mathbf{G}\mathbb{1}\| \le \epsilon\end{aligned}$$

- Linear "distribute" algorithm:

$$\begin{aligned}&\mathbf{G} := \mathbf{I};\\&\mathbf{REPEAT}\\&\quad \mathbf{G}_{old} := \mathbf{G};\\&\quad \mathbf{G} := \left(-\mathbf{A_1} - \mathbf{A_0}\mathbf{G}\right)^{-1}\mathbf{A_2};\\&\mathbf{UNTIL}\|\mathbf{G} - \mathbf{G}_{old}\| \le \epsilon\end{aligned}$$

- logarithmic algorithm:

$$\begin{aligned}&\mathbf{H} := \left(-\mathbf{A_1}\right)^{-1}\mathbf{A_0};\\&\mathbf{L} := \left(-\mathbf{A_1}\right)^{-1}\mathbf{A_2};\\&\mathbf{G} := \mathbf{L};\\&\mathbf{T} := \mathbf{H};\\&\mathbf{REPEAT}\\&\quad \mathbf{U} := \mathbf{HL} + \mathbf{LH};\\&\quad \mathbf{H} := \left(\mathbf{I} - \mathbf{U}\right)^{-1}\mathbf{H}^2;\\&\quad \mathbf{L} := \left(\mathbf{I} - \mathbf{U}\right)^{-1}\mathbf{L}^2;\\&\quad \mathbf{G} := \mathbf{G} + \mathbf{TL};\\&\quad \mathbf{T} := \mathbf{TH};\\&\mathbf{UNTIL}\|\mathbb{1} - \mathbf{G}\mathbb{1}\| \le \epsilon\end{aligned}$$

3/ Summarize the spectral properties (the spectral radius) of the G and R matrices of QBD processes and provide stochastic interpretation.

Solution:

QBD	positive recurrent	null recurrent	transient
$sp(R)$	< 1	1	1
$sp(G)$	1	1	< 1

where $sp()$ denotes the spectral radius.
Matrix R defines a ratio of time spent on level n compared to the time spent on level $n-1$. If the process is positive recurrent then the time spent on level n should be less than the time spent on level $n-1$.

The row sum of matrix G defines probability of reaching level 0 starting from level 1. If the process is positive recurrent then the process returns to level 0 with probability 1. In this case G is a stochastic matrix with $sp(G) = 1$.
In the null recurrent case the process also returns to level 0 with probability 1, but the mean return time infinite. This way $sp(G) = 1$. Due to the fact that the mean return time is infinite the ration of time spent on level n compared to the time spent on level $n-1$ is one, consequently $sp(R) = 1$.

4/ The arrival process of a MAP/M/1 queue is defined by D_0, D_1 and its service intensity is μ.

- *Define the matrix blocks of the QBD description of the queue behaviour.*
- *The stationary phase distribution right after a busy period (when at least one customer is in the system) is α, and the stationary phase distribution right after an idle period is β. α and β satisfy*

$$\alpha = \beta G, \qquad \beta = \alpha(-D_0)^{-1}D1.$$

  Why?
- *Compute α and β.*

Solution:

- $B = \mu I$, $L = D_0 - \mu I$, $F = D_1$, $L' = D_0$.
- The end of the busy period is a jump from level 1 to level 0 and the end of the idle period is a jump from level 0 to level 1.

  Matrix G describes the phase transition during a transition from level 0 to level 1. That is $\alpha = \beta G$.

  The process spends a PH(α, D_0) distributed time at level 0 and after that is jumps to a phase of level 1 according to $D_!$. That is $\beta = \alpha(-D_0)^{-1}D1$.
- α and β are the solutions of the following linear systems of equations:

$$\alpha = \alpha(-D_0)^{-1}D_1G, \quad \alpha\mathbb{1} = 1 \ ,$$

$$\beta = \beta G(-D_0)^{-1}D_1, \quad \beta\mathbb{1} = 1 \ .$$

5/ Define the MAP representation of the output process of the M/PH/1/2 queue with arrival rate λ and service time PH(τ, T).

Solution: The generator of the Markov chain representing the queue behaviour is

$$\mathbf{Q} = \begin{array}{|c|c|c|} \hline -\lambda & \lambda\tau & \\ \hline t & T-\lambda I & \lambda I \\ \hline & t\tau & T \\ \hline \end{array}\,,$$

where $t = -T\mathbb{1}$. The "backward" transitionds of this QBD represents the arrival events of the depature process. That is

$$\mathbf{D_0} = \begin{array}{|c|c|c|} \hline -\lambda & \lambda\tau & \\ \hline & T-\lambda I & \lambda I \\ \hline & & T \\ \hline \end{array}\,, \qquad \mathbf{D_1} = \begin{array}{|c|c|c|} \hline & & \\ \hline t & & \\ \hline & t\tau & \\ \hline \end{array}\,.$$

Exam 3

1/ $X_0, X_1, \ldots$ denote the phase right after arrivals and $T_0 = 0, T_1, \ldots$ denote the arrival instances of a MAP with representation D_0, D_1. Compute

- $Pr(X_1 = j | X_0 = i)$,
- $Pr(T_1 < t | X_0 = i)$,
- *the distribution of the number of arrivals in $(0, t)$ if the initial phase distribution is α,*
- $Pr(X_1 = j, T_1 < t | X_0 = i)$,
- *the stationary inter-arrival time distribution,*
- *the PH representation of the stationary inter-arrival time distribution and interpret the result,*
- *correlation of the first and the second inter-arrival time:*

$$R(T_1, T_2) = \frac{E(T_1 T_2) - E(T_1)E(T_2)}{\sqrt{(E(T_1^2) - E^2(T_1))(E(T_2^2) - E(T_2))}}$$

Solution:

- $Pr(X_1 = j | X_0 = i) = [(-D_0)^{-1} D_1]_{ij}$,
- $Pr(T_1 < t | X_0 = i) = 1 - \epsilon_i e^{D_0 t} \mathbb{1}$, where $\epsilon_i = \{0, \ldots, 0, 1_i, 0, \ldots, 0\}$.
- In z transzform domain: $\hat{P}(z, t) = e^{(D_0 + z D_1)t}$, where $P_{ij}(n, t) = Pr(N(t) = n, J(t) = j | N(0) = 0, J(0) = i)$, and $\hat{P}_{ij}(z, t) = \sum_{n=0}^{\infty} P_{ij}(n, t) z^n$.
- the arrival rate to phase j at time t is $[e^{D_0 t} D_1]_{ij}$, from which

$$Pr(X_1 = j, T_1 < t | X_0 = i) = \int_{\tau=0}^{t} [e^{D_0 \tau} D_1]_{ij} d\tau$$

- The stationary phase distribution at arrivals, α, is the solution of $\alpha = \alpha(-D_0)^{-1} D_1, \alpha \mathbb{1} = 1$. The CDF of the interarrival time distribution is: $1 - \alpha e^{D_0 t} \mathbb{1}$.

- That is the PH representation of the interarrival time distribution is PH(α, D_0).
- $R(T_1, T_2)$ can be computed based on the following relations: $\pi_1 = \pi_0(-D_0)^{-1}D_1$ $E(T_1) = \pi_0(-D_0)^{-1}\mathbb{1}$ $E(T_2) = \pi_1(-D_0)^{-1}\mathbb{1}$, $E(T_1^2) = 2\pi_0(-D_0)^{-2}\mathbb{1}$, $E(T_2^2) = 2\pi_1(-D_0)^{-2}\mathbb{1}$ and
$$E(T_1T_2) = -\sum_i \pi_{0i} \int_{t=0} t \int_{\tau=0} \tau \; [e^{D_0\tau} D_1 \mathbb{1}]_j \; d\tau \; dK_{ij}(t),$$
where $K_{ij}(t) = Pr(X_1 = j, T_1 < t | X_0 = i) = \int_{\tau=0}^{t} [e^{D_0\tau} D_1]_{ij}$.

2/ Describe the transient behaviour of CTMCs based on the embedded behaviour at state transitions.

Solution:

$$\pi_{ij}(t|T_1 = h) = Pr(X(t) = j \mid X(0) = i, T_1 = h) = \begin{cases} \delta_{ij} & h \geq t \\ \displaystyle\sum_{k \in S, k \neq i} \frac{q_{ik}}{-q_{ii}} \pi_{kj}(t-h) & h < t, \end{cases}$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$), and $\dfrac{q_{ik}}{-q_{ii}}$ is the probability that the Markov chain moves from state i to state k. Applying the law of total probability we obtain

$$\begin{aligned} \pi_{ij}(t) &= \int_{h=0}^{\infty} \pi_{ij}(t|T_1 = h) \; (-q_{ii}) \; e^{q_{ii}h} \; dh \\ &= \int_{h=t}^{\infty} \delta_{ij} \; (-q_{ii}) \; e^{q_{ii}h} \; dh + \int_{h=0}^{t} \sum_{k \in S, k \neq i} \frac{q_{ik}}{-q_{ii}} \; \pi_{kj}(t-h) \; (-q_{ii}) \; e^{q_{ii}h} \; dh \\ &= \delta_{ij} \; e^{q_{ii}t} + \int_{h=0}^{t} \sum_{k \in S, k \neq i} \frac{q_{ik}}{-q_{ii}} \; \pi_{kj}(t-h) \; (-q_{ii}) \; e^{q_{ii}h} \; dh \\ &= \delta_{ij} \; e^{q_{ii}t} + \sum_{k \in S, k \neq i} q_{ik} \int_{h=0}^{t} \pi_{kj}(t-h) \; e^{q_{ii}h} \; dh. \end{aligned}$$

4/ In the regular part of a QBD are characterized by B, L, F. Define the condition of stability if
$B = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{bmatrix}$, $L = \begin{bmatrix} \bullet & \mu_2 \\ 0 & \bullet \end{bmatrix}$, $F = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}$.

Solution: The generator of the phase process in the regular part is $Q_J = B + L + F = \begin{bmatrix} -\mu_2 - \lambda_2 & \mu_2 + \lambda_2 \\ \lambda_3 & -\lambda_3 \end{bmatrix}$, from which the stationary phase distributions are $[\frac{\lambda_3}{\lambda_3+\mu_2+\lambda_2}, \frac{\mu_2+\lambda_2}{\lambda_3+\mu_2+\lambda_2}]$ and finally the drift condition is $\frac{\lambda_3}{\lambda_3+\mu_2+\lambda_2}(\lambda_1 + \lambda_2 - \mu_1) + \frac{\mu_2+\lambda_2}{\lambda_3+\mu_2+\lambda_2}(\lambda_3 + \lambda_4 - \mu_1) < 0$.